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ADAPTIVE AND ROBUST SYSTEMS

Comparison of the Two Methods of Identification under Unknown-but-Bounded Disturbances

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Abstract—Considered are stable linear SISO plants subjected to unknown-but-bounded disturbances. For such plants, the potentials of the two methods of active identification are compared—the instrumental variable method and the finite-frequency identification method. The relation between these methods is established in the situation where the measurable input is represented by a test signal in the form of the sum of harmonics, whose number is equal to the state space dimension of the plant. The advantages of the finite-frequency method are twofold: it reduces both the sensitivity of the estimates of the plant coefficients to the errors in experimental data and the effect of the dependence between the measurable input and exogenous disturbance on the accuracy of identification. These extra capabilities are provided by the self-tuning of the frequencies of the test signal.

1. INTRODUCTION

At present, the control theory has at its disposal a number of identification methods for plants specified by linear differential equations. Conventionally, these methods fall into two categories depending on the assumptions on the measurement errors and exogenous disturbances affecting the plant.

The methods of the first class deal with the plants subjected to disturbances of the stochastic nature; i.e., random processes having known statistical characteristics. These are various versions of the method of least squares and the stochastic approximation method; e.g., see well-known monographs [1, 2].

The second class comprises the identification methods under unknown-but-bounded disturbances (whose statistical properties are not known) such as randomized algorithms of [3, 4] and finite-frequency identification, see [5].

A somewhat specific position is occupied by the method of instrumental variables, [6, 7]. It is developed in the framework of the first class; however, in contrast to the other methods in the class, it is applicable to the problems of the second class so that it is reasonable to consider it as a method of that second category.

The identification process can have passive or active forms. In the *passive* identification, the measured input to the plant has the meaning of a control action which depends on the control objectives and is not related to identification of the plant. With such an input, identification might not be possible; hence, *active* identification is often practiced where, in addition to control, the measured input contains an extra component, a so-called test signal aimed at identifying the plant.

The finite-frequency identification method was designed for the needs of active identification. The test signal is represented by the sum of harmonics with automatically tuned (self-tuned) amplitudes and frequencies where the number of harmonics does not exceed the state space dimension of the plant. The self-tuning of amplitudes is carried out to satisfy those requirements on the bounds on the input and output which hold true in the absence of a test signal.

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It is instructive to compare the capabilities of the finite-frequency identification method with those of the other methods in the second class, which are more general and can be used both for passive and active identification. In the randomized algorithms, the test signal is assumed to be a random process with known statistical characteristics; hence, the prespecified tolerances on the plant outputs are hard to guarantee. Accordingly, we will compare the capabilities of the finite-frequency identification method and instrumental variable method as applied to active identification.

The paper is organized as follows. In Section 2, we briefly describe the method of finite-frequency identification and the instrumental variable method. In Section 3, a relation between these methods is established in the situation where the input of the plant is represented by a test signal in the form of the sum of harmonics, whose number is equal to the state space dimension of the plant. Section 4 is devoted to the analysis of the difference between these methods. In Section 5, an example of active identification of a plant by means of the two methods is given and the results of identification are compared.

2. BACKGROUND

2.1. The Identification Problem

Let us consider a completely controllable, asymptotically stable plant specified by the following equation:

$$y^{(n)} + d_{n-1}y^{(n-1)} + \ldots + d_1\dot{y} + d_0 y = k_\gamma u^{(\gamma)} + \ldots + k_1\dot{u} + k_0 u + f, \quad t \geq t_0, \tag{1}$$

where $y(t)$ and $u(t)$ are the measurable output and input, respectively; $y^{(p)}(t)$ and $u^{(q)}(t)$ ($p = \overline{1,n}, q = \overline{1,\gamma}$) are the derivatives of these functions; $f(t)$ is an unknown and unmeasurable bounded disturbance $|f(t)| \leq f^*$, where f^* is a number; the coefficients d_p and k_q ($p = \overline{0,n-1}, q = \overline{0,\gamma}$) are unknown numbers, n is known, and the constant $\gamma < n$ is either given or set to be equal to $n-1$.

The measurable input $u(t)$ consists of the two components: the control $u_{\text{prog}}(t)$ that ensures the control objectives for plant (1), and the test signal $u_{\text{test}}(t)$ aimed at identifying the plant so that $u(t) = u_{\text{prog}}(t) + u_{\text{test}}(t)$. For convenience, we incorporate the function $k_\gamma u^{(\gamma)}_{\text{prog}}(t) + \ldots + k_1\dot{u}_{\text{prog}}(t) + k_0 u_{\text{prog}}(t)$ into the disturbance $f(t)$ and omit the subscript in $u_{\text{test}}(t)$. We then arrive at Eq. (1) where $u(t)$ stands for the test signal.

The identification problem consists in finding estimates $\widehat{d}_p$ and $\widehat{k}_q$ ($p = \overline{0,n-1}, q = \overline{0,\gamma}$) for the coefficients of plant (1) so as to satisfy the following specifications on the relative accuracy of identification:

$$\widehat{d}_p \div d_p \leq \varepsilon^d_p \quad \text{and} \quad \widehat{k}_q \div k_q \leq \varepsilon^k_q, \quad p = \overline{0,n-1}, \quad q = \overline{0,\gamma}. \tag{2}$$

Here, $\div$ is the symbol of computing the relative error, i.e., $a \div b = |a-b|/|b|$ for $b \neq 0$ and $a \div b = |a|$ for $b = 0$; the scalars ε^d_p and ε^k_q ($p = \overline{0,n-1},\ q = \overline{0,\gamma}$) are given.

2.2. The Finite-Frequency Identification Method

2.2.1. Frequency equations of identification. **Definition 1** ([7]). The $2n$ numbers

$$\alpha_r = \operatorname{Re} w(j\omega_r), \quad \beta_r = \operatorname{Im} w(j\omega_r), \quad r = \overline{1,n}, \tag{3}$$

being the values of the transfer function

$$w(s) = \frac{k_\gamma s^\gamma + \ldots + k_1 s + k_0}{s^n + d_{n-1}s^{n-1} + \ldots + d_1 s + d_0} = \frac{k(s)}{d(s)} \tag{4}$$

over the frequencies

$$\omega_r > 0 \quad (r = \overline{1,n}), \quad \omega_p \neq \omega_q \quad (p \neq q), \tag{5}$$

are referred to as the *frequency parameters* of plant (1).

In order to determine experimentally the frequency parameters (3), the test signal

$$u(t) = \sum_{r=1}^{n} \rho_r \sin \omega_r (t - t_u), \qquad t \geq t_u \geq t_0, \tag{6}$$

with amplitudes $\rho_r > 0\,(r = \overline{1,n})$ and given fixed frequencies (5) is fed to the input of plant (1). The output of the plant is then fed to the input of the Fourier filter, whose outputs are taken as the following estimates of the frequency parameters:

$$\begin{aligned} \widehat{\alpha}_r = \alpha_r(\tau) &= \frac{2}{\rho_r \tau} \int_{t_u}^{t_u+\tau} y(t) \sin \omega_r (t - t_u)\, dt, \\ \widehat{\beta}_r = \beta_r(\tau) &= \frac{2}{\rho_r \tau} \int_{t_u}^{t_u+\tau} y(t) \cos \omega_r (t - t_u)\, dt, \end{aligned} \qquad r = \overline{1,n}, \tag{7}$$

where τ is the filtering time.

The estimates of the plant coefficients are found from the estimates of the frequency parameters in the following way. Accounting for the values (3) of the transfer function (4) over the set (5) leads to the system of linear equations

$$k(j\omega_r) - (\alpha_r + j\beta_r)\overline{d}(j\omega_r) = (\alpha_r + j\beta_r)(j\omega_r)^n, \quad r = \overline{1,n}, \tag{8}$$

where $\overline{d}(s) = d(s) - s^n = d_{n-1}s^{n-1} + \ldots + d_1 s + d_0$ and $k(s) = k_\gamma s^\gamma + \ldots + k_1 s + k_0$.

Assertion 1 ([7]). *If plant* (1) *is completely controllable, there exists a unique solution of system* (8) *which does not depend on the choice of frequencies* (5).

Substituting the frequency parameters in (8) with their estimates, we obtain the following *frequency equations of identification*:

$$\widehat{k}(j\omega_r) - (\widehat{\alpha}_r + j\widehat{\beta}_r)\,\widehat{\overline{d}}(j\omega_r) = (\widehat{\alpha}_r + j\widehat{\beta}_r)(j\omega_r)^n, \quad r = \overline{1,n}. \tag{9}$$

Remark 1. The accuracy of solution of system (9) depends on the choice of frequencies (5), since the system matrix

$$\widehat{M} = \left(\begin{array}{cccc|cccc} 1 & 0 & \cdots & \omega_1^\gamma \mathrm{Re}\, j^\gamma & -\widehat{\alpha}_1 & \omega_1\widehat{\beta}_1 & \cdots & -\omega_1^{n-1}(\mathrm{Re}\, j^{n-1}\widehat{\alpha}_1 - \mathrm{Im}\, j^{n-1}\widehat{\beta}_1) \\ 0 & \omega_1 & \cdots & \omega_1^\gamma \mathrm{Im}\, j^\gamma & -\widehat{\beta}_1 & -\omega_1\widehat{\alpha}_1 & \cdots & -\omega_1^{n-1}(\mathrm{Im}\, j^{n-1}\widehat{\alpha}_1 + \mathrm{Re}\, j^{n-1}\widehat{\beta}_1) \\ 1 & 0 & \cdots & \omega_2^\gamma \mathrm{Re}\, j^\gamma & -\widehat{\alpha}_2 & \omega_2\widehat{\beta}_2 & \cdots & -\omega_2^{n-1}(\mathrm{Re}\, j^{n-1}\widehat{\alpha}_2 - \mathrm{Im}\, j^{n-1}\widehat{\beta}_2) \\ 0 & \omega_2 & \cdots & \omega_2^\gamma \mathrm{Im}\, j^\gamma & -\widehat{\beta}_2 & -\omega_2\widehat{\alpha}_2 & \cdots & -\omega_2^{n-1}(\mathrm{Im}\, j^{n-1}\widehat{\alpha}_2 + \mathrm{Re}\, j^{n-1}\widehat{\beta}_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & \omega_n^\gamma \mathrm{Re}\, j^\gamma & -\widehat{\alpha}_n & \omega_n\widehat{\beta}_n & \cdots & -\omega_n^{n-1}(\mathrm{Re}\, j^{n-1}\widehat{\alpha}_n - \mathrm{Im}\, j^{n-1}\widehat{\beta}_n) \\ 0 & \omega_n & \cdots & \omega_n^\gamma \mathrm{Im}\, j^\gamma & -\widehat{\beta}_n & -\omega_n\widehat{\alpha}_n & \cdots & -\omega_n^{n-1}(\mathrm{Im}\, j^{n-1}\widehat{\alpha}_n + \mathrm{Re}\, j^{n-1}\widehat{\beta}_n) \end{array}\right) \tag{10}$$

is composed of the estimates of the frequency parameters, in contrast with the system given by (8). The effect of test frequencies on the conditioning of matrix (10) is studied in [8]; an algorithm for choosing the specific values of frequency that diminish such an effect is also devised in [8].

