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On Convergence of External Ellipsoidal Approximations of the Reachability Domains of Discrete Dynamic Linear Systems[1]

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Abstract—The ellipsoid technique is widely used in the guaranteed estimation for approximation of the reachability domains of dynamic systems. The present paper considered the issues of external ellipsoidal estimation of the current and limiting reachability sets of a stable discrete dynamic linear system. Recurrent estimation algorithms using the criterion of minimum trace of the "weighted" ellipsoid matrix were developed for these systems, and their limiting properties were considered.

1. INTRODUCTION AND FORMULATION OF THE PROBLEM

Wide development of the deterministic or guaranteed approach to estimation, filtration, and identification of the systems [1–3] started in the 1970's as an alternative to the statistical methods and Kalman filtration. This approach assumes that the system errors and perturbations are unknown, but bounded in norm by vectors. This representation of uncertainty in the model is most natural in many applications. Yet owing to their complexity and computational laboriousness, the algorithms based on this description are used insufficiently in applications. For example, the interval formulation often results in *NP*-hard problems. In this connection, further development and simplification of these methods is of prime importance. Here, the method of ellipsoids where the system errors are assumed to satisfy quadratic ellipsoidal constraints and an ellipsoid containing the system phase vector is sought is one of simplest and most convenient approaches to guaranteed estimation. The ellipsoidal technique is a rather popular tool (see [4–8]) used, in particular, to analyze various problems of the control theory.

Let us consider a stationary discrete linear model obeying the equation

$$x_{k+1} = A\,x_k + B\,w_k, \qquad k = 0, 1, 2, \dots, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the system phase vector, $w_k \in \mathbb{R}^m$ is the vector of external perturbations, and A and $B \neq 0$ are real matrices of appropriate sizes. We assume without loss of generality that $\|w_k\| \leq 1$, $k = 0, 1, 2, \dots$, where $\|\cdot\|$ is the Euclidean norm.

We denote the ellipsoid phase space $\mathbb{R}^n$ as a linear transform of the unit sphere

$$E = \{\, c + Sv : \ \|v\| \leq 1 \,\}$$

which with this notation may be degenerate, that is, have no internal points in $\mathbb{R}^n$. Here, the vector c is the center of the ellipsoid and $P = SS^{\mathrm{T}}$ is the symmetrical nonnegatively definite matrix defining its form. The ellipsoid itself will be denoted below by $E(c, P)$.

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In this case, the vector Bw_k belongs to the—possibly degenerate—ellipsoid $E(0, BB^{\mathrm{T}})$ which corresponds to the additive external perturbations acting on the system. We assume that the initial phase vector x_0 is not known precisely, but belongs to a bounded—possibly degenerate—ellipsoid $E_0 = E(c_0, P_0)$. Then,

$$D_k = \Big\{ x_k = A^k x_0 + A^{k-1} B w_0 + A^{k-2} B w_1 + \cdots + AB w_{k-2} + B w_{k-1} : \\ x_0 \in E(c_0, P_0), \quad \|w_j\| \le 1, \quad j = 0, \dots, k-1 \Big\} \tag{2}$$

is the reachability set of system (1) at the instant k. Therefore, D_k is the algebraic sum of $k+1$ ellipsoid which generally is a convex set, rather than an ellipsoid. In the problems of estimation of the states of dynamic systems, this set must be described precisely or approximately. Various algorithms approximating D_k by a class of ellipsoids that are optimal (suboptimal) in a sense were developed for continuous and discrete models [5–7, 9]. In the case of external approximation, optimality is understood in the sense of minimum-size ellipsoid. In the present paper, by the size of the ellipsoid $E(c, P)$ we mean

$$f_V(P) = \operatorname{tr} VP, \tag{3}$$

where V is a symmetrical positively definite "weight" matrix. The minimum-size ellipsoid $f_V(P)$ will be called optimal by the criterion for trace. Much attention is paid in the literature also to the minimum-volume ellipsoids [5, 6] and to other, more general, criteria [9, 10].

Measure (3) is very convenient by virtue of its linearity as the function of the matrix $P = [p_{ij}] \in \mathbb{R}_+^{n\times n}$. Its dependence on the weight matrix enables one to improve the resulting optimal ellipsoidal estimates by a "correct" choice of V. We note that for the weight identity matrix $V = I$, in particular, $f_V(P) = \sum\limits_{i=1}^{n} p_{ii}$ is the sum of the squared lengths of half-axes of $E(c, P)$.

In the general case, the weight matrix is always representable as

$$V = U^{\mathrm{T}} U, \quad \det U \ne 0. \tag{4}$$

Therefore,

$$f_V(P) = \operatorname{tr} UPU^{\mathrm{T}} = \operatorname{tr} \widetilde{P} = f_I(\widetilde{P}), \tag{5}$$

where $\widetilde{P} = UPU^{\mathrm{T}}$ is the matrix of the same ellipsoid in the transformed state space

$$\widetilde{x} = Ux. \tag{6}$$

The matrices A and B of the dynamic system (1) are transformed as follows:

$$\widetilde{A} = UAU^{-1}, \qquad \widetilde{B} = UB. \tag{7}$$

This view of measure (3) allows one to reduce the study of the resulting estimates to the case of the weight identity matrix $V = I$. When constructing estimation algorithms, however, the general expression (3) renders to the estimates an additional, matrix "degree of freedom" whose adequate use improves their accuracy.

Let us denote the spectral radius of the matrix A by

$$\rho_A = \max_{1\le i\le n} |\lambda_i(A)|, \tag{8}$$

where $\lambda_i(A)$ are the eigenvalues of A. The matrix A and system (1) are said to be stable if $\rho_A < 1$. For the stable matrices (and only for them) $\lim\limits_{k\to\infty} A^k = 0$. Therefore, it follows immediately from (1) that stability of A is the necessary and sufficient condition for convergence of the sequence of reachability sets D_k of the original dynamic system to a compactum D_∞ which, consequently, is independent of the initial ellipsoid $E(c_0, P_0)$. It is only natural to raise in this connection the question of constructing external ellipsoidal approximations of the reachability sets and examining their limiting behavior.

Using the trace criterion $f_I(P) = \operatorname{tr} P$, a recurrent locally optimal algorithm of ellipsoidal estimation of the sets D_k was proposed [11, 12] and boundedness of the estimate sequence was proved for $\rho_A < 1$. We note that stability of the matrix A does not guarantee boundedness of a similar sequence of the ellipsoids calculated using the minimum-volume criterion. For continuous dynamic linear systems, similar questions were discussed in [5, 13, 14] which studied for stability the equilibrium points of the differential equations of the locally optimal ellipsoids of stable systems ([5] considered only the variant with the diagonal matrix of dynamics A). Nevertheless, the total picture of global behavior of the external ellipsoidal estimates for the continuous stable systems is rather uncertain. For example, it is not clear whether they will be convergent or at least bounded.

The present paper aims at studying the problem of external ellipsoidal estimation of the reachability sets D_k and D_∞ and obtaining the convergence conditions for the estimation algorithms (optimal in the criterion for trace of the "weighted" matrix of ellipsoid) proposed here for the stable discrete linear systems (1).

2. ESTIMATION OF THE LIMITING REACHABILITY SET

In order to verify the possibility of direct determination of the ellipsoid $E(0, P) \supseteq D_\infty$ that is minimum by the criterion $f_V(P)$, we first of all use the following result encountered in one or another form in [2, 3, 5, 15].

Theorem 1. *Let D_∞ be the limiting reachability set of the stable dynamic system* (1)*. For any fixed $\gamma \in (\rho_A^2, 1)$, D_∞ is contained in the ellipsoid $E(0, P_\gamma)$ with the matrix which is the solution of the Lyapunov equation*

$$P_\gamma = \frac{AP_\gamma A^{\mathrm{T}}}{\gamma} + \frac{BB^{\mathrm{T}}}{1-\gamma}\,. \tag{9}$$

Moreover, for $V > 0$ the function $\varphi(\gamma) = f_V(P_\gamma) = \operatorname{tr} VP_\gamma$ is strictly convex over the interval $\rho_A^2 < \gamma < 1$.

For completeness of presentation, the theorem is proved in the Appendix.

