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Reference-Model Adaptive Control under External Disturbances

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Abstract—A method of constructing an adaptive control for reference-model systems under an unknown bounded external perturbation in the form of sum of an unbounded number of harmonics is designed. This adaptive control ensures that the output of the object tracks the output of the reference model with a given accuracy.

1. INTRODUCTION

The theory of reference-model adaptive control has been developing for the last few decades since the publication of [1, 2] describing the design of an adaptive control algorithm based on the use of a measurable variable and its derivatives. Refinement of this algorithm (for example, in [3]) led to the design of controls without these derivatives. Subsequent works were concerned with the robustness of adaptive control to external disturbance [4], unstructured uncertainty [5], and development of an algorithm for multidimensional systems [6].

Nonmeasurable external disturbance acting on the controlled object may induce large errors in tracking of the output of the reference model by the output of the system. In these papers, control algorithm is designed by the method of Lyapunov functions and does not easily yield to modification so that the tracking error is not greater than a given value.

The method of recurrent objective inequalities [7] is effective in designing adaptive control algorithms for bounded external disturbances. Such an algorithm [8] for reference-model systems ensures a tracking accuracy not greater than a given value dependent on the disturbance bounds.

Using identification of an object and a closed-loop system by the finite-frequency method [9] and synthesis of regulators under external disturbances (sum of an infinite number of harmonics), we design an adaptive control under which tracking error is not greater than a given value.

2. FORMULATION OF THE PROBLEM

Let us consider a minimal-phase system described by the differential equation

$$y^{(n)} + d_{n-1}y^{(n-1)} + \ldots + d_0 y = k_p u^{(p)} + \ldots + k_0 u + f, \quad p < n, \quad t \geq t_0, \tag{1}$$

where $y(t)$, $u(t)$, and $f(t)$ are the measurable output of the system, control, and external disturbance, $y^{(i)}$ and $u^{(j)}$ $(i = \overline{1,n},\ j = \overline{1,p})$ are the derivatives of the output and control, d_i and k_j $(i = \overline{0,n-1},\ j = \overline{0,p})$ are unknown numbers, and n and p are known. For simplicity, we assume that $p = (n-1)$ and the disturbance $f(t)$ is a bounded nonmeasurable paraharmonic function

$$f(t) = \sum_{i=1}^{\infty} f_i \sin\left(\omega_i^f t + \phi_i^f\right), \tag{2}$$

in which ω_i^f and ϕ_i^f $(i = \overline{1,\infty})$ are the unknown frequency and phase, respectively, and the amplitude f_i $(i = \overline{1,\infty})$ is unknown and satisfies the inequality

$$\sum_{i=1}^{\infty} |f_i| \leq f^*, \tag{3}$$

where f^* is a known number.

The desired output of the system is a measurable output $y_m(t)$ of the reference model described by the differential equation

$$y_m^{(n_m)} + d_{m,(n_m-1)} y_m^{(n_m-1)} + \ldots + d_{m,0} y_m = k_{m,p_m} r^{(p_m)} + \ldots + k_{m,0} r, \quad n_m < p_m, \tag{4}$$

where $d_{m,i}$ and $k_{m,j}$ $(i = \overline{0, n_m - 1},\ j = \overline{0, p_m})$ are known numbers and $r(t)$ is a measurable action, which is also a bounded polyharmonic function

$$r(t) = \sum_{i=1}^{\infty} r_i \sin\left(\omega_i^r t + \phi_i^r\right), \tag{5}$$

in which the frequency ω_i^r and phase ϕ_i^r $(i = \overline{1,\infty})$ are unknown, whereas the amplitude r_i $(i = \overline{1,\infty})$ is known and satisfies the inequality

$$\sum_{i=1}^{\infty} |r_i| \leq r^*, \tag{6}$$

where r^* is a known number.

The aim of control $u(t)$ is to force the difference $e(t) = y(t) - y_m(t)$ between the output of system (1) and reference model (4) to satisfy, beginning from a certain instant $t_N > t_0$, the condition

$$|e(t)| \leq e^* + \epsilon(t_N), \qquad t \geq t_N, \tag{7}$$

where e^* is a given number and $\epsilon(t_N)$ is a t_N-dependent number, whose modulus is less than e^*.

Control $u(t)$ is generated by a controller described by the differential equation

$$d_{c,n_c} u^{(n_c)} + d_{c,n_c-1} u^{(n_c-1)} + \ldots + d_{c,0} u = k_{c,p_c} e^{(p_c)} + \ldots + k_{c,0} e, \quad p_c \leq n_c,\ t \geq t_N. \tag{8}$$

Up to instant t_N, control is formed by an adaptive controller described by a differential equation with piecewise constant coefficients

$$\begin{gathered} d_{c,n_c}^{[i]} u^{(n_c)} + d_{c,(n_c-1)}^{[i]} u^{(n_c-1)} + \ldots + d_{c,0}^{[i]} u = k_{c,p_c}^{[i]} e^{(p_c)} + \ldots + k_{c,0}^{[i]} e + v^{[i]}, \\ p_c \leq n_c, t_{i-1} \leq t \leq t_i, \quad i = \overline{1,N}. \end{gathered} \tag{9}$$

In this equation, i $(i = \overline{1,N})$ is the number of the adaptation interval, t_i is the endpoint of the ith interval, the numbers t_i and N are determined in the course of adaptation, and $v^{[i]}(t)$ is a known test signal.