2.2.2. Convergence conditions for the identification process. To formulate the conditions of convergence of estimates (7) to the true values (3), we introduce the following filterability functions (see [9]):

$$\ell_r^\alpha(\tau) = \frac{2}{\rho_r \tau} \int\limits_{t_u}^{t_u+\tau} \overline{y}(t) \sin \omega_r (t - t_u)\, dt, \quad \ell_r^\beta(\tau) = \frac{2}{\rho_r \tau} \int\limits_{t_u}^{t_u+\tau} \overline{y}(t) \cos \omega_r (t - t_u)\, dt, \quad r = \overline{1, n},$$

where $\overline{y}(t)$ denotes the output of the plant in the absence of test signal (6), $u(t) = 0$.

Definition 2 ([9]). A disturbance $f(t)$ is said to be *FF-filterable* over the given set (5) if there exists a filtering time τ^* such that

$$\frac{|\ell_r^\alpha(\tau)|}{|\alpha_r(\tau)|} \le \delta_r^\alpha, \quad \frac{\left|\ell_r^\beta(\tau)\right|}{|\beta_r(\tau)|} \le \delta_r^\beta, \quad r = \overline{1, n}, \quad \tau \ge \tau^*, \tag{11}$$

where δ_r^α and δ_r^β $(r = \overline{1, n})$ are given sufficiently small numbers. If the conditions

$$\lim_{\tau \to \infty} \ell_r^\alpha(\tau) = 0, \quad \lim_{\tau \to \infty} \ell_r^\beta(\tau) = 0, \quad r = \overline{1, n},$$

hold, the disturbance $f(t)$ is said to be *strictly FF-filterable*.

For the sake of simplicity, throughout the paper we consider strictly FF-filterable disturbances $f(t)$. In that case, the filtering errors $\Delta\,\alpha_r(\tau) = \alpha_r(\tau) - \alpha_r$ and $\Delta\,\beta_r(\tau) = \beta_r(\tau) - \beta_r$ $(r = \overline{1, n})$ have the following properties:

$$\lim_{\tau \to \infty} \Delta\,\alpha_r(\tau) = 0, \quad \lim_{\tau \to \infty} \Delta\,\beta_r(\tau) = 0, \quad r = \overline{1, n}.$$

In order to analyze the rate of convergence, we consider bounded, unmeasurable polyharmonic disturbances $f(t)$ having the form

$$f(t) = \sum_{\mu=0}^{\infty} f_\mu \sin(\omega_\mu^f t + \varphi_\mu^f). \tag{12}$$

Here, ω_μ^f and φ_μ^f $(\mu = \overline{0, \infty})$ are unknown frequencies and phases, respectively, and the amplitudes f_μ are unknown numbers satisfying the inequality

$$\sum_{\mu=0}^{\infty} |f_\mu| \le f^*, \tag{13}$$

where the number f^* is not known. Obviously, for $|\omega_\mu^f| \ne \omega_r$ $(\mu = \overline{0, \infty},\ r = \overline{1, n})$, disturbance (12) is FF-filterable. The following assertion is true.

Assertion 2. *The filtering errors satisfy the following inequalities*:

$$|\Delta\,\alpha_r(\tau)| \le \frac{2}{\tau}(v_r e^{-\sigma\tau} + c_r), \quad |\Delta\,\beta_r(\tau)| \le \frac{2}{\tau}(v_r e^{-\sigma\tau} + c_r), \quad r = \overline{1, n},$$

where the quantities $c_r = \vartheta_r + \xi_r + s_r^* f^*/d^*$, v_r, *and* ϑ_r $(r = \overline{1, n})$ *depend on initial conditions*; $\xi_r = 0$ *for* τ *being multiple of* $2\pi/\omega_\delta$ *and* ω_r *being multiples of* ω_δ, $\omega_\delta = \min(\omega_1, \omega_2, \ldots, \omega_n)$; $\sigma = -\max\{\mathrm{Re}\,[\,\mathrm{roots}\ d\,(s)]\}$ *is the degree of stability*; $d^* = \min\limits_{0 \le \omega \le \infty} |d(j\omega)|$, *and* $s_r^* = \max\limits_{0 \le \mu \le \infty} (1/|\omega_\mu^f + \omega_r| + 1/|\omega_\mu^f - \omega_r|)/|\rho_r|$ $(r = \overline{1, n})$.

The proof of Assertion 2 is relegated to Appendix 1.

We now rearrange Eqs. (9) to the form of *equations for the plant identification errors*:

$$\Delta k(j\omega_r) - (\alpha_r + j\beta_r)\Delta d(j\omega_r) = (\Delta\alpha_r + j\Delta\beta_r)d(j\omega_r) + o(\Delta\alpha_r + j\Delta\beta_r), \quad r = \overline{1,n}, \tag{14}$$

where $\Delta d(j\omega_r) = \widehat{d}(j\omega_r) - d(j\omega_r)$, $\Delta k(j\omega_r) = \widehat{k}(j\omega_r) - k(j\omega_r)$ and $o(\Delta\alpha_r + j\Delta\beta_r) = (\Delta\alpha_r + j\Delta\beta_r)\Delta d(j\omega_r)$ $(r = \overline{1,n})$. The matrices M of systems (8) and (14) coincide and have the form (10), in which the estimates of the frequency parameters are to be replaced with their true values. The deviations of the desired coefficients Δd_p and Δk_q $(p = \overline{0, n-1},\ q = \overline{0,\gamma})$ are related to the filtering errors $\Delta\alpha_r$ and $\Delta\beta_r$ $(r = \overline{1,n})$ through the following matrix equality:

$$\begin{aligned} &\left[\begin{array}{cccc|cccc} \Delta k_0 & \Delta k_1 & \cdots & \Delta k_\gamma & \Delta d_0 & \Delta d_1 & \cdots & \Delta d_{n-1} \end{array}\right]^{\mathrm{T}} \\ &= -M^{-1}\,\mathrm{diag}\,[\mathrm{Re}\,d(j\omega_1)E_2 + \mathrm{Im}\,d(j\omega_1)J, \ldots, \mathrm{Re}\,d(j\omega_n)E_2 + \mathrm{Im}\,d(j\omega_n)J] \\ &\quad \times \left[\begin{array}{cccccccc} \Delta\alpha_1 & \Delta\beta_1 & \Delta\alpha_2 & \Delta\beta_2 & \cdots & \Delta\alpha_n & \Delta\beta_n \end{array}\right]^{\mathrm{T}} \\ &\quad + \boldsymbol{o}(\Delta\alpha_1, \Delta\beta_1, \Delta\alpha_2, \Delta\beta_2, \ldots, \Delta\alpha_n, \Delta\beta_n), \end{aligned}$$

where $E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. From this relation it follows that if the matrix M is sufficiently well conditioned (this can be secured by the proper choice of the test frequencies), the identification errors will have the same order of magnitude as the filtering errors; i.e., the values $\widehat{d}_p$ and $\widehat{k}_q$ converge to d_p and k_q $(p = \overline{0, n-1},\ q = \overline{0,\gamma})$ in the same way as $\widehat{\alpha}_r$ and $\widehat{\beta}_r$ converge to α_r and β_r $(r = \overline{1,n})$, respectively.

2.2.3. Self-tuning the frequencies and amplitudes of the test signal. Let us sketch the results in [8] related to self-tuning of the frequencies and amplitudes of test signals. To this end, we rewrite the transfer function (4) of the plant in the form

$$w(s) = k_\gamma \frac{\prod\limits_{q=1}^{p_1}(s + \omega_{1,q}) \prod\limits_{q=1}^{p_2}\left(s^2 + 2\xi_{2,q}\omega_{2,q}s + \omega_{2,q}^2\right)}{\prod\limits_{q=1}^{p_3}(s + \omega_{3,q}) \prod\limits_{q=1}^{p_4}\left(s^2 + 2\xi_{4,q}\omega_{4,q}s + \omega_{4,q}^2\right)}.$$

Definition 3 ([10]). The set $L = \{|\omega_{1,1}|, |\omega_{1,2}|, \ldots, |\omega_{1,p_1}|; |\omega_{2,1}|, |\omega_{2,2}|, \ldots, |\omega_{2,p_2}|; \omega_{3,1}, \omega_{3,2}, \ldots, \omega_{3,p_3}; \omega_{4,1}, \omega_{4,2}, \ldots, \omega_{4,p_4}\}$ is referred to as *eigenfrequencies* of plant (1). The lower (ω_l) and upper (ω_u) limits of eigenfrequencies are denoted by $\omega_l = \min L$ and $\omega_u = \max L$.

It is intuitively clear that the test frequencies ω_r $(r = \overline{1,n})$ are to be chosen from the set $[\omega_l, \omega_u]$ of eigenfrequencies. In fact, it is shown in [8] that choosing the test frequencies from the low-frequency band $\omega_r \in (0, \omega_l)$ $(r = \overline{1,n})$ or from the higher-frequency band $\omega_r \in (\omega_u, \infty)$ $(r = \overline{1,n})$ may lead to arbitrarily large identification errors $d_p \div \widehat{d}_p$ and $k_q \div \widehat{k}_q$ $(p = \overline{0, n-1},\ q = \overline{0,\gamma})$ under arbitrarily small relative filtering errors $\alpha_r \div \widehat{\alpha}_r$ and $\beta_r \div \widehat{\beta}_r$ $(r = \overline{1,n})$. Accordingly, an algorithm for self-tuning the frequencies of the test signal was proposed in [8], which yields test frequencies $\omega_r \in [\omega_l, \omega_u]$ $\left(r \in \overline{1,n}\right)$. This algorithm is based on the following result.

Assertion 3 ([8]). *Assume that plant* (1) *is excited by the test signal*

$$u(t) = \rho_1 \sin\omega_1(t - t_u), \tag{15}$$

and its output is fed to the Fourier filter (7) *for* $n = 1$. *Then there exist a sufficiently large filtering time* $\tau = \tau^*$ *and a sufficiently small frequency* $\omega_1 \in (0, \omega_l)$ *such that the number* $\overline{\omega}_l(\tau^*) = \omega_1\,|\alpha_1(\tau^*)/\beta_1(\tau^*)|$ *is close to the lower limit* (ω_l) *of the eigenfrequencies of the plant.*

A similar assertion (for $\omega_1 \in (\omega_u, \infty)$) [8] is equally valid for the upper limit ω_u of eigenfrequencies.