Therefore, the trace-minimal ellipsoid from the one-parameter family $E(0, P_\gamma)$ is the solution of the convex minimization problem and provides a "good" external estimate of the limiting reachability set. This estimate can be calculated directly from the known matrices A and B without analyzing the evolution of the system reachability sets. This result is important, in particular, for suppression of bounded external perturbations, as well as in other problems of the control theory. We also note that the derivative of $f_V(P_\gamma) = \operatorname{tr} VP_\gamma$ with respect to $\gamma \in (\rho_A^2, 1)$ is calculated fairly easily because the matrix $P'_\gamma = dP_\gamma/d\gamma$ is a single solution of the following Lyapunov equation:

$$P'_\gamma = \frac{AP'_\gamma A^{\mathrm{T}}}{\gamma} - \frac{P_\gamma}{\gamma} + \frac{BB^{\mathrm{T}}}{\gamma(1-\gamma)^2}\,.$$

Example 1. Let the stable system (1) with the matrices $A = \operatorname{diag}\{0.2;\ 0.8\}$ and $B = \operatorname{diag}\{1;\ 0.2\}$ be given. For any choice of the initial ellipsoid including x_0, the system reachability domains D_k

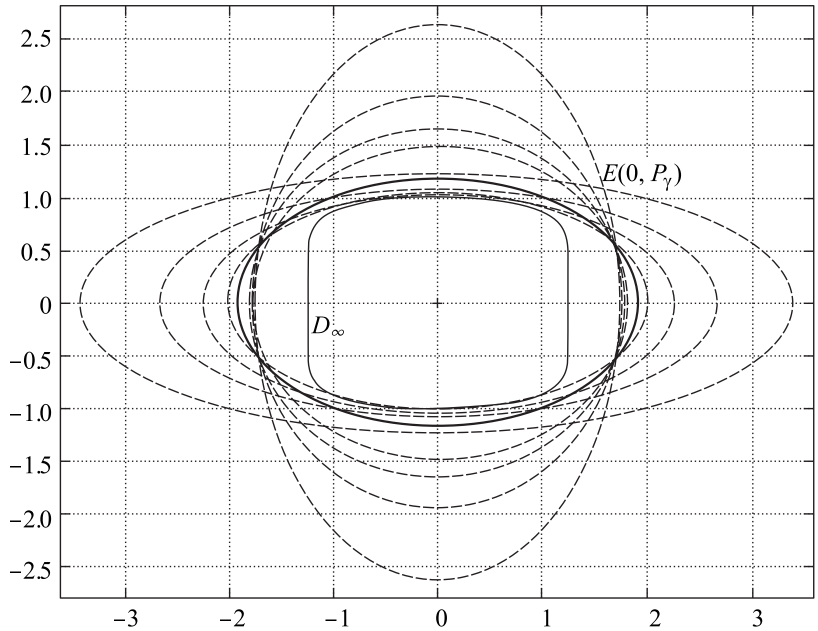

Fig. 1. Family of ellipsoids comprising the limiting reachability set.

approach the same compactum D_∞ whose boundary is shown in Fig. 1 by the thin solid line. This compactum is contained in the one-parameter family of ellipsoids $E(0, P_\gamma)$, where P_γ is the solution of the Lyapunov equation (9) for any fixed $\gamma \in (\rho_A^2, 1)$. Some ellipsoids of this family are shown in Fig. 1 by the dashed lines. The trace-minimal (for $V = I$) ellipsoid from this family (bold line) is $E(0, P)$ with the matrix $P \simeq \mathrm{diag}\{3.6797;\ 1.3727\}$. As can be seen from the figure, it is not an optimal external estimate of the limiting reachability set D_∞ among all ellipsoids comprising D_∞.

3. ESTIMATE OF THE REACHABILITY SETS D_k AND D_∞

To construct more precise ellipsoidal estimates, we make use of the following important result concerning approximation of the ellipsoid sum. Let $\mathcal{A}_N$ be the set of all possible vectors $\alpha = (\alpha_1, \ldots, \alpha_N)^{\mathrm{T}} \in \mathbb{R}^N$ such that all $\alpha_i > 0$ and $\sum\limits_{i=1}^{N} \alpha_i = 1$.

Lemma 1. *Let S_N be the algebraic sum N of the ellipsoids $E(c_i, Q_i)$ and the matrices $Q_i \geq 0$, $Q_i \neq 0$ for all values of $i = 1, \ldots, N$. Then,*

(1) *for any $\alpha \in \mathcal{A}_N$, the ellipsoid $E(c, P(\alpha))$ with the parameters*

$$c = \sum_{i=1}^{N} c_i, \qquad P(\alpha) = \sum_{i=1}^{N} \alpha_i^{-1} Q_i \tag{10}$$

includes S_N and

(2) *the function $\varphi(\alpha) = f_V(P(\alpha)) = \mathrm{tr}\, VP(\alpha)$ is strictly convex on the simplex $\mathcal{A}_N$; here, $V > 0$ is the weight matrix.*

See the proof of the lemma in [7]. Its (1) was first formulated by Schweppe in his monograph [2]. We note that the trace-minimal ellipsoid in the family $E(c, P(\alpha))$ generally needs not be minimal in the entire class of ellipsoids containing the sum S_N. Nevertheless, for a "correct" choice of the weight matrix V its distinction from the minimum one will be insignificant (see Section 5 below). The following remark applies to the special case of the sum of two ellipsoids.

Remark 1. If $N = 2$, then the one-parameter family of ellipsoids $E(c, P(\alpha))$ of (10) contains the trace-minimal ellipsoid among the entire class of ellipsoids which contain the set S_N. Therefore, minimization in the scalar parameter $\gamma = \alpha_1$ of the convex function $f_V(P(\alpha)) = \operatorname{tr} VQ_1/\gamma + \operatorname{tr} VQ_2/(1-\gamma)$ over the interval $\gamma \in (0,1)$ provides a trace-minimal ellipsoid containing S_N.

For more detail on approximation of the sum of two ellipsoids see [5, 6, 9]. For $N \geq 2$, we consider the algebraic sum S_N of the nondegenerate coaxial ellipsoids $E(0, Q_i)$ under the conditions of Lemma 1. As the following lemma asserts, the ellipsoid that comprises the sum S_N and is minimum in the (corresponding) trace criterion also belongs to the family $E(0, P(\alpha))$ (10) (for $c_i \equiv 0$).

Lemma 2. *Let $N \geq 2$ and the nondegenerate ellipsoids $E(0, Q_i)$, $i = 1, \ldots, N$, be coaxial, that is, all the transformed matrices $\widetilde{Q}_i = UQ_iU^{\mathrm{T}}$ be diagonal and have positive diagonal elements for some nondegenerate matrix U. We assume that $V = U^{\mathrm{T}}U$ and introduce the vector $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathcal{A}_N$, simplex $\mathcal{A}_N$, and matrix function $P(\alpha)$ similar to Lemma 1. Then, the ellipsoid $E(0, P(\alpha))$ corresponding to the solution of the minimization problem*

$$\min_{\alpha \in \mathcal{A}_N} \operatorname{tr} VP(\alpha) = \min_{\alpha \in \mathcal{A}_N} \sum_{i=1}^{N} \alpha_i^{-1} \operatorname{tr} VQ_i, \tag{11}$$

is minimum (by the same trace criterion $\operatorname{tr} VP$) also among all possible ellipsoids containing the algebraic sum $S_N = \sum\limits_{i=1}^{N} E(0, Q_i)$.

The proof of the lemma is given in the Appendix.

Since according to (1) the reachability set D_k of system (1) is the sum of $N = k+1$ ellipsoids with the centers

$$d_0 = A^k c_0, \qquad d_i = 0, \quad i = 1, \ldots, k \tag{12}$$

and, correspondingly, the matrices

$$Q_0 = A^k P_0 (A^{\mathrm{T}})^k, \qquad Q_i = A^{k-i} BB^{\mathrm{T}} (A^{\mathrm{T}})^{k-i}, \quad i = 1, \ldots, k, \tag{13}$$

its external ellipsoidal estimate can be constructed directly by Lemma 1 by minimizing $\operatorname{tr} VP(\alpha)$ on the simplex $\mathcal{A}_N$, which leads to the following optimal values of the parameters α_i:

$$\alpha_i = (\operatorname{tr} VQ_i)^{1/2} \left(\sum_{j=0}^{k} (\operatorname{tr} VQ_j)^{1/2} \right)^{-1}. \tag{14}$$

For these values,

$$\operatorname{tr} VP(\alpha) = \left(\left[\operatorname{tr} VA^k P_0 (A^{\mathrm{T}})^k \right]^{1/2} + \sum_{i=0}^{k-1} \left[\operatorname{tr} VA^i BB^{\mathrm{T}} (A^{\mathrm{T}})^i \right]^{1/2} \right)^2. \tag{15}$$

As follows from lemma 1 and (1), (12), and (13), the resulting ellipsoid $E(0, P(\alpha))$ contains the reachability set D_k of system (1) for the initial $x_0 \in E(0, P_0)$, $P_0 \geq 0$. As is corroborated by numerous examples, it provides a good external approximation of the set D_k for "correct" choice of the weight matrix V. Additionally, for any predefined $k \geq 1$, the matrix $P(\alpha)$ of this ellipsoid can be calculated iteratively. One can readily see that this calculation of $P(\alpha) = \widehat{P}_k$ obeys the formulas

$$W_i = AW_{i-1}A^{\mathrm{T}}, \quad i = 1, \dots, k, \quad W_0 = P_0, \tag{16}$$

$$S_i = AS_{i-1}A^{\mathrm{T}}, \quad \mu_i = (\operatorname{tr} VS_i)^{1/2}, \quad i = 1, \dots, k, \quad S_0 = BB^{\mathrm{T}}, \tag{17}$$

$$\nu_i = \nu_{i-1} + \mu_i, \quad i = 1, \dots, k, \quad \nu_0 = (\operatorname{tr} VW_k)^{1/2}, \tag{18}$$

$$\alpha_i = \mu_i/\nu_k, \quad i = 1, \dots, k, \quad \alpha_0 = \nu_0/\nu_k, \tag{19}$$

$$\widehat{P}_i = \widehat{P}_{i-1} + \alpha_i^{-1} S_i, \quad i = 1, \dots, k, \quad \widehat{P}_0 = \alpha_0^{-1} W_k. \tag{20}$$

The following theorem illustrates the limiting behavior of the matrix $\widehat{P}_k$ for $k \to \infty$.