The coefficients of controller (8) upon completion of adaptation at the instant t_N take the values $d_{c,i} = d_{c,i}^{[N]}$ and $k_{c,j} = k_{c,j}^{[N]}$ $(i = \overline{0, n_c}\ j = \overline{0, p_c})$.

Our problem now is to design an adaptation algorithm for the coefficients of controller (9) such that, beginning from a certain instant t_N, the difference between the outputs $e(t)$ of the system and reference model satisfies the tracking accuracy condition (7).

3. CONTROLLER FOR A KNOWN SYSTEM

Assuming that the coefficients of Eq. (1) are known, let us design a controller (8) such that the tracking accuracy condition (7) is satisfied.

Applying the Laplace transformation under zero initial conditions, we find Eqs. (1), (4), and (8) take the form

$$d(s)y = k(s)u + f, \quad d_m(s)y_m = k_m(s)r, \tag{10}$$

$$d_c(s)u = k_c(s)e, \tag{11}$$

where

$$d(s) = s^n + \sum_{i=0}^{n-1} d_i s^i, \quad k(s) = \sum_{i=0}^{p} k_i s^i, \quad d_m(s) = s^{n_m} + \sum_{i=0}^{n_m-1} d_{m,i} s^i,$$

$$k_m(s) = \sum_{i=0}^{p_m} k_{m,i} s^i, \quad d_c(s) = \sum_{i=0}^{n_c} d_{c,i} s^i, \quad k_c(s) = \sum_{i=0}^{p_c} k_{c,i} s^i.$$

Multiplying the first equation in (10) by $d_m(s)$ and the second equation by $d(s)$, and then subtracting them, we obtain the equation for the *extended* system. It describes the difference between the motions of system (1) and reference model (4):

$$\widetilde{d}(s)e = \widetilde{k}(s)u + h(s)r + d_m(s)f, \tag{12}$$

where

$$\widetilde{d}(s) = d(s)d_m(s), \quad \widetilde{k}(s) = k(s)d_m(s), \quad h(s) = -d(s)k_m(s).$$

Eliminating the variable $u(t)$ from the controller Eq. (11), we obtain the equation of the closed-loop system

$$d_z(s)e = h_z(s)r + m_z(s)f, \tag{13}$$

where

$$d_z(s) = \widetilde{d}(s)d_c(s) - \widetilde{k}(s)k_c(s), \quad h_z(s) = d_c(s)h(s), \quad m_z(s) = d_c(s)d_m(s). \tag{14}$$

Let us rewrite Eq. (13) as

$$e = T_{er}(s)r + T_{ef}(s)f, \tag{15}$$

where the transfer functions interconnecting tracking error with defining and perturbing actions are of the form

$$T_{er}(s) = \frac{h_z(s)}{d_z(s)} = \frac{d_c(s)k_m(s)d(s)}{d_m(s)[d(s)d_c(s) - k(s)k_c(s)]}, \tag{16}$$

$$T_{ef}(s) = \frac{m_z(s)}{d_z(s)} = \frac{d_c(s)d_m(s)}{d_m(s)[d(s)d_c(s) - k(s)k_c(s)]}. \tag{17}$$

The ratio of numerators of these transfer functions satisfies the inequality

$$\frac{|k_m(j\omega)d(j\omega)|}{|d_m(j\omega)|} \geq \frac{f^*}{r^*}, \quad 0 \leq \omega < \infty, \tag{18}$$

or the inequality

$$\max_{0\le\omega<\infty} \frac{|k_m(j\omega)d(j\omega)|}{|d_m(j\omega)|} \le \frac{f^*}{r^*}. \tag{19}$$

Let the system and reference model be such that inequality (18) holds. Consider controller (11) with the polynomials

$$d_c(s) = k(s)d_m(s), \quad k_c(s) = d(s)d_m(s) - \delta_1(s), \tag{20}$$

where $\delta_1(s)$ is the Hurwitz polynomial defined by the identity

$$\delta_1(-s)\delta_1(s) = d(-s)d(s)[d_m(-s)d_m(s) + q_{11}k_m(-s)k_m(s)], \tag{21}$$

in which q_{11} is a positive number.

Assertion 3.1. *If system* (1) *and reference model* (4) *have property* (18) *and the coefficient* q_{11} *in identity* (21) *satisfies the condition*

$$q_{11} \ge \frac{4r^{*2}}{e^{*2}}, \tag{22}$$

then the difference between the outputs of the system (*closed by controller* (11), (20)) *and reference model satisfies the tracking accuracy condition* (7).

The proof of this assertion is given in the Appendix.

Assuming that inequality (19) holds, let us consider controller (11) with polynomials

$$d_c(s) = k(s), \quad k_c(s) = d(s) - \delta_2(s), \tag{23}$$

where $\delta_2(s)$ is the Hurwitz polynomial defined by the identity

$$\delta_2(-s)\delta_2(s) = d(-s)d(s) + q_{22}, \tag{24}$$

in which q_{22} is a positive number.