In the process of identification, the inputs and outputs of plant (1) are bounded:

$$|y(t)| \leq y^*, \quad |u(t)| \leq u^*, \quad t \geq t_u, \tag{16}$$

where y^* and u^* are given numbers; moreover, in the absence of a test signal, the output of the plant satisfies the inequality

$$|\overline{y}(t)| < y^*, \quad t \geq t_0. \tag{17}$$

The difference $y^* - \max|\overline{y}(t)| = \varepsilon_y$ defines the tolerance ε_y on the component of the plant output excited by the test signal.

In the process of self-tuning the test frequencies and identification, the test signal is taken in the form (15), where the amplitude ρ is automatically tuned in such a way as to satisfy constraints (17) imposed on the inputs and outputs of plant (1), see [8].

2.2.4. The classical frequency method. The classical frequency method originated in [11, 12] is one of the first identification tool for linear continuous time control plants. This method is aimed at finding estimates for the coefficients of the transfer function (4) of the plant from the estimates of its frequency parameters (3) by minimizing the function

$$\Im = \sum_{r=1}^{p} \left| \frac{\widehat{k}(j\omega_r)}{\widehat{d}(j\omega_r)} - \left(\widehat{\alpha}_r + j\widehat{\beta}_r\right) \right|^2, \tag{18}$$

where the number p of specific values of the frequency response is large enough (basically, $p \to \infty$). The minimization of (18) is a nonlinear least squares problem; the reader is referred to [13] for a survey of results in this difficult problem.

In contrast to this classical method, the finite-frequency identification method operates with the values of the frequency response whose number is equal to the state space dimension of the plant; i.e., $p = n$, and hence, it is referred to as the finite-frequency method as opposed to the classical infinite-frequency method.

As a result, the identification reduces to solving a system of linear frequency equations, and "averaging" of the errors of filtering is based on accurate filtering (within the classical approach, this problem is solved by increasing the number of values of the frequency response).

2.3. The Method of Instrumental Variables

This method was elaborated in [6] for plant identification in discrete time. For ease of comparison with the finite-frequency identification method, we formulate the continuous-time counterpart of the instrumental variable method, which preserves all its basic ideas presented in [2].

We first give a definition on the relation between the inputs $u(t)$ and $f(t)$. Let us consider the two functions $\mu(t)$ and $\nu(t)$ and introduce the scalar

$$(\mu, \nu)_{[t_0, t_1]} = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} \mu(t)\nu(t)\, dt.$$

In what follows, we omit the segment $[t_u, \infty]$ in the subscript at (μ, ν) and assume that the functions $\mu(t)$ and $\nu(t)$ have the following property:

$$(\mu, \nu) = (\mu, \nu)_{[t_u, \infty]} = \lim_{\tau \to \infty} \frac{1}{\tau} \int_{t_u}^{t_u + \tau} \mu(t)\nu(t)\, dt.$$

Definition 4. The inputs $u(t)$ and $f(t)$ of plant (1) are said to be *dependent* if $(u,\, f) \neq 0$; otherwise, they are called *independent* (i.e., if $(u,\, f) = 0$).

We note that the identification problem may not possess a solution for dependent inputs $u(t)$ and $f(t)$; therefore, up to Section 4 these inputs are assumed to be independent. In the finite-frequency identification method, the condition of independence is expressed in the form of the strict FF-filterability requirement for the disturbance $f(t)$, which is assumed satisfied throughout the paper.

We now consider the equation

$$\check{y}^{(n)} + \check{d}_{n-1}\check{y}^{(n-1)} + \ldots + \check{d}_1\,\dot{\check{y}} + \check{d}_0\check{y} = \check{k}_\gamma u^{(\gamma)} + \ldots + \check{k}_1\dot{u} + \check{k}_0 u, \tag{19}$$

whose coefficients $\check{d}_p$ and $\check{k}_q$ $(p = \overline{0, n-1},\ q = \overline{0, \gamma})$ are given numbers such that the polynomials in the numerator and denominator of the transfer function

$$\check{w}(s) = \frac{\check{k}_\gamma s^\gamma + \ldots + \check{k}_1 s + \check{k}_0}{s^n + \check{d}_{n-1}s^{n-1} + \ldots + \check{d}_1 s + \check{d}_0}$$

from $u(t)$ to $\check{y}(t)$ are coprime. The solutions $\check{y}^{(r)}(t)$ $(r = \overline{0, n-1})$ of Eq. (19) are adopted as the instrumental variables.

Multiplying Eq. (1) by $\check{y}(t)$ and integrating the resulting expression from t_u to $t_u + \tau$, we then multiply it by $1/\tau$ and take the limit as $\tau \to \infty$ to obtain

$$\lim_{\tau\to\infty}\frac{1}{\tau}\int\limits_{t_u}^{t_u+\tau} y^{(n)}(t)\check{y}(t)\,dt + \sum_{p=0}^{n-1}\left[\lim_{\tau\to\infty}\frac{1}{\tau}\int\limits_{t_u}^{t_u+\tau} y^{(p)}(t)\check{y}(t)\,dt\right] d_p$$
$$= \sum_{q=0}^{\gamma}\left[\lim_{\tau\to\infty}\frac{1}{\tau}\int\limits_{t_u}^{t_u+\tau} u^{(q)}(t)\check{y}(t)\,dt\right] k_q + \lim_{\tau\to\infty}\frac{1}{\tau}\int\limits_{t_u}^{t_u+\tau} f(t)\check{y}(t)\,dt,$$

or, in the compact form:

$$\left(y^{(n)},\, \check{y}\right) + \sum_{p=0}^{n-1}\left(y^{(p)},\, \check{y}\right) d_p = \sum_{q=0}^{\gamma}\left(u^{(q)},\, \check{y}\right) k_q, \tag{20}$$

where $(f,\, \check{y}) = 0$ because $(f, u) = 0$. Replacing the function $\check{y}(t)$ in (20) successively with its derivatives $\check{y}^{(r)}(t)$ for $r = \overline{1, n-1}$ and the functions $u^{(r)}(t)$ for $r = \overline{0, \gamma}$, we arrive at the system

$$\begin{cases} \displaystyle\sum_{q=0}^{\gamma}\left(u^{(q)},\, u^{(r)}\right) k_q - \sum_{p=0}^{n-1}\left(y^{(p)},\, u^{(r)}\right) d_p = \left(y^{(n)},\, u^{(r)}\right), & r = \overline{0, \gamma} \\ \displaystyle -\sum_{q=0}^{\gamma}\left(u^{(q)},\, \check{y}^{(r)}\right) k_q + \sum_{p=0}^{n-1}\left(y^{(p)},\, \check{y}^{(r)}\right) d_p = -\left(y^{(n)},\, \check{y}^{(r)}\right), & r = \overline{0, n-1}. \end{cases} \tag{21}$$

In the derivation of the formulas above, the equalities $\left(f,\, u^{(r)}\right) = 0$ $(r = \overline{1, \gamma})$ and $\left(f,\, \check{y}^{(r)}\right) = 0$ $(r = \overline{1, n-1})$ were used (they follow from $(f, u) = 0$).

This system yields unique values for d_p and k_q $(p = \overline{0, n-1},\, q = \overline{0, \gamma})$, provided that the input $u(t)$ satisfies the standard sufficient excitation condition, see [2].

3. THE RELATION BETWEEN THE IDENTIFICATION METHODS

Let us establish a linkage between Eqs. (8) (which coincide with the frequency equations (9) as $\tau \to \infty$)) and Eqs. (21) of the instrumental variable method for the case when the function $u(t)$ in (21) is represented by the sum of harmonics (6). For this purpose, we write down Eqs. (8) in the form

$$\sum_{q=0}^{\gamma} \Omega^q \boldsymbol{i} k_q + \sum_{p=0}^{n-1} \Omega^p \boldsymbol{h} d_p = -\Omega^n \boldsymbol{h}, \tag{22}$$

where

$$\Omega = \operatorname{diag}(\omega_1, \omega_2, \ldots, \omega_n) \otimes J, \quad \boldsymbol{i} = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots & 1 & 0 \end{bmatrix}^{\mathrm{T}},$$
$$\boldsymbol{h} = \begin{bmatrix} -\alpha_1 & -\beta_1 & -\alpha_2 & -\beta_2 & \cdots & -\alpha_n & -\beta_n \end{bmatrix}^{\mathrm{T}}.$$

Pre-multiplying system (22) by the matrix $R = \operatorname{diag}(\rho_1, \rho_2, \ldots, \rho_n) \otimes E_2$, we rewrite it in a more compact form

$$M_{\mathrm{ff}}(\boldsymbol{\alpha}, \boldsymbol{\beta})\boldsymbol{\theta} = \boldsymbol{v}_{\mathrm{ff}}(\boldsymbol{\alpha}, \boldsymbol{\beta}), \tag{23}$$

where the following notation is used: $M_{\mathrm{ff}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = RM$, $M = (\boldsymbol{i} \;\; \Omega\boldsymbol{i} \;\; \cdots \;\; \Omega^\gamma \boldsymbol{i} \,|\, \boldsymbol{h} \;\; \Omega\boldsymbol{h} \;\; \cdots \;\; \Omega^{n-1}\boldsymbol{h})$ corresponds to matrix (10) in which the estimates of the frequency parameters are replaced with their true values, $\boldsymbol{v}_{\mathrm{ff}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = -R\Omega^n \boldsymbol{h}$, and $\boldsymbol{\theta} = \begin{bmatrix} k_0 & k_1 & \cdots & k_\gamma \,|\, d_0 & d_1 & \cdots & d_{n-1} \end{bmatrix}^{\mathrm{T}}$.

To represent Eqs. (21) in the similar form, we need the lemma below.

Lemma 1. *Under condition* (6), *Eqs.* (21) *take the form*

$$\begin{cases} \displaystyle\sum_{q=0}^{\gamma} \boldsymbol{i}^{\mathrm{T}} \Omega^{r\mathrm{T}} R^{\mathrm{T}} R \Omega^q \boldsymbol{i} k_q + \sum_{p=0}^{n-1} \boldsymbol{i}^{\mathrm{T}} \Omega^{r\mathrm{T}} R^{\mathrm{T}} R \Omega^p \boldsymbol{h} d_p = -\boldsymbol{i}^{\mathrm{T}} \Omega^{r\mathrm{T}} R^{\mathrm{T}} R \Omega^n \boldsymbol{h}, \quad r = \overline{0, \gamma} \\ \displaystyle\sum_{q=0}^{\gamma} \check{\boldsymbol{h}}^{\mathrm{T}} \Omega^{r\mathrm{T}} R^{\mathrm{T}} R \Omega^q \boldsymbol{i} k_q + \sum_{p=0}^{n-1} \check{\boldsymbol{h}}^{\mathrm{T}} \Omega^{r\mathrm{T}} R^{\mathrm{T}} R \Omega^p \boldsymbol{h} d_p = -\check{\boldsymbol{h}}^{\mathrm{T}} \Omega^{r\mathrm{T}} R^{\mathrm{T}} R \Omega^n \boldsymbol{h}, \quad r = \overline{0, n-1}, \end{cases} \tag{24}$$

where

$$\check{\boldsymbol{h}} = \begin{bmatrix} -\check{\alpha}_1 & -\check{\beta}_1 & -\check{\alpha}_2 & -\check{\beta}_2 & \cdots & -\check{\alpha}_n & -\check{\beta}_n \end{bmatrix}^{\mathrm{T}},$$
$$\check{\alpha}_r = \operatorname{Re} \check{w}(j\omega_r), \quad \check{\beta}_r = \operatorname{Im} \check{w}(j\omega_r) \quad (r = \overline{1, n}).$$

The proof of Lemma 1 is given in Appendix 2.