Theorem 2. *Let the A be stable. Then, for any initial value $P_0 \geq 0$, the matrix $\widehat{P}_k$ determined by the algorithm* (16)–(20) *coincides with $P(\alpha)$* (10), (14) *and tends to the finite limit $\widehat{P}_\infty \geq 0$ for $k \to \infty$. At that,*

$$\widehat{P}_\infty = \sum_{j=0}^{\infty} \frac{A^j BB^{\mathrm{T}} (A^{\mathrm{T}})^j}{\widehat{\alpha}_j}, \tag{21}$$

where $\widehat{\alpha}_j$ are as follows:

$$\widehat{\alpha}_j = [\operatorname{tr} VA^j BB^{\mathrm{T}} (A^{\mathrm{T}})^j]^{1/2} \left(\sum_{i=0}^{\infty} [\operatorname{tr} VA^i BB^{\mathrm{T}} (A^{\mathrm{T}})^i]^{1/2} \right)^{-1}, \tag{22}$$

and the trace

$$\operatorname{tr} V\widehat{P}_\infty = \left(\sum_{j=0}^{\infty} \left[\operatorname{tr} VA^j BB^{\mathrm{T}} (A^{\mathrm{T}})^j \right]^{1/2} \right)^2. \tag{23}$$

For a sufficiently large k, algorithm (16)–(20) is, therefore, an iterative method of approximation of the limiting reachability set. Theorem 2 is proved in the Appendix on the basis of another iterative representation of the matrix $P(\alpha)$ (10), (14) which is as follows. Let the sequence

$$P_{i+1} = \frac{AP_iA^{\mathrm{T}}}{\gamma_i} + \frac{BB^{\mathrm{T}}}{1-\gamma_i}, \quad i = 0, 1, \dots, k-1, \tag{24}$$

be given for some $P_0 \geq 0$. Here, the parameters $\gamma_i \in (0, 1)$, $i = 0, 1, \dots, k-1$. It follows from (24) that

$$\begin{aligned} P_k = {} & \frac{A^k P_0 (A^{\mathrm{T}})^k}{\gamma_0 \gamma_1 \dots \gamma_{k-1}} + \frac{A^{k-1} BB^{\mathrm{T}} (A^{\mathrm{T}})^{k-1}}{(1-\gamma_0)\gamma_1 \dots \gamma_{k-1}} + \frac{A^{k-2} BB^{\mathrm{T}} (A^{\mathrm{T}})^{k-2}}{(1-\gamma_1)\gamma_2 \dots \gamma_{k-1}} \\ & + \dots + \frac{ABB^{\mathrm{T}} A^{\mathrm{T}}}{(1-\gamma_{k-2})\gamma_{k-1}} + \frac{BB^{\mathrm{T}}}{1-\gamma_{k-1}}, \end{aligned} \tag{25}$$

and, consequently, $P_k = P(\alpha)$ for the parameter vector $\alpha = (\alpha_0, \dots, \alpha_k)^{\mathrm{T}} \in \mathcal{A}_{k+1}$ defined by the relations

$$\alpha_0 = \prod_{i=0}^{k-1} \gamma_i, \qquad \alpha_j = (1-\gamma_{j-1}) \prod_{i=j}^{k-1} \gamma_i, \quad j = 1, \dots, k. \tag{26}$$

Since they define the one-one relation between the hypercube $(0,1)^k$ and the simplex $\mathcal{A}_{k+1}$, choice in algorithm (24) of the parameters $\gamma_0, \dots, \gamma_{k-1}$ from the condition

$$(\gamma_0, \dots, \gamma_{k-1}) = \operatorname*{argmin}_{0<\gamma_0,\dots,\gamma_{k-1}<1} \operatorname{tr} VP_k \tag{27}$$

leads to the same optimal matrix $P(\alpha) = P_k$ as that determined by the algorithm (16)–(20).

Lemma 3. *For any $P_0 \geq 0$, solution of* (27) *is representable as*

$$\gamma_i = \frac{[\operatorname{tr} VA^{k-i}P_i(A^{\mathrm{T}})^{k-i}]^{1/2}}{[\operatorname{tr} VA^{k-i}P_i(A^{\mathrm{T}})^{k-i}]^{1/2} + [\operatorname{tr} VA^{k-i-1}BB^{\mathrm{T}}(A^{\mathrm{T}})^{k-i-1}]^{1/2}}, \qquad i = 0, 1, \dots, k-1, \tag{28}$$

where the matrices P_i are recurrently recalculated by (24) *beginning from P_0. The resulting value of P_k coincides with $P(\alpha)$* (10), (13), (14), *as well as with the result $\widehat{P}_k$ of the algorithm* (16)–(20)*:*

$$P_k = \widehat{P}_k = (\operatorname{tr} VP_k)^{1/2} \left(\frac{A^k P_0 (A^{\mathrm{T}})^k}{[\operatorname{tr} VA^k P_0 (A^{\mathrm{T}})^k]^{1/2}} + \sum_{i=0}^{k-1} \frac{A^i BB^{\mathrm{T}} (A^{\mathrm{T}})^i}{[\operatorname{tr} VA^i BB^{\mathrm{T}} (A^{\mathrm{T}})^i\,]^{1/2}} \right). \tag{29}$$

Lemma 3 is proved in the Appendix.

We note that method (24), (28) is unsuitable for recurrent use because each time when seeking a new estimate at the next, $(k+1)$st step one has to recalculate all the preceding values of the parameters $\gamma_0, \dots, \gamma_k$ and matrices $P_1, \dots, P_k$. For smaller k, however, this seems possible.

On the other hand, by means of (24) one can readily demonstrate that the external ellipsoidal estimate $E(0, P_\gamma)$ of the set D_∞, which is minimal by the trace criterion and is obtained from Theorem 1, is always inferior in accuracy of approximation to the corresponding limiting estimate $E(0, \widehat{P}_\infty)$ of Theorem 2. Indeed, if one takes $\gamma_i \equiv \gamma \in (\rho_A^2, 1)$ in (24), then, by virtue of the Lyapunov theorem, we get

$$P_\gamma = \lim_{k\to\infty} P_k, \tag{30}$$

and since the minimum does not decrease with contraction of the admissible set,

$$\operatorname{tr} VP_k \geq \operatorname{tr} V\widehat{P}_k, \tag{31}$$

where the matrix $\widehat{P}_k$ is the result of algorithm (16)–(20) or, which is the same, the algorithm (24), (27). Hence, with regard for Theorem 2 we get the following result by passing to the limit for $k \to \infty$.

Theorem 3. *Let the matrix A be stable and P_γ and $\widehat{P}_\infty$ be as defined in Theorems 1 and 2, respectively. For any weight matrix $V > 0$, the following inequality is valid:*

$$\min_{\gamma\in(\rho_A^2,1)} \operatorname{tr} VP_\gamma \geq \operatorname{tr} V\widehat{P}_\infty. \tag{32}$$

4. LOCALLY OPTIMAL RECURRENT ALGORITHM

For the original system (1), a straightforward computational recurrent algorithm for ellipsoidal estimation can also be constructed on the basis of Lemma 1 and (24) (see [5, 7]) by approximating at each step the sum of only two ellipsoids. Indeed, we consider for some initial c_0 and $P_0 \geq 0$ the sequence of ellipsoids $E_k = E(c_k, P_k)$ with

$$c_{k+1} = Ac_k, \qquad P_{k+1} = \frac{AP_kA^{\mathrm{T}}}{\gamma_k} + \frac{BB^{\mathrm{T}}}{1-\gamma_k}, \qquad k = 0, 1, 2, \dots, \tag{33}$$

where the parameter γ_k is the solution of the one-dimensional convex problem of optimization

$$\gamma_k = \operatorname*{argmin}_{0<\gamma_k<1} \operatorname{tr} V P_{k+1}. \tag{34}$$

Then, $E_k \supseteq D_k$ for any $k = 1, 2, 3, \ldots$. As compared with the direct method (16)–(20), the recurrent algorithm (33), (34) requires straightforward computations. Moreover, according to Remark 1, at each step it produces locally optimal ellipsoids. In the global sense, however, optimality as a rule is not kept, and accuracy of the recurrent estimates is sometimes much inferior to the accuracy of the direct nonrecurrent method (16)–(20).

We note that the centers of recurrent ellipsoids $c_k = A^k c_0$ converge to zero for $k \to \infty$. Therefore, the asymptotics of these estimates depends completely on the behavior of their matrices.