Assertion 3.2. *If a system and its reference model have property* (19) *and the coefficient* q_{22} *in identity* (24) *satisfies the condition*

$$q_{22} \ge \frac{4f^{*2}}{e^{*2}}, \tag{25}$$

then the difference between the outputs of the system (*closed by controller* (11), (23)) *and its reference model satisfies the tracking accuracy condition* (7).

The proof of the assertion is given in the Appendix.

In the sequel, we refer to $\delta_1(s)$ and $\delta_2(s)$ as *factorized* polynomials.

Note that for polynomials (20) and (23) of the controller and the polynomials of the equation of the closed-loop system (13) are of the structure

$$d_z(s) = k(s)d_m(s)\delta_1(s), \quad h_z = k(s)d_m(s)k_m(s)d(s), \tag{26}$$

or

$$d_z(s) = k(s)d_m(s)\delta_2(s), \quad h_z = k(s)k_m(s)d(s). \tag{27}$$

4. THE FIRST ADAPTATION INTERVAL (IDENTIFICATION OF SYSTEM)

A Solution Approach

Assertions 3.1 and 3.2 are helpful in designing a controller that solves our problem if the coefficients of system (1) are known. If unknown, they must be identified and polynomials (20) or (23) of the controller must be found using the estimates of these coefficients. Since the output of the system is not compared with the output of the reference model in identification, identification time must be minimal. For this purpose, in the first adaptation interval, we must find rough estimates for the coefficients of the system that are adequate for designing a stabilizing controller (controller ensuring the asymptotic stability of the system). In the second and subsequent adaptation intervals, identification of the system is continued, but with the difference between the outputs of the reference model and system closed by a controller constructed at the previous adaptation interval.

Fourier Filter and Frequency Identification Equations

Adaptation algorithms based on finite-frequency identification are described in [9, 10], in which adaptation aim is to design a model control. The aim, expressed by inequality (7), leads to certain properties of the algorithm, though the algorithm, its parameters, and convergence conditions are preserved. Therefore, below we briefly describe the adaptation algorithm, mentioning its specific properties induced by aim (7).

Let us describe the Fourier filter and frequency identification equations underlying the design of the algorithm. Let us consider a *generalized* system described by the differential equation

$$\eta^{(\gamma)} + \theta_1\eta^{(\gamma-1)} + \ldots + \theta_\gamma\eta = \theta_{\gamma+1}u^{(\nu)} + \ldots + \theta_{\gamma+\nu+1}u + f, \tag{28}$$

where $\theta = [\theta_1, \ldots, \theta_{\gamma+\nu+1}]$ is a vector of unknown coefficients and $\eta(t)$ is the measurable output. These equations are the same as Eq. (1) if $\gamma = n$, $\nu = p$, $\eta = y$, $\theta_i = d_{n-i}$ $(i = \overline{1,\gamma}, \gamma = n)$, and $\theta_{\gamma+i+1} = k_{\nu-i}$ $(i = \overline{0,\nu}, \nu = p)$.

The Fourier filter is of the form

$$\alpha_k^\eta(\tau) = \frac{2}{\rho_k\tau}\int\limits_{t_F}^{t_F+\tau} e^{\lambda(t-t_0)}\eta(t)\sin\omega_k(t-t_0)dt, \tag{29}$$

$$\alpha_{\gamma+k}^\eta(\tau) = \frac{2}{\rho_k\tau}\int\limits_{t_F}^{t_F+\tau} e^{\lambda(t-t_0)}\eta(t)\cos\omega_k(t-t_0)dt \quad (k = \overline{1,\gamma}),$$

where ρ_k, ω_k $(k = \overline{1,\gamma})$, and λ are given positive numbers (amplitude, frequency, and exponent of the test signal), which can be found experimentally [11]; τ and t_F are numbers that take the values $\tau = qT_b$, $t_F = \tilde{q}T_b$, $T_b = \dfrac{2*\pi}{\omega_b}$, $\omega_b = \min(\omega_1, \ldots, \omega_\gamma)$; $q = 1, 2, \ldots$, $\tilde{q}$ is a given number, and τ and t_F are called the filtering time and filtering start instant, respectively. In the sequel, we use the vector $\alpha^\eta(\tau) = [\alpha_1^\eta(\tau), \ldots, \alpha_{2\gamma}^\eta(\tau)]$.

Frequency identification equations are of the form

$$-[\alpha_k^\eta(\tau) + j\alpha_{\gamma+k}^\eta(\tau)]\sum_{i=1}^{\gamma} s_k^{\gamma-i}\theta_i(\tau) + \sum_{i=0}^{\nu} s_k^{\nu-i}\theta_{\gamma+i+1}(\tau) = [\alpha_k^\eta(\tau) + j\alpha_{\gamma+k}^\eta(\tau)]s_k^\gamma, \tag{30}$$
$$s = \lambda + j\omega_k, k = \overline{1,\gamma},$$

where $\theta_i(\tau)$, $i = \overline{1, \gamma+\nu+1}$, are the estimates of the coefficients of system (28).

Identification process terminates at the instant $\overline{t} = \overline{q}T_b$ when the necessary conditions for its convergence

$$\frac{|\theta_i(qT_b) - \theta_i[(q-1)T_b]|}{|\theta_i[(q-1)T_b]|} \leq \epsilon_i^\theta, \quad \theta_i[(q-1)T_b]| \neq 0 \quad (i = \overline{1, \gamma + \nu + 1}), \tag{31}$$

are satisfied, where ϵ_i^θ $(i = \overline{1, \gamma + \nu + 1})$ are sufficiently small numbers.