Let us now rewrite Eqs. (24) in the compact form similar to that of (23), namely:

$$M_{\mathrm{iv}}(\check{\boldsymbol{\alpha}}, \check{\boldsymbol{\beta}};\, \boldsymbol{\alpha}, \boldsymbol{\beta})\boldsymbol{\theta} = \boldsymbol{v}_{\mathrm{iv}}(\check{\boldsymbol{\alpha}}, \check{\boldsymbol{\beta}};\, \boldsymbol{\alpha}, \boldsymbol{\beta}). \tag{25}$$

The following assertion is immediate.

Assertion 4. *For Eqs.* (23) *and* (25), *the following is true*:

$$\begin{cases} M_{\mathrm{iv}}(\check{\boldsymbol{\alpha}}, \check{\boldsymbol{\beta}};\, \boldsymbol{\alpha}, \boldsymbol{\beta}) = M_{\mathrm{ff}}^{\mathrm{T}}(\check{\boldsymbol{\alpha}}, \check{\boldsymbol{\beta}}) M_{\mathrm{ff}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \\ \boldsymbol{v}_{\mathrm{iv}}(\check{\boldsymbol{\alpha}}, \check{\boldsymbol{\beta}};\, \boldsymbol{\alpha}, \boldsymbol{\beta}) = M_{\mathrm{ff}}^{\mathrm{T}}(\check{\boldsymbol{\alpha}}, \check{\boldsymbol{\beta}}) \boldsymbol{v}_{\mathrm{ff}}(\boldsymbol{\alpha}, \boldsymbol{\beta}), \end{cases} \tag{26}$$

where

$$M_{\mathrm{ff}}^{\mathrm{T}}(\check{\boldsymbol{\alpha}}, \check{\boldsymbol{\beta}}) = \begin{pmatrix} \boldsymbol{i} & \Omega\boldsymbol{i} & \cdots & \Omega^\gamma \boldsymbol{i} \,|\, \check{\boldsymbol{h}} & \Omega\check{\boldsymbol{h}} & \cdots & \Omega^{n-1}\check{\boldsymbol{h}} \end{pmatrix}^{\mathrm{T}} R^{\mathrm{T}}. \tag{27}$$

Indeed, factoring the identical terms out of system (24) we obtain

$$\begin{cases} \boldsymbol{i}^{\mathrm{T}} \Omega^{r\mathrm{T}} R^{\mathrm{T}} R \left(\sum\limits_{q=0}^{\gamma} \Omega^{q} \boldsymbol{i} k_q + \sum\limits_{p=0}^{n-1} \Omega^{p} \boldsymbol{h} d_p = -\Omega^{n} \boldsymbol{h} \right), & r = \overline{0, \gamma} \\ \check{\boldsymbol{h}}^{\mathrm{T}} \Omega^{r\mathrm{T}} R^{\mathrm{T}} R \left(\sum\limits_{q=0}^{\gamma} \Omega^{q} \boldsymbol{i} k_q + \sum\limits_{p=0}^{n-1} \Omega^{p} \boldsymbol{h} d_p = -\Omega^{n} \boldsymbol{h} \right), & r = \overline{0, n-1}, \end{cases}$$

and composing the matrix M_{ff} (27), we arrive at

$$M_{\mathrm{ff}}^{\mathrm{T}}(\check{\boldsymbol{\alpha}}, \check{\boldsymbol{\beta}}) \left[M_{\mathrm{ff}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \boldsymbol{\theta} = \boldsymbol{v}_{\mathrm{ff}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \right].$$

Using (25) this yields relations (26).

4. THE DIFFERENCE BETWEEN THE IDENTIFICATION METHODS

4.1. Sensitivity to Inaccuracies in Experimental Data

To analyze the difference between the two methods, we consider the following two types of measurable inputs $u(t)$:

(1) $u(t) = u_{\mathrm{prog}}(t)$, where $u_{\mathrm{prog}}(t)$ is the control signal aimed at achieving the control objectives; such an input is not pertinent to the plant identification so that we are in the situation of *passive* identification;

(2) $u(t) = u_{\mathrm{test}}(t)$, where $u_{\mathrm{test}}(t)$ is a test signal which is specified either *a priori* (in the design of experiment, see [2]) or in the process of identification according to the identification objectives. In that case, identification is said to be *active*, see [4].

By a number of reasons, specification (2) often cannot be satisfied by means of passive identification. First, the measurable input and the disturbance are dependent; moreover, the independence of $u_{\mathrm{prog}}(t)$ and $f(t)$ cannot be checked, since the disturbance is not known both before and in the process of identification. Second, the control $u_{\mathrm{prog}}(t)$ is not "sufficiently excited" (for instance, $u_{\mathrm{prog}}(t) = \mathrm{const}$ or $u_{\mathrm{prog}}(t) = \sin \omega t$ for $n > 1$). On top of that, even in the absence of the two obstacles above, the identification time may turn out to be too large.

On the other hand, using self-tuning of the test signal $u_{\mathrm{test}}(t)$ within the active identification framework, objectives (2) can almost always be achieved.

Obviously, finite-frequency identification is an active identification method; in its present form it cannot be applied to passive identification, although certain research is being carried out in this direction. The instrumental variable method is a more general tool, and it can be used both in active and passive identification. Therefore, these methods can only be compared as applied to active identification.

The finite-frequency identification method is specially designed for the needs of active identification, and it has the following advantages over the method of instrumental variables (when the latter is applied to active identification).

First, in the finite-frequency identification method, the inputs $u(t)$ and $f(t)$ are checked for independence by checking the FF-filterability condition (11). This offers a way to tune the test frequencies in order to satisfy these conditions. Another approach is based on the compensation of the parameter components caused by the dependence between the inputs $u(t)$ and $f(t)$. These approaches are discussed in Section 4.2 below. With the instrumental variable method, it is impossible to check if the inputs $u(t)$ and $f(t)$ are independent, since $f(t)$ is not measured.

Second, using the self-tuning procedure in the finite-frequency identification method, the values of the test frequencies can be chosen from within the set of eigenfrequencies of the plant. The lack of such kind of self-tuning in the method of instrumental variables may result in the fact that the *a priori* chosen test frequencies do not belong to this set. As a result, this leads to a high sensitivity of solutions of Eqs. (24) to the variations of the coefficients (see discussion in Subsection 2.2.3), and hence, the identification time may turn out to be too large (in particular, an example in Section 5 illustrates this phenomenon).

4.2. The Effect of Dependent Test Signal and Exogenous Disturbance

Definition 5. A disturbance $f(t)$ is said to be *stationary frequency-dependent* if the following inequalities are valid over the neighboring segments $[t_u,\, t_u+\tau^*]$ and $[t_u+\tau^*,\, t_u+2\tau^*]$:

$$\left.\begin{aligned} &\left|\frac{2}{\rho_r}(\overline{y},\, s_r)_{[t_u,\, t_u+\tau^*]} - \frac{2}{\rho_r}(\overline{y},\, s_r)_{[t_u+\tau^*,\, t_u+2\tau^*]}\right| \le \varepsilon_r^s, \\ &\left|\frac{2}{\rho_r}(\overline{y},\, c_r)_{[t_u,\, t_u+\tau^*]} - \frac{2}{\rho_r}(\overline{y},\, c_r)_{[t_u+\tau^*,\, t_u+2\tau^*]}\right| \le \varepsilon_r^c, \end{aligned}\right. \quad r = \overline{1, n}, \tag{28}$$

where $s_r(t) = \sin\omega_r(t-t_u)$, $c_r(t) = \cos\omega_r(t-t_u)\,(r=\overline{1,n})$, $\tau^* > 0$ is a given sufficiently large number, and $\varepsilon_r^s > 0$ and $\varepsilon_r^c > 0\,(r=\overline{1,n})$ are given sufficiently small numbers.

Let a disturbance $f(t)$ be stationary frequency-dependent for ε_r^s and ε_r^c $(r=\overline{1,n})$ small enough as compared to α_r and β_r $(r=\overline{1,n})$. In that case, there are two ways to diminish the effect of this dependence on the accuracy of identification, see [14].

(I) *Tuning the test frequencies.* Let us measure and store the input of plant (1) for $u(t)=0$ and check the conditions

$$\left|\frac{2}{\rho_r}(\overline{y},\, s_r)_{[t_u,\, t_u+\tau^*]}\right| \le \varepsilon_r^s, \quad \left|\frac{2}{\rho_r}(\overline{y},\, c_r)_{[t_u,\, t_u+\tau^*]}\right| \le \varepsilon_r^c, \quad r=\overline{1,n}, \tag{29}$$

which provide a small effect of a stationary frequency-dependent disturbance on the accuracy of identification. We then replace the frequencies $\omega_{r_1},\, \omega_{r_2}, \ldots, \omega_{r_{n'}}$ $(n' \le n)$ that violate these conditions with the new values $\omega'_{r_1},\, \omega'_{r_2}, \ldots, \omega'_{r_{n'}}$, and check Ineqs. (29) for them, etc. Such a replacement is to be performed until the set of n frequencies satisfying conditions (29) is formed.

(II) *Conjugate tests.* This method is based on the two consecutive experiments of length τ^*. In the first experiment, the test signal has the form (6), and in the second one it has the opposite sign:

$$u(t) = -\sum_{r=1}^{n} \rho_r \sin\omega_r(t-t_F), \quad t_F \le t \le t_F+\tau^*, \quad t_F = t_u+\tau^*.$$

Let $\alpha_r^+(\tau^*),\, \beta_r^+(\tau^*)$ $(r=\overline{1,n})$ and $\alpha_r^-(\tau^*),\, \beta_r^-(\tau^*)$ $(r=\overline{1,n})$ be the estimates of the frequency parameters obtained from the first and second experiments, respectively. We store them and find the following estimates which are not biased by the quantities $\ell_r^\alpha(\tau^*)$ and $\ell_r^\beta(\tau^*)\,(r=\overline{1,n})$:

$$\left.\begin{aligned} \alpha_r(\tau^*) &= \frac{\alpha_r^+(\tau^*) - \alpha_r^-(\tau^*)}{2} = \alpha_r + \varepsilon_r^\alpha(\varepsilon_r^s, \tau^*), \\ \beta_r(\tau^*) &= \frac{\beta_r^+(\tau^*) - \beta_r^-(\tau^*)}{2} = \beta_r + \varepsilon_r^\beta(\varepsilon_r^c, \tau^*), \end{aligned}\right. \quad r=\overline{1,n},$$

Here, $\varepsilon_r^\alpha(\varepsilon_r^s,\tau^*)$ and $\varepsilon_r^\beta(\varepsilon_r^c,\tau^*)$ $(r=\overline{1,n})$ are sufficiently small numbers which depend on the right-hand sides of Ineqs. (28) and the difference between the initial conditions in the two above-mentioned experiments.