We assume without loss of generality that the initial phase vector x_0 of system (1) belongs to the bounded ellipsoid $E(0, P_0)$, $P_0 \geq 0$ centered at zero. Solution of the minimization problem (34) provides locally optimal parameters

$$\gamma_k = \frac{(\operatorname{tr} V A P_k A^{\mathrm{T}})^{1/2}}{(\operatorname{tr} V A P_k A^{\mathrm{T}})^{1/2} + (\operatorname{tr} V B B^{\mathrm{T}})^{1/2}} \tag{35}$$

and enables one to set down explicitly the algorithm (33), (34):

$$P_{k+1} = \frac{\alpha_k + \beta}{\alpha_k} A P_k A^{\mathrm{T}} + \frac{\alpha_k + \beta}{\beta} B B^{\mathrm{T}}, \tag{36}$$

where

$$\alpha_k = \sqrt{\operatorname{tr} V A P_k A^{\mathrm{T}}}, \qquad \beta = \sqrt{\operatorname{tr} V B B^{\mathrm{T}}}\,. \tag{37}$$

At that, as follows from Lemma 1, $E(0, P_k) \supseteq D_k$, $k = 1, 2, 3, \ldots$.

Lemma 4. *The matrix sequence* (36) *with the coefficients* (37) *is bounded if and only if the matrix A is stable.*

Proof see in [11].

So, let the system matrix A be stable, that is, $\rho_A < 1$. Existence of a set of limiting points of the sequence follows from its boundedness. We consider the limiting points of sequence (36) and present the sufficient condition for its global convergence to the equilibrium point. Since $P_0 \neq 0$, also $P_k \neq 0$, $\forall\, k = 1, 2, 3, \ldots$. Then we assume that

$$Q_k = \frac{P_k}{\sqrt{\operatorname{tr} V P_k}}\,. \tag{38}$$

Since $\operatorname{tr} V P_{k+1} = (\alpha_k + \beta)^2$ follows from (36), (37), for the transformed matrix sequence we obtain the following recurrent equation that better yields to examination:

$$Q_{k+1} = \nu_k A Q_k A^{\mathrm{T}} + C, \qquad \nu_k = \left(\frac{\operatorname{tr} V Q_k}{\operatorname{tr} V A Q_k A^{\mathrm{T}}}\right)^{1/2}, \tag{39}$$

where $C = BB^{\mathrm{T}}/\beta \geq 0$, $\beta \neq 0$. Transformation (38) establishes a one-one correspondence between the nonnegatively definite matrices P_k and Q_k of the sequences (36) and (39), respectively. Consequently, their limiting behaviors are identical. In particular, according to Lemma 4, stability of the matrix A is the necessary and sufficient condition for boundedness also of the sequence Q_k. In what follows, we consider the asymptotic characteristics of the transformed recurrent algorithm (39).

4.1. Equilibrium

The equilibrium points of Eq. (39), if any, are the solutions of the equation

$$Q = \nu_Q AQA^{\mathrm{T}} + C, \qquad \nu_Q = \left(\frac{\operatorname{tr} VQ}{\operatorname{tr} VAQA^{\mathrm{T}}}\right)^{1/2}. \tag{40}$$

Theorem 4. *For any nonnegatively definite matrix $C \neq 0$, there exists a unique solution of the nonlinear matrix Eq.* (40) *among the nonnegatively definite matrices Q if and only if the matrix A is stable. At that, the value of the parameter ν_Q lies within the interval $1 < \nu_Q < 1/\rho_A^2$.*

See the proof in the Appendix.

Therefore, the recurrent Eq. (39) has a single equilibrium point Q^* that can be determined by numerical solution of Eq. (40). Subsequent application of the transformation inverse to (38), that is, the passage to the matrices P, leads to calculation of the matrix P^* of the equilibrium ellipsoid containing the limiting system reachability set D_∞. It would be of definite interest to study this equilibrium point for local stability, but it meets with some difficulties, and we omit it here and focus on global convergence.

4.2. Convergence

We formulate the sufficient convergence conditions for the recurrent algorithm (39) whose validity is based on the already established uniqueness of the equilibrium point and the proof of lack of other limiting points of the sequence $\{Q_k\}$.

Lemma 5. *Let the matrix $C \geq 0$, $C \neq 0$. For any stable matrix $A \in \mathbb{R}^{n\times n}$, there will be a weight matrix $V = U^{\mathrm{T}}U > 0$ such that its transformed matrix $\widetilde{A} = UAU^{-1}$ has the greatest singular value $\sigma_{\max}(\widetilde{A}) < 1$, and the solution $Q = Q^*$ of Eq.* (40) *satisfies the inequality*

$$\nu_{Q^*} \leq \sigma_{\max}^{-2}(\widetilde{A}). \tag{41}$$

Theorem 5. *Let the matrix A be stable, $C \geq 0$, $C \neq 0$, and the weight matrix $V = U^{\mathrm{T}}U > 0$ be chosen so that the transformed matrix $\widetilde{A} = UAU^{-1}$ has $\sigma_{\max}(\widetilde{A}) < 1$ and the solution $Q = Q^*$ of the Eq.* (40) *satisfies inequality* (41)*. Then, for an arbitrary initial $Q_0 \geq 0$ the matrix sequence $\{Q_k\}$ generated by algorithm* (39) *converges to the equilibrium point Q^* which is uniquely defined by the Eqs.* (40)*.*

Lemma 5 and Theorem 5 are proved in the Appendix. Therefore, the locally optimal algorithm (39) and, consequently, algorithm (36), (37) in the original variables generate a converging sequence of the ellipsoidal external estimates of the sets D_k containing D_∞ at the limit for $k \to \infty$. It can be seen, however, from relations (9) and (36) that the limiting matrix P_∞ belongs to the family $\{P_\gamma\}$ from Theorem 1, and, therefore, generally speaking the ellipsoid $E(0, P_\infty)$ is a less precise estimate of the set D_∞ as compared with the optimal ellipsoid $E(0, P_\gamma)$, that is,

$$\lim_{k\to\infty} \operatorname{tr} VP_k \geq \min_{\gamma\in(\rho_A^2,1)} \operatorname{tr} VP_\gamma. \tag{42}$$

We note that only substantially more rigid sufficient conditions for convergence of the ellipsoidal estimates are known from the literature for similar problems. For the continuous-time dynamic systems, for example, Chernous'ko [5] studied asymptotic behavior of the approximating ellipsoids only in the class of diagonal matrices, which enables one to decompose the original system into n simpler subsystems obeying scale equations. Conditions for local convergence of ellipsoids in the neighborhood of the equilibrium points were given in [13, 14]. Therefore, the results of Theorem 5 are stronger than those mentioned above.

Convergence of the recurrent ellipsoidal algorithm (39) and, consequently, of (36) is another attractive feature of the trace criterion when used to determine the minimum-size ellipsoid as compared with other criteria such as the minimum-size criterion (see [11]).

5. CHOICE OF THE WEIGHT MATRIX

Let us consider for simplicity a special case where the matrix A is not degenerate and has n different eigenvalues. Then (see, for example, [16]), there exists a real similarity transformation $\widetilde{A} = UAU^{-1}$ driving A to the block-diagonal form. Namely, for the matrix $\widetilde{A}$, only the diagonal 1×1 blocks of real eigenvalues $\lambda_j(A)$ and 2×2 blocks corresponding to the complex-conjugate pairs of nonreal eigenvalues $\lambda_j(A) = |\lambda_j(A)| \exp\{\cos\varphi_j \pm i \sin\varphi_j\}$ and representable as

$$|\lambda_j(A)| \begin{pmatrix} \cos\varphi_j & \sin\varphi_j \\ -\sin\varphi_j & \cos\varphi_j \end{pmatrix}$$

are nonzero. Hence, an arbitrary kth degree of $\widetilde{A}^k$, $k \geq 1$, repeats the block-diagonal structure of the matrix $\widetilde{A}$, the matrix $\widetilde{A}^k(\widetilde{A}^{\mathrm{T}})^k$ being diagonal. Namely,

$$\widetilde{A}^k(\widetilde{A}^{\mathrm{T}})^k = \mathrm{diag}\left\{ |\lambda_1(A)|^{2k}, \ldots, |\lambda_n(A)|^{2k} \right\}. \tag{43}$$

We also assume that the transformed matrix $\widetilde{B} = UB$ is such that the decomposition

$$\widetilde{B}\widetilde{B}^{\mathrm{T}} = \sum_{k=0}^{m} \nu_k \widetilde{A}^k(\widetilde{A}^{\mathrm{T}})^k \tag{44}$$

is valid for some $m \geq 0$ and $\nu_k \in \mathbb{R}$, $k = 0, \ldots, m$. Let in addition the matrix of the ellipsoid $E(0, \widetilde{P}_0)$ (initial states in the transformed space $\widetilde{x} = Ux$) also admit representation (44), that is, the decomposition

$$\widetilde{P}_0 = \sum_{k=0}^{l} \mu_k \widetilde{A}^k(\widetilde{A}^{\mathrm{T}})^k \tag{45}$$

be valid for some $l \geq 0$ and $\nu_k \in \mathbb{R}$, $k = 1, \ldots, l$. We note, that by virtue of (43) the conditions (44), (45) necessarily mean that the matrices $\widetilde{B}\widetilde{B}^{\mathrm{T}}$ and $\widetilde{P}_0$ are diagonal. Therefore, according to (13), diagonal are also the matrices

$$\widetilde{Q}_0 = \widetilde{A}^N \widetilde{P}_0 (\widetilde{A}^{\mathrm{T}})^N, \qquad \widetilde{Q}_k = \widetilde{A}^k \widetilde{B}\widetilde{B}^{\mathrm{T}} (\widetilde{A}^{\mathrm{T}})^k, \quad k \geq 1,$$

of the ellipsoids that are the addends of the reachability set $\widetilde{D}_{N-1}$, and in the transformed space the ellipsoids $E(0, \widetilde{Q}_k)$ are coaxial.