The exponent $\lambda \geq 0$ must satisfy the condition

$$\lambda \geq \max(Re\overline{s}_1, \ldots, Re\overline{s}_\gamma), \tag{32}$$

where $\overline{s}_i$, $i \subset \overline{1, \gamma}$, are the roots of the polynomial $\theta(s) = s^{(\gamma)} + \sum\limits_{i=1}^{\gamma-1} \theta_i s^i$ with positive real part. The number λ can be experimentally determined. If system (28) is asymptotically stable, then $\lambda = 0$.

Identification of a system. The aim in the first interval is to identify system (1) with some accuracy that aids in finding the stabilizing controller (9) for the second adaptation interval.

This aim is achieved by **Procedure 4.1** consisting of the following operations:

(a) System (1) is excited by a test signal

$$u(t) = \exp^{\lambda(t-t_0)} \sum_{k=1}^{n} \rho_k \sin \omega_k (t - t_0) \tag{33}$$

of given amplitude ρ_k, frequency ω_k $(k = \overline{1, n})$, and exponent λ satisfying condition (32).

(b) The output of system (1) is applied to the inputs of the Fourier filter (29), where $\gamma = n$. The filter output is the vector $\alpha^y(\tau)$ of estimates of the frequency parameters of the system at the instants $\tau = qT_b$, $q = 1, 2, \ldots$.

(c) For every τ, the frequency Eqs. (30), where $\gamma = n$, $\nu = p$, and $\alpha^\eta(\tau) = \alpha^y(\tau)$, are solved to find the vector $\theta^{[1]}(\tau) = [d_{n-1}^{[1]}(\tau), \ldots, d_0^{[1]}(\tau), k_p^{[1]}(\tau), \ldots, k_0^{[1]}(\tau)]$ of estimates of the coefficients of system (1). These coefficients are used in verifying the necessary conditions (31). Let these conditions hold at instant $\tau_1 = q_1 T_b$. Then the polynomials of the identified system are formed

$$d^{[1]}(s) = s^n + \sum_{i=0}^{n-1} d_i^{[1]}(\tau_1) s^i, \quad k^{[1]}(s) = \sum_{i=0}^{p} k_i^{[1]}(\tau_1) s^i. \tag{34}$$

(d) Verification of conditions (18) and (19). Depending on which of them is satisfied, either polynomials (20) or (23) for the controller are formed:

$$d_c^{[2]}(s)u = k_c^{[2]}(s)e + v^{[2]}, \quad t \geq t_1, \quad t_1 = t_0 + \tau_1. \tag{35}$$

In this equation,

$$d_c^{[2]}(s) = k^{[1]}(s) d_m(s), \quad k_c^{[2]}(s) = d^{[1]}(s) d_m(s) - \delta_1^{[1]}(s), \tag{36}$$

or

$$d_c^{[2]}(s) = k^{[1]}(s), \quad k_c^{[2]}(s) = d^{[1]}(s) - \delta_2^{[1]}(s) \tag{37}$$

and $v^{[2]}(t) = \sum\limits_{k=1}^{n_z} \rho_k^{[2]} \sin \omega_k (t - t_1)$, where $\rho_k^{[2]}$ and $\omega_k^{[2]}$ $(k = \overline{1, n_z})$ are given numbers.

The polynomials $\delta_1^{[1]}(s)$ and $\delta_2^{[1]}(s)$ are determined from formulas (21) and (24) for $d(s) = d^{[1]}(s)$ and $k^(s) = k^{[1]}(s)$, respectively.

(e) The *characteristic polynomial* of the closed-loop system

$$\check{d}_z^{[2]}(s) = d^{[1]}(s)d_m(s)d_c^{[2]}(s) - k^{[1]}(s)d_m(s)k_c^{[2]}(s) \tag{38}$$

is formed. It, like polynomials (26) and (27), has the structure $\check{d}_z^{[2]}(s) = k^{[1]}(s)d_m(s)\delta_1^{[1]}(s)$ or $\check{d}_z^{[2]}(s) = k^{[1]}(s)d_m(s)\delta_2^{[1]}(s)$.

5. SECOND ADAPTATION INTERVAL

Identification of a Closed-Loop System and Adaptation Termination Conditions

The aim in the second adaptation interval is to identify system (1) exactly when it is closed by the stabilizing controller (35). Then, the estimates thus obtained are used to design a new controller that ensures the tracking accuracy condition (7) or a close condition.

In this interval, system (1) is closed by controller (35). System (1), (4), (35) is described, upon elimination of the variable $u(t)$, by an equation of the type (13)

$$d_z^{[2]}(s)e = k(s)d_m(s)v^{[2]} + \widetilde{f}^{[2]}, \quad t \geq t_1, \tag{39}$$

where $\widetilde{f}^{[2]} = -d_c^{[2]}(s)d(s)k_m(s)r + d_c^{[2]}(s)d_m(s)f$ and

$$d_z^{[2]}(s) = d(s)d_m(s)d_c^{[2]}(s) - k(s)d_m(s)k_c^{[2]}(s). \tag{40}$$

Polynomial (40) contains the coefficients of system (1), whereas polynomial (38) contains the estimates of these coefficients.