5. EXAMPLES

We consider a completely controllable, asymptotically stable plant specified by the equation

$$d_3\dddot{y} + d_2\ddot{y} + d_1\dot{y} + d_0 y = k_1\dot{u} + k_0 u + f, \tag{30}$$

where the exogenous disturbance $f(t)$ is a bounded function. The problem is to estimate the coefficients d_3, d_2, d_1, d_0, k_1, and k_0 of plant (30).

Remark 2. In the experiments on the identification of plant (30), we used the model discussed in [15] with the numerical values of the coefficients given by

$$d_3 = 0.2, \quad d_2 = 1.24, \quad d_1 = 5.24, \quad d_0 = 1; \qquad k_1 = -0.4, \quad k_0 = 1; \tag{31}$$

the disturbance $f(t) = \operatorname{sign}(\sin 2.75t)$, and sample time $h = 0.01$ s.

The transfer function of plant (30) with coefficients (31) has the form

$$w(s) = \frac{-0.4s + 1}{(5s+1)(0.04s^2 + 0.24s + 1)} = -2\frac{s - 2.5}{(s+0.2)(s^2 + 6s + 25)}.$$

The experiments were carried out using MATLAB. The routines `d111sefad` from the ADAPLAB-M package ([16]) and `iv4` from the System Identification toolbox ([17]) were used for active identification of plant (30) on the basis of finite-frequency identification and the instrumental variable method, respectively. The procedure `iv4` yields the discrete-time model, which we convert into continuous time in order to compare adequately with the results obtained with `d111sefad`.

The first experiment. Using the `d111sefad` routine leads to the following results:

(a) the estimates

$$\widehat{\widehat{\omega}}_l = 0.176, \qquad \widehat{\widehat{\omega}}_u = 7.94$$

of the lower and upper limits for the test frequencies were found. The initial value of frequency for computing the lower limit by means of `d111sefad` was taken to be $\omega_{\text{init}} = 0.05$;

(b) from these estimates, the following test frequencies were obtained:

$$\omega_1 = 0.176, \qquad \omega_2 = 1.23, \qquad \omega_3 = 7.94; \tag{32}$$

(c) using frequencies (32) and the amplitudes

$$\rho_1 = 3, \qquad \rho_2 = 3, \qquad \rho_3 = 3, \tag{33}$$

the test signal (6) was shaped, which was then used to obtain the following estimates of the coefficients of plant (30):

$$\widehat{d}_3 = 0.209, \quad \widehat{d}_2 = 1.23, \quad \widehat{d}_1 = 5.26, \quad \widehat{d}_0 = 1; \quad \widehat{k}_1 = -0.407, \quad \widehat{k}_0 = 0.999.$$

Finally, the transfer function of the identified plant has the form

$$\widehat{w}(s) = -1.94\frac{s - 2.455}{(s + 0.198)(s^2 + 5.69s + 24.04)}, \tag{34}$$

and the identification time is equal to $\tau = 972$ s.

The second experiment. Use of the MATLAB procedure `iv4` with the test signal having the same frequencies (32) and amplitudes (33) over the same time of identification ($\tau = 972$ s) resulted in the transfer function close to (34).

The third experiment. We now skip the self-tuning procedure for the test frequencies and set $\omega_r = 5^{r-1}\omega_{\text{init}}$ $(r = \overline{1,3})$:

$$\omega_1 = 0.05, \quad \omega_2 = 0.25, \quad \omega_3 = 1.25, \tag{35}$$

while remaining amplitudes (33) the same values.

Application of the `iv4` routine for test frequencies (35) gave the following result:

$$\widehat{w}(s) = \frac{-0.0024441(s+655.9)(s-2.736)}{(s+0.2136)(s^2+5.642s+21.46)} \approx \frac{-1.59(s-2.736)}{(s+0.2136)(s^2+5.642s+21.46)}.$$

The identification time was equal to $\tau = 5\,650$ s, and varying this time period did not improve the accuracy of identification using this routine.

The fourth experiment. For frequencies (35), amplitudes (33), and filtering time $\tau = 5\,650$ s, the routine `d111sefad` yielded the result

$$\widehat{w}(s) = \frac{-1.604s+4.181}{s^3+5.485s^2+21.96s+4.182} = \frac{-1.604(s-2.607)}{(s+0.2001)(s^2+5.285s+20.9)},$$

which is also close to the true transfer function; moreover, as τ increases, we have $\widehat{w}(s) \to w(s)$.

6. CONCLUSIONS

The comparative study of the two methods of active identification under unknown-but-bounded disturbances demonstrated

—*the relation between the methods*: for the case when the test signal is represented by the sum of harmonics (6) with the given amplitudes and frequencies, the identification equations (22) and (24) (for computing the coefficients of the identified plant) are related to each other via a nonsingular matrix (see Assertion 4);

—*the different capabilities of the methods*:

(1) the results of identification obtained with the finite-frequency method is less sensitive to errors in the experimental data (frequency parameters) because of the self-tuning of the frequencies of the test signal (6);

(2) the finite-frequency method provides checking if there is a dependence between the test signal and the exogenous disturbance; this allows for diminishing its effect on the results of identification by varying the frequencies of the test signal or by compensating the component caused by such a dependence.

APPENDIX 1

Proof of Assertion 2. For the test signal and exogenous disturbance of the forms (6) and (12), respectively, Eq. (1) has solution $y(t) = y_*(t) + y_u(t) + y_f(t)$ such that

$$y_*(t) = \boldsymbol{c}^{\mathrm{T}} e^{A(t-t_0)} \left[\boldsymbol{x}(t_0) + \sum_{r=1}^{n} \rho_r\omega_r(E_n\omega_r^2 + A^2)^{-1}\boldsymbol{b}\right], \tag{A.1}$$

$$y_u(t) = \sum_{r=1}^{n} \rho_r \Big[\alpha_r \sin\omega_r(t-t_0) + \beta_r\cos\omega_r(t-t_0)\Big], \tag{A.2}$$

$$y_f(t) = \sum_{\mu=0}^{\infty} \frac{f_\mu}{|d(j\omega_\mu^f)|} \sin(\omega_\mu^f t + \psi_\mu^f), \tag{A.3}$$

where A, $\boldsymbol{b}$ and $\boldsymbol{c}$ are the parameters of the Cauchy form of Eq. (1): $\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{b}u + \boldsymbol{m}f$ $y = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$. Hereafter, for ease of exposition, we consider $t_u = t_0$.

The component (A.1) can be represented in the form

$$y_*(t) = \sum_{\nu=1}^{n} e^{\lambda_\nu^*(t-t_0)}[\rho_\nu^s \sin\omega_\nu^*(t-t_0) + \rho_\nu^c \cos\omega_\nu^*(t-t_0)] \tag{A.4}$$

under the simplifying assumption that the $n \times n$-matrix A has all distinct eigenvalues $s_\nu^* = \lambda_\nu^* + j\omega_\nu^*$ $(\nu = \overline{1,n})$.

Substituting this component into (7) instead of $y(t)$ and taking the integral leads to

$$\alpha_r^*(\tau) = \frac{2}{\rho_r\tau}\sum_{\nu=1}^{n}\left(v_\nu^\alpha e^{\lambda_\nu^*\tau} + \vartheta_\nu^\alpha\right), \quad \beta_r^*(\tau) = \frac{2}{\rho_r\tau}\sum_{\nu=1}^{n}\left(v_\nu^\beta e^{\lambda_\nu^*\tau} + \vartheta_\nu^\beta\right), \quad r = \overline{1,n},$$

where

$$\begin{gathered}
v_\nu^\alpha = \rho_\nu^s(\rho_\nu^{ss}\varphi_\nu^{ss} - \rho_\nu^{sc}\varphi_\nu^{sc} - \rho_\nu^{cs}\varphi_\nu^{cs} + \rho_\nu^{cc}\varphi_\nu^{cc}) + \rho_\nu^c(\rho_\nu^{ss}\varphi_\nu^{cs} - \rho_\nu^{sc}\varphi_\nu^{cc} + \rho_\nu^{cs}\varphi_\nu^{ss} - \rho_\nu^{cc}\varphi_\nu^{sc}),\\
v_\nu^\beta = \rho_\nu^s(\rho_\nu^{ss}\varphi_\nu^{sc} + \rho_\nu^{sc}\varphi_\nu^{ss} - \rho_\nu^{cs}\varphi_\nu^{cc} - \rho_\nu^{cc}\varphi_\nu^{cs}) + \rho_\nu^c(\rho_\nu^{ss}\varphi_\nu^{cc} + \rho_\nu^{sc}\varphi_\nu^{cs} + \rho_\nu^{cs}\varphi_\nu^{sc} + \rho_\nu^{cc}\varphi_\nu^{ss}),\\
\vartheta_\nu^\alpha = \rho_\nu^c\rho_\nu^{sc} - \rho_\nu^s\rho_\nu^{cc}, \quad \vartheta_\nu^\beta = \rho_\nu^s\rho_\nu^{cs} - \rho_\nu^c\rho_\nu^{ss},\\
\rho_\nu^{ss} = \lambda_\nu^*\frac{\lambda_\nu^{2*} + \omega_\nu^{2*} + \omega_r^2}{(\lambda_\nu^{2*} + \omega_\nu^{2*} + \omega_r^2)^2 - (2\omega_\nu^*\omega_r)^2}, \quad \rho_\nu^{sc} = \omega_r\frac{\lambda_\nu^{2*} - \omega_\nu^{2*} + \omega_r^2}{(\lambda_\nu^{2*} + \omega_\nu^{2*} + \omega_r^2)^2 - (2\omega_\nu^*\omega_r)^2},\\
\rho_\nu^{cs} = \omega_\nu^*\frac{\lambda_\nu^{2*} + \omega_\nu^{2*} - \omega_r^2}{(\lambda_\nu^{2*} + \omega_\nu^{2*} + \omega_r^2)^2 - (2\omega_\nu^*\omega_r)^2}, \quad \rho_\nu^{cc} = 2\frac{\lambda_\nu^*\omega_\nu^*\omega_r}{(\lambda_\nu^{2*} + \omega_\nu^{2*} + \omega_r^2)^2 - (2\omega_\nu^*\omega_r)^2},\\
\varphi_\nu^{ss} = \sin\omega_\nu^*\tau\sin\omega_r\tau, \quad \varphi_\nu^{sc} = \sin\omega_\nu^*\tau\cos\omega_r\tau,\\
\varphi_\nu^{cs} = \cos\omega_\nu^*\tau\sin\omega_r\tau, \quad \varphi_\nu^{cc} = \cos\omega_\nu^*\tau\cos\omega_r\tau.
\end{gathered}$$