According to Lemma 2, the aforementioned situation is most favorable for using algorithm (16)–(20) of ellipsoidal estimation of the reachability set $\widetilde{D}_{N-1}$ which is a sum of N coaxial ellipsoids. At that, the weight matrix is defined by the aforementioned matrix of transformation of U according to $V = U^{\mathrm{T}}U$, and application of the trace criterion $f_I(\widetilde{P}) = \mathrm{tr}\,\widetilde{P}$ to the transformed phase space is equivalent to using the weighted-trace criterion $f_V(P) = \mathrm{tr}\,VP$ in the original phase space. We note that in algorithm (16)–(20) all computations are carried out in the original phase space, and no transformation is done. At that, by virtue of Theorem 2, the algorithm leads to an ellipsoid that is optimal by the weighted-trace criterion among all ellipsoids containing the reachability set D_{N-1}. Naturally, V is independent of N and is defined only by the matrix A.

If condition (44) is not satisfied, then for the above choice of the weight matrix V one cannot assert that the ellipsoid obtained by algorithm (16)–(20) is globally optimal. The same is valid also for condition (45) whose role, however, is not so important for sufficiently great N and vanishes at all at the limit for $N \to \infty$. At the same time, accuracy of the ellipsoid estimates remains sufficiently high and, as a rule, is much better than in the case of $V = I$.

For an arbitrary (stable) matrix A, there always exists the real similarity transformation $\widetilde{A} = U_\varepsilon A U_\varepsilon^{-1}$ driving A to the block-diagonal form with any predefined precision $\varepsilon > 0$ [16]. In this case, it is recommended to take the weight matrix equal to $V = U_\varepsilon U_\varepsilon^{\mathrm{T}}$. This general recommendation is applicable also to the locally optimal algorithm (39) because choice of a sufficiently small $\varepsilon > 0$ secures satisfaction of the conditions of Theorem 5 guaranteeing convergence of the generated ellipsoids.

6. NUMERICAL EXAMPLES

We begin with a simple example where the weight identity matrix $V = I$ can be naturally taken to construct the recurrent estimates by the trace criterion.

Example 2. Let the following dynamic linear system with the matrices

$$A = \begin{pmatrix} 0.2 & 0.3 \\ 0 & 0.8 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

be given. The matrix $A^{\mathrm{T}}A$ is stable (its spectral radius is $\rho_{A^{\mathrm{T}}A} = 0.7352$); therefore, we take $V = I$. Lemma 5 readily enables one to verify whether condition (41) is satisfied. The ellipsoids $E(0, P_k)$ generated by the locally-optimal algorithm (36) converge to the limiting one with the matrix

$$P_\infty \simeq \begin{pmatrix} 9.4400 & 10.4167 \\ 10.4167 & 26.1727 \end{pmatrix}$$

under any initial conditions. In particular, Fig. 2 depicts asymptotic behavior of these estimates for the case where the initial ellipsoid is a unit circle. We note that the limiting ellipsoid $E(0, P_\infty)$

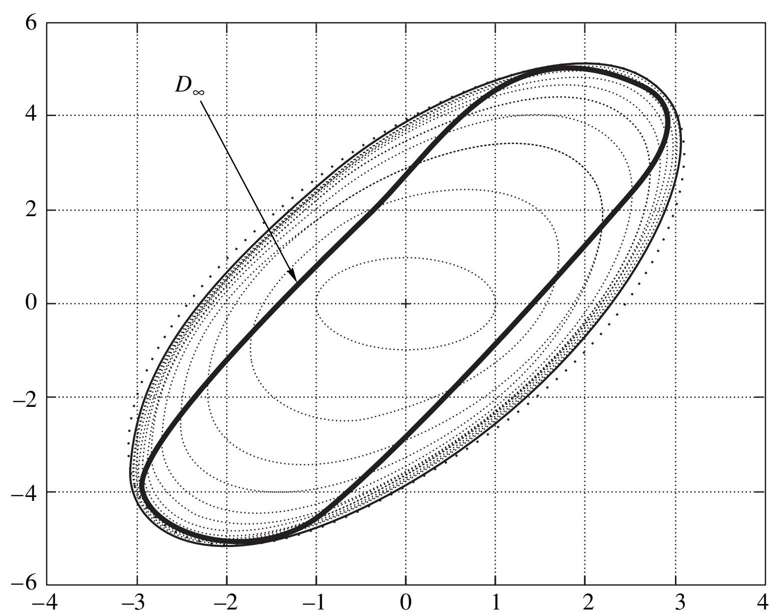

Fig. 2. Convergence of the estimates for $\rho_{A^{\mathrm{T}}A} < 1$ and $V = I$.

(thin solid line) approximates very well the limiting reachability set D_∞ whose boundary is shown by bold line. At that, it actually does not differ from the minimum ellipsoid $E(0,P)$ (point line in Fig. 2) belonging to the one-parameter family (9) obtained by Theorem 1. In the case at hand,

$$P \simeq \begin{pmatrix} 9.5761 & 9.5182 \\ 9.5182 & 25.0540 \end{pmatrix},$$

and $\operatorname{tr} P_\infty = 35.6127$ and $\operatorname{tr} P = 34.6301$. Therefore, for the case of the stable matrix $A^{\mathrm{T}}A$ one can assume that the recurrent algorithm (36) is a simple and effective iterative method of determining the external ellipsoidal estimate of the limiting reachability set D_∞ of the dynamic system (1).

Computational experiments demonstrate that for $V = I$ (36) seems to converge also under a simpler requirement of stability of the matrix A. However, we still cannot prove this fact. On the other hand, as will be illustrated by the following example, its practical value would be insignificant because for $\rho_{A^{\mathrm{T}}A} \gg 1$ the size of the limiting ellipsoid (for $V = I$) turns out to be unsatisfactory as compared with the limiting reachability set D_∞.

Example 3. Let us consider

$$A = \begin{pmatrix} 0.2 & 5 \\ 0 & 0.8 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The system matrix A is stable, but has an appreciable off-diagonal element, and on this account $\rho_{A^{\mathrm{T}}A} \gg 1$. The limiting reachability set for this case is shown by bold line in Fig. 3. Computations demonstrate that for $V = I$ and any initial ellipsoid $E(0,P_0)$, $P_0 \geq 0$, the recurrent algorithm (36) converges to the limiting ellipsoid $E(0,P_\infty)$ with the matrix

$$P_\infty \simeq \begin{pmatrix} 5.8764 & 0.7680 \\ 0.7680 & 0.1578 \end{pmatrix} \times 10^3.$$

The ellipsoid $E(0,P_\infty)$ is shown by thin solid line in Fig. 3. One can see that this estimate is very overstated in comparison with the trace-minimal ellipsoid $E(0,P)$ (point line) obtained

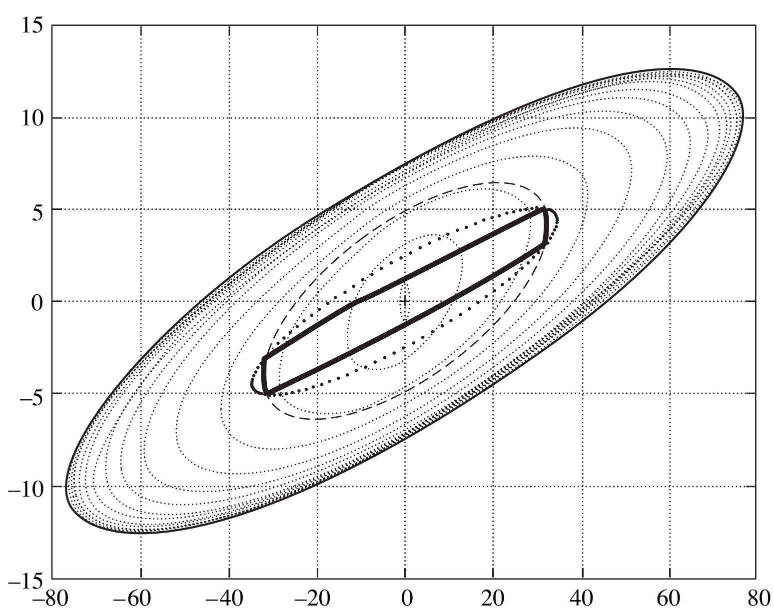

Fig. 3. Behavior of estimates for $\rho_{A^{\mathrm{T}}A} \gg 1$ and $V = I$.