Since system (1), (4), (35) is assumed to be asymptotically stable (unstable case is described in the next section), $\lambda = 0$ in the test signal $v^{[2]}$.

"System" (39) is identified with help of procedure 4.1.

Operation (b) of procedure produces the vector $\alpha^{e[2]}(\tau)$, called the estimates for the frequency parameters of the closed-loop system. Using these estimates, we can easily compute new vectors $\alpha^{y[2]}(\tau)$ of estimates for the frequency parameters of system (1) by the formulas of [10].

Operation (c) is implemented twice: first for the vectors $\alpha^{e[2]}(\tau)$ and then for the vectors $\alpha^{y[2]}(\tau)$. Thus we obtain the vectors

$$\theta^{[2]}(\tau) = \left[d_{n-1}^{[2]}(\tau), \ldots, d_0^{[2]}(\tau), k_p^{[2]}(\tau), \ldots, k_0^{[2]}(\tau)\right], \tag{41}$$

$$\theta_z^{[2]}(\tau) = \left[d_{z,n_z}^{[2]}(\tau), \ldots, d_{z,0}^{[2]}(\tau)\right]. \tag{42}$$

The second adaptation interval terminates at the instant $t_2 = t_1 + \tau_2$ when the following conditions are satisfied:

(i) the necessary conditions (32) for vectors (41) and (42),

(ii) inequality

$$\tau_2 \geq \tau_1 + \tau^*, \tag{43}$$

where τ^* is a given number,

(iii) the condition

$$\frac{|\check{d}_{z,i}^{[2]} - d_{z,i}^{[2]}(\tau_2)|}{|\check{d}_{z,i}^{[2]}|} \leq \epsilon_{z,i} \quad (i = \overline{0, n_z - 1}), \quad \check{d}_{z,i}^{[2]} \neq 0, \tag{44}$$

for the closeness of the coefficients of the expected and identified characteristic polynomials of the system, where $\epsilon_{z,i}$ $(i = \overline{0, n_z - 1})$ are sufficiently small given numbers, and

(iv) objective condition (7).

If inequalities (44) and (7) hold, then the adaptation process terminates: $N = 2$ and controller (35) (for $v^{[2]}(t) = 0$, $t \geq t_2$) is the unknown controller (8).

Otherwise, operation (d), in which the vector $\theta^{[2]}(\tau_2)$ is used, gives controller polynomials $d_c^{[3]}(s)$ and $k_c^{[3]}(s)$ for the third adaptation interval, and operation (e) generates a new expected characteristic polynomial, etc.

Note that aim (7) may not attained even in large adaptation intervals, because inequalities (31) and (44) defining the duration of adaptation intervals contain the numbers ϵ_i^{θ} $(i = \overline{1, \gamma + \nu + 1})$ and $\epsilon_{z,i}$ $(i = \overline{0, n_z - 1})$. Therefore, these numbers must be reduced if such a situation arises, and reduction of numbers is repeated until the aim is attained.

Refinement of Conditions (44) *for the Closeness of Polynomials of the System*

There is an important fact in identification of "system" (39), viz., its transfer function interconnecting the tracking error with test signal contains reducible polynomials (though system (1) does not contain such polynomials) and reducible polynomials cannot be identified since the finite-frequency identification method is based on the values of this transfer function at test frequencies.

Let us consider two types of such polynomials.

First, the polynomial $d_m(s)$, according to (40), is contained in both parts of Eq. (39) of the system. To avoid this known polynomial from identification, we must take $\nu = p$ and $\gamma = (n + n_m + p)$ (for the controller with polynomial (36)) or $\gamma = (n + p)$ (for the controller with polynomial (37)) in the frequency Eqs. (30).

Second, system (1) and the reference model (4) may be such that the polynomials $k(s)$ and $d_m(s)$ have a common unknown polynomial factor $l(s)$ of unknown degree n_l. According to (39) and (40), this polynomial for the controller with polynomials (36) is reducible. In this case, the determinant in frequency Eqs. (30) is zero. Reducing γ and ν in these equations by one and computing their determinant (if it is nonzero), we obtain $n_l = 1$. Otherwise, we once again reduce γ and ν by one until we obtain γ and ν for which the determinant is nonzero. In real computations, condition number of the matrix of frequency equations is used instead of this determinant.

Third, the identified and true polynomials $k^{[1]}(s)$ and $k(s)$ may have a common unknown polynomial factor $q(s)$. In this case, γ and ν are to be reduced as described above.