Since $\lambda_\nu^* \le -\sigma < 0$ $(\nu = \overline{1,n})$, where $\sigma = -\max\limits_{1\le\nu\le n}(\operatorname{Re}\lambda_\nu(A))$ is the degree of stability, we have

$$\left.\begin{aligned}
|\alpha_r^*(\tau)| &\le \frac{2}{|\rho_r|\tau}\left(\sum_{\nu=1}^{n}|v_\nu^\alpha||e^{\lambda_\nu^*\tau}| + \left|\sum_{\nu=1}^{n}\vartheta_\nu^\alpha\right|\right) \le \frac{2}{\tau}(v_r e^{-\sigma\tau} + \vartheta_r),\\
|\beta_r^*(\tau)| &\le \frac{2}{|\rho_r|\tau}\left(\sum_{\nu=1}^{n}|v_\nu^\beta||e^{\lambda_\nu^*\tau}| + \left|\sum_{\nu=1}^{n}\vartheta_\nu^\beta\right|\right) \le \frac{2}{\tau}(v_r e^{-\sigma\tau} + \vartheta_r),
\end{aligned}\right\} \quad r = \overline{1,n}.$$

Here, it is denoted

$$\begin{gathered}
v_r = (v_1^\rho + v_2^\rho + \ldots + v_n^\rho)/|\rho_r|,\\
v_\nu^\rho = (|\rho_\nu^s| + |\rho_\nu^c|)(|\rho_\nu^{ss}| + |\rho_\nu^{sc}| + |\rho_\nu^{cs}| + |\rho_\nu^{cc}|) \ge \max(|v_\nu^\alpha|, |v_\nu^\beta|),\\
\vartheta_r = \max(|\vartheta_1^\alpha + \vartheta_2^\alpha + \ldots + \vartheta_n^\alpha|, |\vartheta_1^\beta + \vartheta_2^\beta + \ldots + \vartheta_n^\beta|)/|\rho_r|.
\end{gathered}$$

The scalars v_r and ϑ_r $(r = \overline{1,n})$ depend on the initial state vector $\boldsymbol{x}(t_0)$ which forms the amplitudes ρ_ν^s and ρ_ν^c $(\nu = \overline{1,n})$ of solution (A.4):

$$\begin{aligned}
y_*^{(\gamma)}(t_0) &= \sum_{\nu=1}^{n}[\rho_\nu^s \operatorname{Im}(\lambda_\nu^* + j\omega_\nu^*)^\gamma + \rho_\nu^c \operatorname{Re}(\lambda_\nu^* + j\omega_\nu^*)^\gamma] = \sum_{\nu=1}^{n}\rho_\nu^*(\lambda_\nu^* + j\omega_\nu^*)^\gamma\\
&= \boldsymbol{c}^{\mathrm{T}}A^\gamma\left[\boldsymbol{x}(t_0) + \sum_{r=1}^{n}\rho_r\omega_r(E_n\omega_r^2 + A^2)^{-1}\boldsymbol{b}\right], \quad \gamma = \overline{0,n-1}.
\end{aligned}$$

From the equation above, we obtain

$$\left[\begin{array}{cccc} \rho_1^* & \rho_2^* & \cdots & \rho_n^* \end{array}\right]^{\mathrm{T}} = S^{-*}\mathcal{O}\left[\boldsymbol{x}(t_0) + \sum_{r=1}^{n} \rho_r\omega_r(E_n\omega_r^2 + A^2)^{-1}\boldsymbol{b}\right],$$

where

$$S^* = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1^* + j\omega_1^* & \lambda_2^* + j\omega_2^* & \cdots & \lambda_n^* + j\omega_n^* \\ \vdots & \vdots & \ddots & \vdots \\ (\lambda_1^* + j\omega_1^*)^{n-1} & (\lambda_2^* + j\omega_2^*)^{n-1} & \cdots & (\lambda_n^* + j\omega_n^*)^{n-1} \end{pmatrix}, \quad \mathcal{O} = \begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{c}A \\ \vdots \\ \boldsymbol{c}A^{n-1} \end{pmatrix}.$$

Clearly, for $\omega_\nu^* = 0$, we have $\rho_\nu^s = 0$ and $\rho_\nu^c = \rho_\nu^*$. Otherwise, if $\omega_{\nu'}^* \neq 0$, there exists $\omega_{\nu''}^* \neq 0$ such that $\lambda_{\nu'}^* = \lambda_{\nu''}^*$ and $\omega_{\nu'}^* = -\omega_{\nu''}^*$. In that case we have $\rho_{\nu'}^c = \rho_{\nu''}^c = \operatorname{Re}\rho_{\nu'}^* = \operatorname{Re}\rho_{\nu''}^*$ and $\rho_{\nu'}^s = -\rho_{\nu''}^s = -\operatorname{Im}\rho_{\nu'}^* = \operatorname{Im}\rho_{\nu''}^*$.

The component (A.2) can be expressed in terms of the frequency parameters, with the Fourier filters being its projectors on the appropriate pairs of functions orthogonal in L_2. With these functions, the frequency parameters can be found:

$$\begin{aligned} \alpha_r &= \lim_{\tau\to\infty} \frac{2}{\rho_r\tau} \int_{t_0}^{t_0+\tau} y_u(t)\sin\omega_r(t-t_0)\,dt, \\ \beta_r &= \lim_{\tau\to\infty} \frac{2}{\rho_r\tau} \int_{t_0}^{t_0+\tau} y_u(t)\cos\omega_r(t-t_0)\,dt, \end{aligned} \qquad r = \overline{1,n}. \tag{A.5}$$

Indeed, replacing $y(t)$ with (A.2) in (7), we obtain

$$\begin{aligned} \alpha_r^u(\tau) &= \sum_{\nu=1}^{n} \left[\alpha_\nu\xi_{\nu r}^{ss}(\tau) + \beta_\nu\xi_{\nu r}^{cs}(\tau)\right], \\ \beta_r^u(\tau) &= \sum_{\nu=1}^{n} \left[\alpha_\nu\xi_{\nu r}^{sc}(\tau) + \beta_\nu\xi_{\nu r}^{cc}(\tau)\right], \end{aligned} \qquad r = \overline{1,n}, \tag{A.6}$$

where

$$\begin{aligned} \xi_{\nu r}^{ss}(\tau) &= \frac{2}{\tau}\frac{\rho_\nu}{\rho_r} \int_{t_0}^{t_0+\tau} \sin\omega_\nu(t-t_0)\sin\omega_r(t-t_0)\,dt, \\ \xi_{\nu r}^{cs}(\tau) &= \frac{2}{\tau}\frac{\rho_\nu}{\rho_r} \int_{t_0}^{t_0+\tau} \cos\omega_\nu(t-t_0)\sin\omega_r(t-t_0)\,dt, \\ \xi_{\nu r}^{sc}(\tau) &= \frac{2}{\tau}\frac{\rho_\nu}{\rho_r} \int_{t_0}^{t_0+\tau} \sin\omega_\nu(t-t_0)\cos\omega_r(t-t_0)\,dt, \\ \xi_{\nu r}^{cc}(\tau) &= \frac{2}{\tau}\frac{\rho_\nu}{\rho_r} \int_{t_0}^{t_0+\tau} \cos\omega_\nu(t-t_0)\cos\omega_r(t-t_0)\,dt, \end{aligned} \qquad \nu = \overline{1,n}, \quad r = \overline{1,n}.$$

Computing the limits of the tabulated integrals gives $\lim\limits_{\tau\to\infty}\xi_{rr}^{ss}(\tau) = \lim\limits_{\tau\to\infty}\xi_{rr}^{ss}(\tau) = 1$, $\lim\limits_{\tau\to\infty}\xi_{\nu\neq r}^{ss}(\tau) = \lim\limits_{\tau\to\infty}\xi_{\nu r}^{cs}(\tau) = \lim\limits_{\tau\to\infty}\xi_{\nu r}^{sc}(\tau) = \lim\limits_{\tau\to\infty}\xi_{\nu\neq r}^{cc}(\tau) = 0$ ($\nu = \overline{1,n}$, $r = \overline{1,n}$), which confirms the validity of (A.5).

As τ increases, the rate of decrease of the difference between (A.6) and (A.5) is given by

$$\begin{aligned} \alpha_r^u(\tau) - \alpha_r &= \frac{1}{\tau}\sum_{\nu=1}^{n}\left[\alpha_\nu \overline{\xi}_{\nu r}^{ss}(\tau) + \beta_\nu \xi_{\nu r}^{cs}(\tau)\right], \\ \beta_r^u(\tau) - \beta_r &= \frac{1}{\tau}\sum_{\nu=1}^{n}\left[\alpha_\nu \xi_{\nu r}^{sc}(\tau) + \beta_\nu \overline{\xi}_{\nu r}^{cc}(\tau)\right], \end{aligned} \quad r = \overline{1,n}.$$

The absolute values of the functions $\overline{\xi}_{\nu r}^{ss}(\tau) = \xi_{\nu r}^{ss}(\tau) - e_{\nu r}, \xi_{\nu r}^{cs}(\tau), \xi_{\nu r}^{sc}(\tau)$, and $\overline{\xi}_{\nu r}^{cc}(\tau) = \xi_{\nu r}^{cc}(\tau) - e_{\nu r}$ have the form

$$\begin{aligned} |\overline{\xi}_{\nu r}^{ss}(\tau)| &= |(\sin(\omega_\nu - \omega_r)\tau/(\omega_\nu - \omega_r) - \sin(\omega_\nu + \omega_r)\tau/(\omega_\nu + \omega_r))\rho_\nu/\rho_r| \\ &\le \overline{\xi}_{\nu r}^{ss} = (1/(|\omega_\nu - \omega_r|) + 1/(|\omega_\nu + \omega_r|))|\rho_\nu/\rho_r|; \\ |\overline{\xi}_{\nu r}^{cc}(\tau)| &= |(\sin(\omega_\nu - \omega_r)\tau/(\omega_\nu - \omega_r) + \sin(\omega_\nu + \omega_r)\tau/(\omega_\nu + \omega_r))\rho_\nu/\rho_r| \\ &\le \overline{\xi}_{\nu r}^{cc} = (1/(|\omega_\nu - \omega_r|) + 1/(|\omega_\nu + \omega_r|))|\rho_\nu/\rho_r|; \\ |\xi_{\nu r}^{cs}(\tau)| &= |(\cos(\omega_\nu - \omega_r)\tau/(\omega_\nu - \omega_r) - \cos(\omega_\nu + \omega_r)\tau/(\omega_\nu + \omega_r) - 2\omega_r/(\omega_\nu^2 - \omega_r^2))\rho_\nu/\rho_r| \\ &\le \xi_{\nu r}^{cs} = (1/(|\omega_\nu - \omega_r|) + 1/(|\omega_\nu + \omega_r|) + 2|\omega_r|/(|\omega_\nu^2 - \omega_r^2|))|\rho_\nu/\rho_r|; \\ |\xi_{\nu r}^{sc}(\tau)| &= |(\cos(\omega_\nu - \omega_r)\tau/(\omega_\nu - \omega_r) - \cos(\omega_\nu + \omega_r)\tau/(\omega_\nu + \omega_r) + 2\omega_\nu/(\omega_\nu^2 - \omega_r^2))\rho_\nu/\rho_r| \\ &\le \xi_{\nu r}^{sc} = (1/(|\omega_\nu - \omega_r|) + 1/(|\omega_\nu + \omega_r|) + 2|\omega_\nu|/(|\omega_\nu^2 - \omega_r^2|))|\rho_\nu/\rho_r| \end{aligned}$$