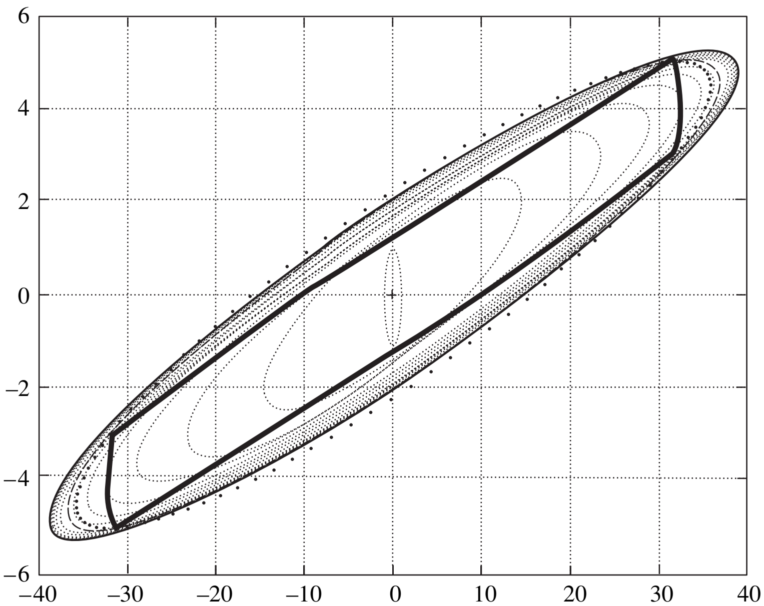

Fig. 4. Behavior of estimates for $\rho_{A^{\mathrm{T}}A} \gg 1$ and $V = UU^{\mathrm{T}}$ (46).

by Theorem 1 and the ellipsoid $E(0, \widehat{P}_\infty)$ (dashed line) where the matrix $\widehat{P}_\infty$ was taken from Theorem 2.

Therefore, if $\rho_A \ll \|A\|$, that is, the spectral matrix norm exceeds very much (a few times) its spectral radius, then for great k it is not advisable to do external approximation of the reachability sets of the dynamic system by the recurrent algorithm (36) with the weight identity matrix $V = I$.

Example 4. Under the conditions of Example 3, all calculations are repeated for the weight matrix

$$V = \begin{pmatrix} 1.0 & -8.3333 \\ -8.3333 & 139.8889 \end{pmatrix} = U^{\mathrm{T}}U \quad \text{and} \quad U = \begin{pmatrix} 1.0 & -8.3333 \\ 0 & 8.3931 \end{pmatrix}. \tag{46}$$

At that, the matrix U drives A to the diagonal matrix $\widetilde{A} = UAU^{-1} = \mathrm{diag}\{0.2; 0.8\}$. In Fig. 4, the set D_∞ and its corresponding ellipsoids are denoted as their counterparts in Example 3. One can see how strongly the accuracy of estimation is improved—in particular, by the locally-optimal algorithm (36)—owing to correct choice of the weight matrix V.

7. CONCLUSIONS

Proposed was a one-parameter family of external ellipsoidal estimates of the limiting reachability set for the stable dynamic linear system, the optimal estimate being determined by solving the one-dimensional convex optimization problem. Asymptotic behavior of the recurrent estimation algorithms was studied for the discrete-time systems. The sufficient condition for convergence of the trace-criterion locally optimal recurrent algorithm was proved. As was shown, the estimates by this method may be very overstated. In this connection, it is suggested to make use of the trace criterion with a weight matrix which, being correctly chosen, can provide more effective ellipsoidal approximations of the reachability domains.

APPENDIX

Proof of Theorem 1. It follows from (1) that the support function of the limiting reachability set D_∞ is as follows:

$$\varphi_\infty(d) \doteq \sup_{x\in D_\infty} d^{\mathrm{T}}x = \sum_{i=0}^{\infty} \|(A^iB)^{\mathrm{T}}d\|, \quad d \in \mathbb{R}^n. \tag{A.1}$$

Therefore, D_∞ is a set of points $x \in \mathbb{R}^n$ satisfying the inequality $d^{\mathrm{T}}x \le \varphi_\infty(d)$ for any vector $d \in \mathbb{R}^n$. Hence, in particular, it is a closed convex set symmetrical to the origin.

Let us consider the ellipsoid $E(0, P_\gamma)$ with a matrix satisfying the Lyapunov equation (9) for any $\gamma \in (\rho_A^2, 1)$. Since the matrix $A/\sqrt{\gamma}$ is stable, by the Lyapunov theorem this equation has unique solution $P_\gamma \ge 0$ because $BB^{\mathrm{T}} \ge 0$. Additionally,

$$P_\gamma = \sum_{k=0}^{\infty} \frac{A^kBB^{\mathrm{T}}(A^k)^{\mathrm{T}}}{\gamma^k(1-\gamma)}.$$

Then, the support function of the ellipsoid $E(0, P_\gamma)$ is as follows:

$$\varphi_{E(0,P_\gamma)}(d) = \sqrt{d^{\mathrm{T}}P_\gamma\, d} = \left(\sum_{k=0}^{\infty} \frac{d^{\mathrm{T}}A^kBB^{\mathrm{T}}(A^k)^{\mathrm{T}}d}{\gamma^k(1-\gamma)}\right)^{1/2}$$
$$\ge \left(\inf_{\{\alpha_k\}} \sum_{k=0}^{\infty} \frac{\|(A^kB)^{\mathrm{T}}d\|^2}{\alpha_k}\right)^{1/2} = \sum_{k=0}^{\infty} \|(A^kB)^{\mathrm{T}}d\| = \varphi_\infty(d),$$

where inf is taken over all sequences $\{\alpha_k\}_{k\ge 0}$ of the positive numbers satisfying the normalization condition $\sum\limits_{i=0}^{\infty} \alpha_i = 1$. We note that $\alpha_k = (1-\gamma)\gamma^k$ is one of such sequences. Therefore, the inequality $\varphi_{E(0,P_\gamma)}(d) \ge \varphi_\infty(d)$ for the support functions is valid for any vector $d \in \mathbb{R}^n$ and any γ from the interval $(\rho_A^2, 1)$. Consequently, $E(0, P_\gamma) \supseteq D_\infty$.

Now, since $\operatorname{tr} VA^kBB^{\mathrm{T}}(A^k)^{\mathrm{T}} \ge 0$ and this inequality is strict for some $k \ge 0$, the function

$$\varphi(\gamma) \doteq \operatorname{tr} VP_\gamma = \sum_{k=0}^{\infty} \frac{\operatorname{tr} VA^kBB^{\mathrm{T}}(A^k)^{\mathrm{T}}}{\gamma^k(1-\gamma)}$$

is the sum of the strictly convex functions over the interval $(\rho_A^2, 1)$. Indeed,

$$\frac{1}{\gamma^k(1-\gamma)} = \frac{1}{1-\gamma} + \sum_{j=1}^{k} \gamma^{-j},$$

and the functions $(1-\gamma)^{-1}$ and γ^{-j} are strictly convex for any $j \ge 1$. Consequently, $\varphi(\gamma)$ is strictly convex, which completes the proof.

Proof of Lemma 2. We assume without loss of generality that the matrices Q_i are diagonal with positive diagonal elements and U and V are the identity matrices (because transformation (6) of the state space $\widetilde{x} = Ux$ with the original matrix U leads namely to this case). The support function $\varphi_P(d)$ of the nondegenerate ellipsoid $E(0, P)$ is as follows

$$\varphi_P(d) = \max_{x\in E(0,P)} x^{\mathrm{T}}d = \sqrt{d^{\mathrm{T}}Pd}, \quad d \in \mathbb{R}^n, \tag{A.2}$$

and for the diagonal matrices P we get for $d = e_n = (1, \dots, 1)^{\mathrm{T}}$ that $\varphi_P(e_n) = \sqrt{\operatorname{tr} P}$. Similar to (A.1), by virtue of diagonality of Q_i the support function of the sum S_N for $d = e_n$ provides

$$\varphi_S(e_n) = \sum_{i=1}^{N} \sqrt{d^{\mathrm{T}} Q_i d} = \sum_{i=1}^{N} \sqrt{\operatorname{tr} Q_i}. \tag{A.3}$$

In the problem of minimization of the trace $\operatorname{tr} P$ on the set of all ellipsoids $E(c, P)$ including the sum S_N of the ellipsoids $E(0, Q_i)$ with the diagonal matrices Q_i, one can readily see that it suffices to confine oneself to the central ellipsoids with the diagonal matrices P. Indeed, each of the ellipsoids $E(0, Q_i)$ is invariant to any (orthogonal) transformation $\widetilde{x} = U_{\pm} x$ with $U_{\pm} = \operatorname{diag}\{\pm 1, \dots, \pm 1\}$; consequently, their sum S_N features the same invariance. Additionally, $\operatorname{tr} U_{\pm} P U_{\pm}^{\mathrm{T}} = \operatorname{tr} P$ for any matrix P by virtue of orthogonality of $U_{\pm}$. Therefore, being unique (see [7, 10]), the minimum ellipsoid $E(c, P)$ features the same invariance and, consequently, the vector c and the matrix P satisfy the equations

$$c = U_{\pm} c, \quad P = U_{\pm} P U_{\pm}^{\mathrm{T}} \tag{A.4}$$

for any of the 2^n matrices $U_{\pm}$. Hence, $c = 0$, and P is a diagonal matrix.