To avoid the determination of the degrees of the polynomials $l(s)$ and $q(s)$, we take $\nu = 0$ and $\gamma = n + n_m$ (or $\gamma = n$) in the frequency Eqs. (30). Solving them, we obtain the factorized polynomials $\delta_1(s)$ (or $\delta_2(s)$). Then the closeness condition (44) is replaced by the condition for the closeness of the assumed and identified factorized polynomials

$$\frac{|\delta_{1,j}^{[i]} - \delta_{1,j}^{[i]}(\tau_i)|}{|\delta_{1,j}^{[i]}|} \leq \epsilon_{1,j}, \quad \text{or} \quad \frac{|\delta_{2,k}^{[i]} - \delta_{2,k}^{[i]}(\tau_i)|}{|\delta_{2,k}^{[i]}|} \leq \epsilon_{2,k}, \quad \delta_{1,j}^{[i]} \neq 0, \quad \delta_{2,k}^{[i]} \neq 0$$
$$(i = \overline{2, N}, \quad j = \overline{1, n + n_m}, \quad k = \overline{1, n}), \tag{45}$$

where $\epsilon_{1,j}$ and $\epsilon_{2,k}$ $(j = \overline{1, n + n_m}$, $k = \overline{1, n})$ are sufficiently small given numbers.

6. CONVERGENCE OF THE ADAPTATION PROCESS

Asymptotically Stable Closed-Loop System

For the adaptation process of an asymptotically stable system to converge (for the closeness conditions (44) and objective condition (7) to hold), it is sufficient that the adaptation intervals be extensible and the perturbation $f(t)$ be strictly filterable by a Fourier filter [10], because any accuracy of identification of system (1) can be attained under these conditions.

The first condition implies that adaptation interval lengths satisfy the inequality

$$\tau_i \geq \tau_{i-1} + \tau^* \quad (i = \overline{2, N}), \tag{46}$$

where τ^* is a given number. The condition of strict Fourier filterability implies that the test signal, external disturbance, and defining signal be not of the same frequency:

$$\omega_i \neq \omega_k^f, \quad \omega_i \neq \omega_j^r \quad (i \neq k, \quad i \neq j, \quad k = \overline{1, \infty}, \quad j = \overline{1, \infty}). \tag{47}$$

These inequalities can be experimentally verified [10].

If the assumption that the closed-loop is asymptotically stable on all adaptation intervals is satisfied, then conditions (46) and (47) are sufficient for the adaptation process to converge.

Unstable Closed-Loop System

Let system (39) be unstable on the second adaptation interval. After verifying this fact experimentally, controller (35) is disconnected and operations of the first interval are repeated on the third interval, which, according to (46), has a large length. The stability of the closed-loop system on the next adaptation interval is verified. If it is unstable, the controller is disconnected and this process is repeated until such a duration of identification of system (1) is obtained, in which an identification accuracy sufficient for finding the stabilizing controller is attained.

It is not a simple matter to realize this algorithm, because there are constraints on the permissible inputs and outputs of system (1)

$$|y(t)| \leq y^*, \quad |u(t)| \leq u^*, \quad t \geq t_0, \tag{48}$$

where y^* and u^* are given numbers.

In this case, **Procedure 6.1** consisting of the following operations is constructed.

(a1) Procedure 4.1, which terminates at instant t_1 when the limiting value $|y(t_1)| = y^*$ or $|u(t_1)| = u^*$ is attained.

(b1) The degree $\lambda^{[2]}$ of instability of "system" (39) is determined in the second adaptation interval, which consists of several subintervals in which $\lambda^{[2]}$ is determined by a simple algorithm (not described here).

(c1) If $\lambda^{[2]} \neq 0$, then the operations described in Section 4 are implemented for $\lambda = \lambda^{[2]}$ in the third adaptation interval. Operation (c) terminates when the limiting permissible value of y^* or u^* is attained, and operation (d) generates a controller for the next adaptation interval, in which the degree of instability of the closed-loop system is determined once again. This process is repeated until a stabilizing controller ($\lambda^{[N_1+1]} = 0$) is obtained in some N_1th adaptation interval.

Termination of the Adaptation Process

Let us determine the termination instant t_N of the adaptation process. Every adaptation interval on which the system is closed by a stabilizing controller begins with the verification of the objective condition (7) in a certain time. For example, verification in the second adaptation interval is implemented on the subinterval $[t_1, t_1+t_2^*]$, where t_2^* is a given positive number. Let $f(t) = r(t) = 0$ in this subinterval. These time functions are bounded at other instants. Hence functions (2) and (5) can be expanded as Taylor series on the interval $[t_0, t^*]$ (t^* is the instant when the operation of the system terminates). Since $f(t) = r(t) = 0$ ($[t_1 \le t \le t_1 + t_2^*]$), aim (7) is attained with any stabilizing controller. If conditions (i)–(iv) hold at some instant $t_2 = t_1+t_2^*+\tau_2$, then the adaptation process also terminates. But the objective condition (7) may be violated at some instant $t > t_2$. Then operation (d) is implemented, which generates controller polynomials for the third adaptation interval. This implies that adaptation intervals may be of the form $t_i - t_{i-1} = t_i^* + \tau_i + t_i^{**}$ ($i \ge 2$), where $t_{i-1} + t_i^* + \tau_i$ is the instant at which adaptation termination conditions were satisfied and t_i is the instant at which the objective condition (7) was violated. Hence the objective condition (7) is not violated, beginning from the instant t_N. Hence we have

Assertion 6.1. *Adaptation process converges if the expandability condition* (46) *for adaptation intervals holds, test frequencies satisfy inequalities* (47)*, and an adaptation interval* N_1 *yielding a stabilizing controller under constraints* (48) *exists.*

7. AN EXAMPLE

Let us consider the minimal-phase system described by the equation

$$\ddot{y} + d_1\dot{y} + d_0 y = k_1\dot{u} + k_0 u + f, \tag{49}$$

where d_1, d_0, k_1, and k_0 are unknown numbers, $f(t)$ is a polyharmonic disturbance of the type (2), and the sum of its amplitudes is bounded by $f^* = 5$.