for $\nu \ne r$, and

$$\begin{aligned} |\overline{\xi}_{rr}^{ss}(\tau)| &= |-\sin\omega_r\tau\cos\omega_r\tau/\omega_r| \le \overline{\xi}_{rr}^{ss} = 1/|\omega_r|; \\ |\overline{\xi}_{rr}^{cc}(\tau)| &= |\sin\omega_r\tau\cos\omega_r\tau/\omega_r| \le \overline{\xi}_{rr}^{cc} = 1/|\omega_r|; \\ |\xi_{rr}^{cs}(\tau)| = |\xi_{rr}^{sc}(\tau)| &= |\sin^2\omega_r\tau/\omega_r| \le \xi_{rr}^{cs} = \xi_{rr}^{sc} = 1/|\omega_r| \end{aligned}$$

otherwise. It is seen that they are bounded by the quantities $\overline{\xi}_{\nu r}^{ss}$, $\xi_{\nu r}^{cs}$, $\xi_{\nu r}^{sc}$, and $\overline{\xi}_{\nu r}^{cc}$ ($\nu = \overline{1,n},\ r = \overline{1,n}$). Similarly,

$$|\alpha_r^u(\tau) - \alpha_r| \le \frac{2}{\tau}\xi_r, \quad |\beta_r^u(\tau) - \beta_r| \le \frac{2}{\tau}\xi_r, \quad r = \overline{1,n},$$

where $\xi_r = \dfrac{1}{2}\max\left(\sum\limits_{\nu=1}^{n}\left(|\alpha_\nu|\overline{\xi}_{\nu r}^{ss} + |\beta_\nu|\xi_{\nu r}^{cs}\right), \sum\limits_{\nu=1}^{n}\left(|\alpha_\nu|\xi_{\nu r}^{sc} + |\beta_\nu|\overline{\xi}_{\nu r}^{cc}\right)\right)$ $(r = \overline{1,n})$.

It is also clear that for τ being multiple of $2\pi/\omega_\delta$ and ω_r $(r = \overline{1,n})$ being multiples of ω_δ, we have $\xi_r = 0$ $(r = \overline{1,n})$, where $\omega_\delta = \min(\omega_1, \omega_2, \ldots, \omega_n)$ and

$$\alpha_r^u(\tau) = \alpha_r, \quad \beta_r^u(\tau) = \beta_r, \quad r = \overline{1,n}.$$

The outputs of the Fourier filter excited by component (A.3) have the form

$$\alpha_r^f(\tau) = \frac{2}{\rho_r\tau}\int\limits_{t_0}^{t_0+\tau} y_f(t)\sin\omega_r(t - t_0)\,dt = \frac{2}{\rho_r\tau}\sum_{\mu=0}^{\infty}\frac{f_\mu}{|d(j\omega_\mu^f)|}$$
$$\times \int\limits_{t_0}^{t_0+\tau}\sin(\omega_\mu^f t + \psi_\mu^f)\sin\omega_r(t - t_0)\,dt = \frac{2}{\rho_r\tau}\sum_{\mu=0}^{\infty}\frac{f_\mu}{|d(j\omega_\mu^f)|}s_{\mu r}^{\alpha}, \quad r = \overline{1,n},$$
$$\beta_r^f(\tau) = \frac{2}{\rho_r\tau}\int\limits_{t_0}^{t_0+\tau} y_f(t)\cos\omega_r(t - t_0)\,dt = \frac{2}{\rho_r\tau}\sum_{\mu=0}^{\infty}\frac{f_\mu}{|d(j\omega_\mu^f)|}$$
$$\times \int\limits_{t_0}^{t_0+\tau}\sin(\omega_\mu^f t + \psi_\mu^f)\cos\omega_r(t - t_0)\,dt = \frac{2}{\rho_r\tau}\sum_{\mu=0}^{\infty}\frac{f_\mu}{|d(j\omega_\mu^f)|}s_{\mu r}^{\beta}, \quad r = \overline{1,n},$$

where

$$s_{\mu r}^{\alpha} = -\frac{\sin[(\omega_\mu^f + \omega_r)(t_0 + \tau) + \psi_\mu^f] - \sin[(\omega_\mu^f + \omega_r)t_0 + \psi_\mu^f]}{2(\omega_\mu^f + \omega_r)} + \frac{\sin[(\omega_\mu^f - \omega_r)(t_0 + \tau) + \psi_\mu^f] - \sin[(\omega_\mu^f - \omega_r)t_0 + \psi_\mu^f]}{2(\omega_\mu^f - \omega_r)},$$

$$s_{\mu r}^{\beta} = -\frac{\cos[(\omega_\mu^f + \omega_r)(t_0 + \tau) + \psi_\mu^f] - \cos[(\omega_\mu^f + \omega_r)t_0 + \psi_\mu^f]}{2(\omega_\mu^f + \omega_r)} - \frac{\cos[(\omega_\mu^f - \omega_r)(t_0 + \tau) + \psi_\mu^f] - \cos[(\omega_\mu^f - \omega_r)t_0 + \psi_\mu^f]}{2(\omega_\mu^f - \omega_r)}.$$

Let us denote

$$d^* = \min_{0 \le \omega \le \infty} |d(j\omega)|,$$

$$s_r^* = \max_{0 \le \mu \le \infty} (1/|\omega_\mu^f + \omega_r| + 1/|\omega_\mu^f - \omega_r|)/|\rho_r| \quad (r = \overline{1,n}).$$

With (13) in mind, we have

$$\left.\begin{aligned} |\alpha_r^f(\tau)| &\le \frac{2}{|\rho_r|\tau} \sum_{\mu=0}^{\infty} \frac{1}{|d(j\omega_\mu^f)|} |f_\mu||s_{\mu r}^{\alpha}| \le \frac{2}{\tau}\frac{s_r^*}{d^*} \sum_{\mu=0}^{\infty} |f_\mu| \le \frac{2}{\tau}\frac{s_r^*}{d^*} f^*, \\ |\beta_r^f(\tau)| &\le \frac{2}{|\rho_r|\tau} \sum_{\mu=0}^{\infty} \frac{1}{|d(j\omega_\mu^f)|} |f_\mu||s_{\mu r}^{\beta}| \le \frac{2}{\tau}\frac{s_r^*}{d^*} \sum_{\mu=0}^{\infty} |f_\mu| \le \frac{2}{\tau}\frac{s_r^*}{d^*} f^*, \end{aligned}\right\} \quad r = \overline{1,n}.$$

Hence, the filtering errors satisfy the following inequalities:

$$|\Delta\, \alpha_r(\tau)| \le \frac{2}{\tau}(v_r e^{-\sigma\tau} + c_r), \quad |\Delta\, \beta_r(\tau)| \le \frac{2}{\tau}(v_r e^{-\sigma\tau} + c_r), \quad r = \overline{1,n},$$

where $c_r = \vartheta_r + \xi_r + s_r^* f^*/d^*$.

APPENDIX 2

Proof of Lemma 1. Equation (1) with test signal (6) possesses the solution

$$y(t) = y_0(t) + \sum_{r=1}^{n} \rho_r \Big[\alpha_r \sin\omega_r(t - t_0) + \beta_r \cos\omega_r(t - t_0)\Big] + y_f(t),$$

where

$$y_0(t) = \boldsymbol{c}^{\mathrm{T}} e^{A(t-t_0)} \left[\boldsymbol{x}(t_0) + \sum_{r=1}^{n} \rho_r \omega_r (E_n \omega_r^2 + 1A^2)^{-1} \boldsymbol{b}\right],$$

$$y_f(t) = \boldsymbol{c}^{\mathrm{T}} \int_{t_0}^{t} e^{A(t-\tau)} \boldsymbol{m} f(\tau)\, d\tau.$$

Here A, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{m}$ are the parameters of the Cauchy form of Eq. (1): $\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{b}u + \boldsymbol{m}f$, $y = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$. Equation (19) with test signal (6) has the similar solution:

$$\check{y}(t) = \check{y}_0(t) + \sum_{r=1}^{n} \rho_r \Big[\check{\alpha}_r \sin\omega_r(t - t_0) + \check{\beta}_r \cos\omega_r(t - t_0)\Big].$$

We next substitute the quantities $u^{(p)}(t) = \sum\limits_{r=1}^{n} \rho_r \omega_r^p \Big[\operatorname{Re} j^p \sin\omega_r(t-t_0) + \operatorname{Im} j^p \cos\omega_r(t-t_0)\Big]$, $y^{(p)}(t) = y_0^{(p)}(t) + \sum\limits_{r=1}^{n} \rho_r \omega_r^p \Big[(\operatorname{Re} j^p \alpha_r - \operatorname{Im} j^p \beta_r)\sin\omega_r(t-t_0) + (\operatorname{Im} j^p \alpha_r + \operatorname{Re} j^p \beta_r)\cos\omega_r(t-t_0)\Big] + y_f^{(p)}(t)$ and $\breve{y}^{(p)}(t) = \breve{y}_0^{(p)}(t) + \sum\limits_{r=1}^{n} \rho_r \omega_r^p \Big[(\operatorname{Re} j^p \breve{\alpha}_r - \operatorname{Im} j^p \breve{\beta}_r)\sin\omega_r(t-t_0) + (\operatorname{Im} j^p \breve{\alpha}_r + \operatorname{Re} j^p \breve{\beta}_r)\cos\omega_r(t-t_0)\Big]$ into the appropriate expressions to obtain