Taking into consideration the aforementioned, we assume that the vector $c = 0$ and consider the diagonal matrices $P > 0$ for which

$$\psi(P) = \min_{d \in \mathbb{R}^n} (\varphi_P(d) - \varphi_S(d)) \geq 0. \tag{A.5}$$

Then,

$$\min_{\psi(P) \geq 0} \operatorname{tr} P \geq \min_{\varphi_P(e_n) \geq \varphi_S(e_n)} \operatorname{tr} P = \varphi_S^2(e_n) = \left(\sum_{i=1}^{N} \sqrt{\operatorname{tr} Q_i} \right)^2 \tag{A.6}$$

in the class of diagonal matrices P. But the right-hand side of (A.6) coincides with the minimum in (10):

$$\min_{\alpha \in \mathcal{A}_N} \operatorname{tr} P(\alpha) = \left(\sum_{i=1}^{N} \sqrt{\operatorname{tr} Q_i} \right)^2. \tag{A.7}$$

With regard for Lemma 1, the proof is completed.

Proof of Theorem 2. Let the matrices $\widehat{P}_k$ and P_k be obtained by algorithms (16)–(20) and (24), (27), respectively. The sequences of the matrices $\widehat{P}_k = P_k$ are bounded because

$$\operatorname{tr} V P_\gamma \geq \limsup_{k \to \infty} \operatorname{tr} V \widehat{P}_k = \limsup_{k \to \infty} \operatorname{tr} V P_k. \tag{A.8}$$

follows from relations (30), (31) and Lemma 3 for an arbitrary $\gamma \in (\rho_A^2, 1)$. We consider now expression (15)

$$(\operatorname{tr} V P_k)^{1/2} = \left[\operatorname{tr} V A^k P_0 (A^{\mathrm{T}})^k \right]^{1/2} + \sum_{i=0}^{k-1} \left[\operatorname{tr} V A^i B B^{\mathrm{T}} (A^{\mathrm{T}})^i \right]^{1/2}.$$

The first term in its right-hand side tends to zero for $k \to \infty$ because the matrix A is stable. The second term in the right-hand side is a series of positive elements whose value increases monotonically with k and is bounded. Hence convergence follows for $\operatorname{tr} V P_k$:

$$(\operatorname{tr} V P_k)^{1/2} \underset{k \to \infty}{\longrightarrow} \sum_{i=0}^{\infty} \left[\operatorname{tr} V A^i B B^{\mathrm{T}} (A^{\mathrm{T}})^i \right]^{1/2}. \tag{A.9}$$

To prove convergence of the matrices P_k we set down (29) as

$$P_k = \frac{A^k P_0 (A^{\mathrm{T}})^k}{\alpha_0} + (\operatorname{tr} V P_k)^{1/2} \sum_{i=0}^{k-1} \frac{A^i B B^{\mathrm{T}} (A^{\mathrm{T}})^i}{[\operatorname{tr} V A^i B B^{\mathrm{T}} (A^{\mathrm{T}})^i]^{1/2}}, \tag{A.10}$$

where $\alpha_0 = (\operatorname{tr} V A^k P_0 (A^{\mathrm{T}})^k)^{1/2} / (\operatorname{tr} V P_k)^{1/2}$. Its first term $A^k P_0 (A^{\mathrm{T}})^k / \alpha_0$ is a nonnegatively definite matrix that with k tends to the zero matrix since the trace

$$\operatorname{tr} V (A^k P_0 (A^{\mathrm{T}})^k) \,/\, \alpha_0 = \sqrt{\operatorname{tr} V P_k \cdot \operatorname{tr} (A^k P_0 (A^{\mathrm{T}})^k)} \underset{k \to \infty}{\longrightarrow} 0,$$

because $A^k \to 0$ and $(A^{\mathrm{T}})^k \to 0$ for $k \to \infty$. To demonstrate convergence of the matrices P_k, it now suffices to prove convergence of the sums appearing in the right-hand side of (A.10). Yet these sums do not decrease monotonically and are bounded; consequently, they converge. Therefore, we proved convergence of P_k to the finite limit

$$\widehat{P}_\infty = \left(\sum_{i=0}^{\infty} \left[\operatorname{tr} V A^i B B^{\mathrm{T}} (A^{\mathrm{T}})^i \right]^{1/2} \right) \left(\sum_{i=0}^{\infty} \frac{A^i B B^{\mathrm{T}} (A^{\mathrm{T}})^i}{[\operatorname{tr} V A^i B B^{\mathrm{T}} (A^{\mathrm{T}})^i]^{1/2}} \right),$$

which coincides with (21), (22). Hence, the expression for the trace

$$\operatorname{tr} V \widehat{P}_\infty = \left(\sum_{i=0}^{\infty} \left[\operatorname{tr} V A^i B B^{\mathrm{T}} (A^{\mathrm{T}})^i \right]^{1/2} \right)^2 < \infty$$

follows immediately, which completes the proof.

Proof of Lemma 3. The equalities $P_k = \widehat{P}_k = P(\alpha)$ were obtained immediately before the formulation of this lemma, and (29) stems directly from (10) and (13)–(15). Let us prove (28).

One can minimize (27) by passing consecutively all γ_i in the reverse order, that is, first by γ_{k-1}, then by γ_{k-2}, and so on. Indeed, the matrix P_k from (3) is representable as

$$P_k = \frac{A P_{k-1} A^{\mathrm{T}}}{\gamma_{k-1}} + \frac{B B^{\mathrm{T}}}{1 - \gamma_{k-1}},$$

where P_{k-1} is no more dependent on γ_{k-1}. Minimization of $\operatorname{tr} V P_k$ by γ_{k-1} provides a unique solution (by virtue of convexity, see [7, 9]), the solution being determined analytically. We get

$$\min_{0<\gamma_0,\ldots,\gamma_{k-2},\gamma_{k-1}<1} \operatorname{tr} V P_k = \left[\left(\min_{0<\gamma_0,\ldots,\gamma_{k-2}<1} \operatorname{tr} V A P_{k-1} A^{\mathrm{T}} \right)^{1/2} + \beta_0 \right]^2, \tag{A.11}$$

where $\beta_0 = \sqrt{\operatorname{tr} V B B^{\mathrm{T}}}$, and the optimal value

$$\gamma_{k-1} = \frac{\left(\operatorname{tr} V A P_{k-1} A^{\mathrm{T}} \right)^{1/2}}{(\operatorname{tr} V A P_{k-1} A^{\mathrm{T}})^{1/2} + (\operatorname{tr} V B B^{\mathrm{T}})^{1/2}}.$$

Now, by representing

$$A P_{k-1} A^{\mathrm{T}} = \frac{A^2 P_{k-2} (A^{\mathrm{T}})^2}{\gamma_{k-2}} + \frac{A B B^{\mathrm{T}} A^{\mathrm{T}}}{1 - \gamma_{k-2}}$$

and minimizing $\operatorname{tr} V A P_{k-1} A^{\mathrm{T}}$ with respect to γ_{k-2}, we similarly obtain

$$\min_{0<\gamma_0,\ldots,\gamma_{k-3},\gamma_{k-2}<1} \operatorname{tr} V A P_{k-1} A^{\mathrm{T}} = \left[\left(\min_{0<\gamma_0,\ldots,\gamma_{k-3}<1} \operatorname{tr} V A^2 P_{k-2} (A^{\mathrm{T}})^2 \right)^{1/2} + \beta_1 \right]^2, \tag{A.12}$$

where $\beta_1 = \sqrt{\operatorname{tr} VABB^{\mathrm{T}}A^{\mathrm{T}}}$, and the optimal value

$$\gamma_{k-2} = \frac{\left(\operatorname{tr} VA^2P_{k-2}(A^{\mathrm{T}})^2\right)^{1/2}}{(\operatorname{tr} VA^2P_{k-2}(A^{\mathrm{T}})^2)^{1/2} + (\operatorname{tr} VABB^{\mathrm{T}}A^{\mathrm{T}})^{1/2}}.$$

The remaining $\gamma_{k-3}, \ldots, \gamma_0$ are consecutively determined in the same manner. Consequently, for $i = 0, 1, \ldots, k-1$ all γ_i satisfy (28), which proves the lemma.

Proof of Theorem 4. It suffices to carry out the proof for the case of the weight identity matrix $V = I$. Indeed, the formulation of this theorem comes (equivalently) namely to this case if one passes to the transformed matrices $\widetilde{A} = UAU^{-1}$, $\widetilde{B} = UB$, and $\widetilde{P} = UPU^{\mathrm{T}}$, where the nondegenerate matrix of the transformation U satisfies $V = U^{\mathrm{T}}U$, see (4)–(7). Therefore, we assume below that $V = I$.