The reference model is described by the equation

$$\ddot{y_m} + 5\dot{y_m} + 6y_m = \dot{r} + r, \tag{50}$$

where the polyharmonic signal of the type (5) is bounded by $r^* = 40$.

Our problem now is to find the coefficients of the controller

$$(d_{c,3}s^3 + d_{c,2}s^2 + d_{c,1}s + d_{c,0})u = (k_{c,3}s^3 + k_{c,2}s^2 + k_{c,1}s + k_{c,0})e, \tag{51}$$

which ensures, beginning from a certain instant t_N, that the difference between the outputs of the system and reference model satisfies the condition

$$|y(t) - y_m(t)| \le 1, \quad t \ge t_N. \tag{52}$$

Remark 7.1. In the experimental results given below, $d_0 = -1$, $d_1 = 0$, $k_1 = 1$, $k_0 = 2$, $f(t) = 5\cos 4.6t$, and $r(t) = 20\cos 2.5t + 20\cos 5t$ are the values of coefficients of the system, external disturbance, and defining signal drawn from [4].

By Eqs. (49) and (50) of the system and reference model, inequalities (18) hold. Therefore, q_{11} can be determined from formula (22).

Below we give the results obtained in the course of adaptation by controller (51) (only the operations a, b, etc. of Procedure 4.1 are shown).

First adaptation interval. (a) system (49) was excited by the test signal $u(t) = 0.1\exp(1,1t)$ $(\sin 2t + \sin 4t)$,

(b)–(c) the estimates

$$d_1^{[1]} = 0.566, \quad d_0^{[1]} = -6, \quad k_1^{[1]} = 3.06, \quad k_0^{[1]} = -1.75 \tag{53}$$

of its coefficients were generated at the instant $\tau_1 = 2T_b$ $\left(T_b = \frac{2\pi}{2}\right)$,

(d) the controller for the second adaptation interval constructed for $q_{11} = 10$ is of the form

$$(3.06s^3 + 13.5s^2 + 9.61s - 10.5)u = -(11s^3 + 72.4s^2 + 154s + 106)e + v^{[2]}, \tag{54}$$

(e) the factorized polynomial of system (49), (50), (54) is

$$\delta_1^{[2]}(s) = (s^4 + 16.6s^3 + 72.5s^2 + 127s + 70). \tag{55}$$

Second adaptation interval. (a) System (49), (59) was excited by the test signal $v^{[2]}(t) = 10^3(\sin 2t + \sin 4t + \sin 5t + \sin 8t + \sin 10t)$,

(b)–(c) estimates

$$\delta_{1,4}^{[2]} = 1,\ \delta_{1,3}^{[2]} = 6.39,\ \delta_{1,2}^{[2]} = 21.1,\ \delta_{1,1}^{[2]} = 50.9,\ \delta_{1,0}^{[2]} = 35.5 \tag{56}$$

$$d_1^{[2]} = -0.126,\ d_0^{[2]} = -0.89,\ k_1^{[2]} = 1.01,\ k_0^{[2]} = 1.87 \tag{57}$$

for the coefficients of the factorized polynomial and system were generated at the instant τ_2.

The coefficients of polynomial (55) and identified coefficients (56) were used in verifying closeness conditions (45) for $\epsilon_{1,j} = 0.5$ ($j = \overline{1,4}$). These conditions were violated. Moreover, modeling of system (49), (50), (54) has shown that tracking accuracy condition (52) was also violated. Therefore adaptation was continued.

(d) The controller for the third adaptation interval constructed with estimates (56) for $q_{11} = 100$ is of the form

$$(1.01s^3 + 6.95s^2 + 15.4s + 11.2)u = -(98s^3 + 287s^2 + 284s + 94.5)e + v^{[3]}. \tag{58}$$

(e) The factorized polynomial of system (49), (50), (58) is

$$\delta_1^{[3]}(s) = (s^4 + 102s^3 + 292s^2 + 279s + 89.2). \tag{59}$$

Third adaptation interval. (a) System (49), (50), (58) was excited by a test signal $v^{[3]} = v^{[2]}$.

(b)–(c) Estimates

$$\delta_{1,4}^{[3]} = 1, \quad \delta_{1,3}^{[3]} = 148, \quad \delta_{1,2}^{[3]} = 418, \quad \delta_{1,1}^{[3]} = 407, \quad \delta_{1,0}^{[3]} = 121 \tag{60}$$

for the coefficients of the factorized polynomial were generated at the instant $\tau_3 = \tau_2$.

Comparing them with the coefficients of polynomial (59), we find that the closeness conditions (45) are satisfied. Modeling of system (49), (50), (58) has shown that tracking accuracy condition (52) is also satisfied. Consequently, the unknown controller is of the form (58), where $v^{[3]} = 0$.

APPENDIX

Proof of Assertion 3.1. Let us express the objective condition (7) in terms of the parameters of system (10), (11). For this purpose, let us write relation (15) as $e = e_r + e_f$, where $e_r = T_{er}(s)r$ and $e_f = T_{ef}(s)f$.