$$\Big(y^{(p)}, \breve{y}^{(r)}\Big) = \sum_{\mu=1}^{n} \rho_\mu \omega_\mu^p \sum_{\nu=1}^{n} \rho_\nu \omega_\nu^r$$
$$\times \Big\{ \Big([\operatorname{Re} j^p \alpha_\mu - \operatorname{Im} j^p \beta_\mu] \sin\omega_\mu(t-t_0),\ [\operatorname{Re} j^r \breve{\alpha}_\nu - \operatorname{Im} j^r \breve{\beta}_\nu] \sin\omega_\nu(t-t_0) \Big)$$
$$+ \Big([\operatorname{Re} j^p \alpha_\mu - \operatorname{Im} j^p \beta_\mu] \sin\omega_\mu(t-t_0),\ [\operatorname{Im} j^r \breve{\alpha}_\nu + \operatorname{Re} j^r \breve{\beta}_\nu] \cos\omega_\nu(t-t_0) \Big)$$
$$+ \Big([\operatorname{Im} j^p \alpha_\mu + \operatorname{Re} j^p \beta_\mu] \cos\omega_\mu(t-t_0),\ [\operatorname{Re} j^r \breve{\alpha}_\nu - \operatorname{Im} j^r \breve{\beta}_\nu] \sin\omega_\nu(t-t_0) \Big)$$
$$+ \Big([\operatorname{Im} j^p \alpha_\mu + \operatorname{Re} j^p \beta_\mu] \cos\omega_\mu(t-t_0),\ [\operatorname{Im} j^r \breve{\alpha}_\nu + \operatorname{Re} j^r \breve{\beta}_\nu] \cos\omega_\nu(t-t_0) \Big) \Big\}$$
$$= \frac{1}{2} \sum_{\mu=1}^{n} \rho_\mu^2 \omega_\mu^{p+r} \Big([\operatorname{Re} j^p \alpha_\mu - \operatorname{Im} j^p \beta_\mu][\operatorname{Re} j^r \breve{\alpha}_\mu - \operatorname{Im} j^r \breve{\beta}_\mu] + [\operatorname{Im} j^p \alpha_\mu + \operatorname{Re} j^p \beta_\mu][\operatorname{Im} j^r \breve{\alpha}_\mu + \operatorname{Re} j^r \breve{\beta}_\mu] \Big)$$
$$= \frac{1}{2} \sum_{\mu=1}^{n} \rho_\mu^2 \omega_\mu^{p+r} \begin{bmatrix} -\breve{\alpha}_\mu & -\breve{\beta}_\mu \end{bmatrix} \times \begin{pmatrix} \operatorname{Re} j^p \operatorname{Re} j^r + \operatorname{Im} j^p \operatorname{Im} j^r & \operatorname{Re} j^p \operatorname{Im} j^r - \operatorname{Im} j^p \operatorname{Re} j^r \\ \operatorname{Im} j^p \operatorname{Re} j^r - \operatorname{Re} j^p \operatorname{Im} j^r & \operatorname{Re} j^p \operatorname{Re} j^r + \operatorname{Im} j^p \operatorname{Im} j^r \end{pmatrix} \times \begin{bmatrix} -\alpha_\mu \\ -\beta_\mu \end{bmatrix}$$
$$= \frac{1}{2} \sum_{\mu=1}^{n} \rho_\mu^2 \omega_\mu^{p+r} \begin{bmatrix} -\breve{\alpha}_\mu & -\breve{\beta}_\mu \end{bmatrix} \times J^{r\mathrm{T}} J^p \times \begin{bmatrix} -\alpha_\mu \\ -\beta_\mu \end{bmatrix}$$
$$= \frac{1}{2} \breve{\boldsymbol{h}}^{\mathrm{T}} \operatorname{diag} \begin{pmatrix} \rho_1^2 \omega_1^{p+r} & \rho_2^2 \omega_2^{p+r} & \cdots & \rho_n^2 \omega_n^{p+r} \end{pmatrix} \otimes J^{r\mathrm{T}} J^p \boldsymbol{h}$$
$$= \frac{1}{2} \breve{\boldsymbol{h}}^{\mathrm{T}} \Big[\operatorname{diag} \begin{pmatrix} \omega_1 & \omega_2 & \cdots & \omega_n \end{pmatrix} \otimes J \Big]^{r\mathrm{T}} \times \Big[\operatorname{diag} \begin{pmatrix} \rho_1 & \rho_2 & \cdots & \rho_n \end{pmatrix} \otimes E_2 \Big]^{\mathrm{T}}$$
$$\times \Big[\operatorname{diag} \begin{pmatrix} \rho_1 & \rho_2 & \cdots & \rho_n \end{pmatrix} \otimes E_2 \Big] \times \Big[\operatorname{diag} \begin{pmatrix} \omega_1 & \omega_2 & \cdots & \omega_n \end{pmatrix} \otimes J \Big]^{p} \boldsymbol{h} = \frac{1}{2} \breve{\boldsymbol{h}}^{\mathrm{T}} \Omega^{r\mathrm{T}} R^{\mathrm{T}} R \Omega^p \boldsymbol{h},$$

$$\Big(y^{(p)},\ u^{(r)}\Big) = \sum_{\mu=1}^{n} \rho_\mu \omega_\mu^p \sum_{\nu=1}^{n} \rho_\nu \omega_\nu^r$$
$$\times \Big\{ \Big([\operatorname{Re} j^p \alpha_\mu - \operatorname{Im} j^p \beta_\mu] \sin\omega_\mu(t-t_0),\ \operatorname{Re} j^r \sin\omega_\nu(t-t_0) \Big)$$
$$+ \Big([\operatorname{Re} j^p \alpha_\mu - \operatorname{Im} j^p \beta_\mu] \sin\omega_\mu(t-t_0),\ \operatorname{Im} j^r \cos\omega_\nu(t-t_0) \Big)$$
$$+ \Big([\operatorname{Im} j^p \alpha_\mu + \operatorname{Re} j^p \beta_\mu] \cos\omega_\mu(t-t_0),\ \operatorname{Re} j^r \sin\omega_\nu(t-t_0) \Big)$$
$$+ \Big([\operatorname{Im} j^p \alpha_\mu + \operatorname{Re} j^p \beta_\mu] \cos\omega_\mu(t-t_0),\ \operatorname{Im} j^r \cos\omega_\nu(t-t_0) \Big) \Big\}$$
$$= \frac{1}{2} \sum_{\mu=1}^{n} \rho_\mu^2 \omega_\mu^{p+r} \Big([\operatorname{Re} j^p \alpha_\mu - \operatorname{Im} j^p \beta_\mu][\operatorname{Re} j^r] + [\operatorname{Im} j^p \alpha_\mu + \operatorname{Re} j^p \beta_\mu][\operatorname{Im} j^r] \Big)$$
$$= -\frac{1}{2} \sum_{\mu=1}^{n} \rho_\mu^2 \omega_\mu^{p+r} \begin{bmatrix} 1 & 0 \end{bmatrix} \times \begin{pmatrix} \operatorname{Re} j^p \operatorname{Re} j^r + \operatorname{Im} j^p \operatorname{Im} j^r & \operatorname{Re} j^p \operatorname{Im} j^r - \operatorname{Im} j^p \operatorname{Re} j^r \\ \operatorname{Im} j^p \operatorname{Re} j^r - \operatorname{Re} j^p \operatorname{Im} j^r & \operatorname{Re} j^p \operatorname{Re} j^r + \operatorname{Im} j^p \operatorname{Im} j^r \end{pmatrix} \times \begin{bmatrix} -\alpha_\mu \\ -\beta_\mu \end{bmatrix}$$
$$= -\frac{1}{2} \boldsymbol{i}^{\mathrm{T}} \Omega^{r\mathrm{T}} R^{\mathrm{T}} R \Omega^p \boldsymbol{h},$$

$$\Big(u^{(q)},\ \breve{y}^{(r)}\Big) = -\frac{1}{2} \breve{\boldsymbol{h}}^{\mathrm{T}} \Omega^{r\mathrm{T}} R^{\mathrm{T}} R \Omega^q \boldsymbol{i},$$

$$\left(u^{(q)},\, u^{(r)}\right) = \sum_{\mu=1}^{n} \rho_\mu \omega_\mu^q \sum_{\nu=1}^{n} \rho_\nu \omega_\nu^r$$

$$\times\Big[\Big(\mathrm{Re}\, j^q \sin\omega_\mu(t-t_0),\, \mathrm{Re}\, j^r \sin\omega_\nu(t-t_0)\Big) + \Big(\mathrm{Re}\, j^q \sin\omega_\mu(t-t_0),\, \mathrm{Im}\, j^r \cos\omega_\nu(t-t_0)\Big)$$

$$+\Big(\mathrm{Im}\, j^q \cos\omega_\mu(t-t_0),\, \mathrm{Re}\, j^r \sin\omega_\nu(t-t_0)\Big) + \Big(\mathrm{Im}\, j^q \cos\omega_\mu(t-t_0),\, \mathrm{Im}\, j^r \cos\omega_\nu(t-t_0)\Big)\Big]$$

$$= \frac{1}{2}\sum_{\mu=1}^{n} \rho_\mu^2 \omega_\mu^{q+r} \left(\mathrm{Re}\, j^q \mathrm{Re}\, j^r + \mathrm{Im}\, j^q \mathrm{Im}\, j^r\right)$$

$$= \frac{1}{2}\sum_{\mu=1}^{n} \rho_\mu^2 \omega_\mu^{q+r} \begin{bmatrix} 1 & 0 \end{bmatrix} \times \begin{pmatrix} \mathrm{Re}\, j^q \mathrm{Re}\, j^r + \mathrm{Im}\, j^q \mathrm{Im}\, j^r & \mathrm{Re}\, j^q \mathrm{Im}\, j^r - \mathrm{Im}\, j^q \mathrm{Re}\, j^r \\ \mathrm{Im}\, j^q \mathrm{Re}\, j^r - \mathrm{Re}\, j^q \mathrm{Im}\, j^r & \mathrm{Re}\, j^q \mathrm{Re}\, j^r + \mathrm{Im}\, j^q \mathrm{Im}\, j^r \end{pmatrix} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2}\boldsymbol{i}^{\mathrm{T}} \Omega^{r\mathrm{T}} R^{\mathrm{T}} R \Omega^q \boldsymbol{i}.$$

Finally, substituting the values

$$\left(u^{(q)},\, u^{(r)}\right) = \frac{1}{2}\, \boldsymbol{i}^{\mathrm{T}} \Omega^{r\mathrm{T}} R^{\mathrm{T}} R \Omega^q \boldsymbol{i}, \quad \left(y^{(p)},\, u^{(r)}\right) = -\frac{1}{2}\, \boldsymbol{i}^{\mathrm{T}} \Omega^{r\mathrm{T}} R^{\mathrm{T}} R \Omega^p \boldsymbol{h},$$

$$\left(u^{(q)},\, \check{y}^{(r)}\right) = -\frac{1}{2}\check{\boldsymbol{h}}^{\mathrm{T}} \Omega^{r\mathrm{T}} R^{\mathrm{T}} R \Omega^q \boldsymbol{i}, \quad \left(y^{(p)},\, \check{y}^{(r)}\right) = \frac{1}{2}\check{\boldsymbol{h}}^{\mathrm{T}} \Omega^{r\mathrm{T}} R^{\mathrm{T}} R \Omega^p \boldsymbol{h}$$

obtained above into system (21) and multiplying the associated equations by 2, we arrive at system (24).

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