Using expressions (2) and (5) for the functions $r(t)$ and $f(t)$, we can write

$$e_r(t) = \sum_{i=1}^{\infty} a_r(\omega_i^r)\sin(\omega_i^r t + \varphi_i^r), \quad e_f(t) = \sum_{i=1}^{\infty} a_f(\omega_i^f)\sin(\omega_i^f t + \varphi_i^f) \quad \text{as} \quad t \to \infty, \tag{61}$$

where $a_r(\omega_i^r) = |T_{er}(j\omega_i^r)|r_i$ and $a_f(\omega_i^f) = |T_{ef}(j\omega_i^f)|f_i$ ($i = \overline{1,\infty}$).

Obviously,

$$|e_r(t)| \le \sum_{i=1}^{\infty} |T_{er}(j\omega_i^r)||r_i|, \quad |e_f(t)| \le \sum_{i=1}^{\infty} |T_{ef}(j\omega_i^f)||f_i|. \tag{62}$$

By virtue of bounds (3) and (6), the objective condition (7) can be expressed as

$$|e(t)| \le |e_r(t)| + |e_f(t)| \le \|T_{er}\| r^* + \|T_{ef}\| f^* \le e^*, \tag{63}$$

where $\|T_{er}\| = \max\limits_{0 \le \omega < \infty} |T_{er}(j\omega)|$ and $\|T_{ef}\| = \max\limits_{0 \le \omega < \infty} |T_{ef}(j\omega)|$. The numbers $\|T_{er}\|$ and $\|T_{ef}\|$ are the H_∞-norms of the transfer functions $T_{er}(s)$ and $T_{ef}(s)$, respectively.

For the objective condition (7) to hold, it is sufficient that

$$\text{(j)} \quad \|T_{er}\| \le \frac{e^*}{2r^*}, \quad \text{(jj)} \quad \|T_{ef}\| \le \frac{e^*}{2f^*}. \tag{64}$$

Let us verify that controller (11), (20) ensures condition (j). Substituting polynomials (20) into expression (16) for the transfer function $T_{er}(s)$, we obtain

$$T_{er}(s) = \frac{k_m(s)d(s)}{\delta_1(s)}. \tag{65}$$

By virtue of identity (21) and inequality $d_m(-j\omega)d_m(j\omega) \ge 0$,

$$T_{er}(-j\omega)T_{er}(j\omega) = \frac{k_m(-j\omega)k_m(j\omega)}{d_m(-j\omega)d_m(j\omega) + q_{11}k_m(-j\omega)k_m(j\omega)} \le \frac{1}{q_{11}}. \tag{66}$$

Hence, by virtue of (22), $\|T_{er}\| \le \dfrac{1}{\sqrt{q_{11}}} \le \dfrac{e^*}{2r^*}$.

Now we verify that controller (11), (20) ensures condition (jj). Substituting polynomials (20) into expression (17) for the transfer function $T_{ef}(s)$, we obtain $T_{ef}(s) = \dfrac{d_m(s)}{\delta_1(s)}$. By virtue of condition (18) and inequality $d_m(-j\omega)d_m(j\omega) \ge 0$,

$$T_{ef}(-j\omega)T_{ef}(j\omega) = \frac{d_m(-j\omega)d_m(j\omega)}{q_{11}d(-j\omega)d(j\omega)k_m(-j\omega)k_m(j\omega)} \le \frac{1}{q_{11}\frac{f^{*2}}{r^{*2}}}. \tag{67}$$

Hence, by virtue of (22), $\|T_{ef}\| \le \dfrac{e^*}{2f^*}$.

Proof of Assertion 3.2. Let us verify that controller (11), (23) ensures condition (jj). Substituting polynomials (23) into expression (17) for the transfer function $T_{ef}(s)$, we obtain

$$T_{ef}(s) = \frac{1}{\delta_2(s)}. \tag{68}$$

By virtue of identity (24) and inequality $d(-j\omega)d(j\omega) \ge 0$,

$$T_{ef}(-j\omega)T_{ef}(j\omega) = \frac{1}{d(-j\omega)d(j\omega) + q_{22}} \le \frac{1}{q_{22}}. \tag{69}$$

Now, by virtue of (25), $\|T_{ef}\| \le \dfrac{1}{\sqrt{q_{22}}} \le \dfrac{e^*}{2f^*}$.

Let us verify that controller (11), (23) ensures condition (j). Substituting polynomials (23) into the expression for the transfer function $T_{er}(s)$, we obtain

$$T_{er}(s) = \frac{k_m(s)d(s)}{d_m(s)\delta_2(s)}.$$

By virtue of condition (19) and inequality $d(-j\omega)d(j\omega) \geq 0$,

$$T_{er}(-j\omega)T_{er}(j\omega) = \frac{k_m(-j\omega)k_m(j\omega)d(-j\omega)d(j\omega)}{q_{22}d_m(-j\omega)d_m(j\omega)} \leq \frac{f^{*2}}{q_{22}r^{*2}}. \quad (70)$$

Hence, by virtue of (25), $\|T_{er}\| \leq \dfrac{e^*}{2r^*}$.

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