Let us consider the scalar equation $\nu^2 = f(\nu)$ with $f(\nu) = \frac{\operatorname{tr} Q(\nu)}{\operatorname{tr} AQ(\nu)A^{\mathrm{T}}}$, where $Q(\nu)$ is the solution of the Lyapunov equation $Q = \nu AQA^{\mathrm{T}} + C$ for some fixed parameter ν such that $0 < \nu < 1/\rho(A)^2$. We put down

$$Q = \nu AQA^{\mathrm{T}} + C, \quad \frac{dQ}{d\nu} = \nu A\frac{dQ}{d\nu}A^{\mathrm{T}} + AQA^{\mathrm{T}},$$
$$\operatorname{tr} Q = \nu \operatorname{tr} AQA^{\mathrm{T}} + \operatorname{tr} C, \quad \operatorname{tr} \frac{dQ}{d\nu} = \nu \operatorname{tr}\left(A\frac{dQ}{d\nu}A^{\mathrm{T}}\right) + \operatorname{tr} AQA^{\mathrm{T}}$$

and note that the matrices $Q(\nu)$ and $\frac{dQ}{d\nu}$ are nonnegatively definite for $0 < \nu < 1/\rho(A)^2$.

Direct: Let Eq. (40) have a unique solution $Q \geq 0$. Let us assume the opposite: $\rho_A \geq 1$. Then, for the Lyapunov equation $Q = \nu AQA^{\mathrm{T}} + C$ with $C \geq 0$ to have a nonnegatively definite solution for a fixed ν, it is necessary and sufficient that $0 < \nu < 1/\rho_A^2 \leq 1$.

However, for any ν such that $0 < \nu \leq 1$, the function $f(\nu) = \frac{\operatorname{tr} Q(\nu)}{\operatorname{tr} AQ(\nu)A^{\mathrm{T}}} = \nu + \frac{\operatorname{tr} C}{\operatorname{tr} AQ(\nu)A^{\mathrm{T}}} > \nu > \nu^2$ and Eq. (40) has no solution, which contradicts the condition. Therefore, $\rho_A < 1$.

Inverse: Let the matrix A be stable. Then, the Lyapunov equation $Q = \nu AQA^{\mathrm{T}} + C$ has a unique nonnegatively definite solution for any fixed value of ν from the interval $0 < \nu < 1/\rho_A^2$. Let us consider the functions $f(\nu)$ and ν^2 over this interval and prove uniqueness of the intersection point of their graphs. As was shown above, $f(\nu) > \nu^2$ on $0 < \nu \leq 1$. Therefore, solutions of (40) may be only on $1 < \nu < 1/\rho_A^2$. For $\nu = 1$, the function $f(\nu) = \nu + \frac{\operatorname{tr} C}{\operatorname{tr} AQ(\nu)A^{\mathrm{T}}} > 1$. On the other hand, $\lim_{\nu\to 1/\rho_A^2 - 0} f(\nu) = \lim_{\nu\to 1/\rho_A^2 - 0}\left(\nu + \frac{\operatorname{tr} C}{\operatorname{tr} AQ(\nu)A^{\mathrm{T}}}\right) = 1/\rho_A^2$, since at that $\operatorname{tr} AQ(\nu)A^{\mathrm{T}}$ tends to infinity. Therefore, $\lim_{\nu\to 1/\rho_A^2 - 0} f(\nu) < \nu^2|_{\nu=1/\rho_A^2} = 1/\rho_A^4$. Hence, $f(\nu)$ and ν^2 have intersection points over this interval. We consider further the derivative

$$\begin{aligned}
\frac{d}{d\nu}f(\nu) &= \frac{d}{d\nu}\left(\frac{\operatorname{tr} Q(\nu)}{\operatorname{tr} AQ(\nu)A^{\mathrm{T}}}\right) = \frac{1}{(\operatorname{tr} AQA^{\mathrm{T}})^2}\left\{\operatorname{tr}\frac{dQ}{d\nu}\operatorname{tr} AQA^{\mathrm{T}} - \operatorname{tr} Q \operatorname{tr}(A\frac{dQ}{d\nu}A^{\mathrm{T}})\right\} \\
&= \frac{1}{(\operatorname{tr} AQA^{\mathrm{T}})^2}\left\{\left(\nu \operatorname{tr}(A\frac{dQ}{d\nu}A^{\mathrm{T}}) + \operatorname{tr} AQA^{\mathrm{T}}\right)\operatorname{tr} AQA^{\mathrm{T}} - \operatorname{tr} Q \operatorname{tr}(A\frac{dQ}{d\nu}A^{\mathrm{T}})\right\} \\
&= \frac{1}{(\operatorname{tr} AQA^{\mathrm{T}})^2}\left\{\left(\nu \operatorname{tr} AQA^{\mathrm{T}} - \operatorname{tr} Q\right)\operatorname{tr}(A\frac{dQ}{d\nu}A^{\mathrm{T}}) + (\operatorname{tr} AQA^{\mathrm{T}})^2\right\} \\
&= \frac{1}{(\operatorname{tr} AQA^{\mathrm{T}})^2}\left\{-\operatorname{tr} C \operatorname{tr}(A\frac{dQ}{d\nu}A^{\mathrm{T}}) + (\operatorname{tr} AQA^{\mathrm{T}})^2\right\} = 1 - \operatorname{tr} C\frac{\operatorname{tr}(A\frac{dQ}{d\nu}A^{\mathrm{T}})}{(\operatorname{tr} AQA^{\mathrm{T}})^2} < 1.
\end{aligned}$$

Consequently, the difference $\nu^2 - f(\nu)$ will be here a monotone function because

$$\frac{d}{d\nu}\left(\nu^2 - f(\nu)\right) = 2\nu - \frac{d}{d\nu}\left(\frac{\operatorname{tr} Q(\nu)}{\operatorname{tr} AQ(\nu)A^{\mathrm{T}}}\right) > 2\nu - 1 > 0$$

for all $\nu > 1/2$. Consequently, uniqueness of the intersection point of the functions ν^2 and $f(\nu)$, as well as of the solution of Eq. (40) was demonstrated, which proves the theorem.

Proof of Lemma 5. It is common knowledge (see, for example, [16]) that for any $\varepsilon > 0$ there exists a nondegenerate real matrix of similarity transformation U for which $\rho_A > \sigma_{\max}(\widetilde{A}) - \varepsilon$. In particular, if all eigenvalues $\lambda_i(A)$ of the matrix A are simple, that is, of multiplicity 1, it is possible to provide $\varepsilon = 0$ by reducing the matrix A to the real block-diagonal form such that $\widetilde{A}^{\mathrm{T}}\widetilde{A} = \operatorname{diag}\{|\lambda_1(A)|^2, \dots, |\lambda_n(A)|^2\}$ and, consequently, $\rho_A = \sigma_{\max}(\widetilde{A})$. In the general case, $\varepsilon > 0$ may be made arbitrarily small. Therefore, for a stable matrix A, one can both make the matrix $\widetilde{A}^{\mathrm{T}}\widetilde{A}$ stable and satisfy inequality (41) with regard for Teorem 4, which proves the lemma.

To prove Theorem 5, we need some auxiliary results.

Lemma A.1. *Let A, $Q = Q^{\mathrm{T}}$, and $Q_0 = Q_0^{\mathrm{T}}$ be real $(n\times n)$ matrices and the parameter $\beta \geq \operatorname{tr} Q_0$, the matrices A and Q_0 being fixed. Then, for all $Q \geq Q_0$ such that $\operatorname{tr} Q = \beta$, valid is the inequality*

$$\operatorname{tr} AQA^{\mathrm{T}} \leq \operatorname{tr} AQ_0A^{\mathrm{T}} + (\beta - \operatorname{tr} Q_0)\,\sigma^2_{\max}(A) \tag{A.13}$$

which turns into equality for

$$Q = Q_0 + (\beta - \operatorname{tr} Q_0) H \operatorname{diag}\{1, 0, \dots, 0\} H^{\mathrm{T}}. \tag{A.14}$$

Here, H is an orthogonal matrix driving the symmetrical matrix $A^{\mathrm{T}}A$ to the diagonal form where the element $(1,1)$ is the greatest eigenvalue, that is,

$$H^{\mathrm{T}}(A^{\mathrm{T}}A)H = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}, \qquad \lambda_1 = \sigma^2_{\max}(A). \tag{A.15}$$

Proof. The case of $\beta = \operatorname{tr} Q_0$ is trivial because the admissible set of matrices Q consists of a single point $Q = Q_0$; therefore, we assume below that $\beta > \operatorname{tr} Q_0$. We change variables

$$Z = Q - Q_0 \tag{A.16}$$

and make use of the theory of duality of the problems of convex programming (see, for example, [17]). Since $\operatorname{tr} AZA^{\mathrm{T}} = \langle A^{\mathrm{T}}A, Z\rangle$ and $\operatorname{tr} Z = \langle I, Z\rangle$, for $\beta_0 = \beta - \operatorname{tr} Q_0 > 0$ we get that

$$\max_{\substack{\operatorname{tr} Z = \beta_0 \\ Z \geq 0}} \operatorname{tr} AZA^{\mathrm{T}} = \min_{\lambda I \geq A^{\mathrm{T}}A} \beta_0\,\lambda = \beta_0\,\sigma^2_{\max}(A). \tag{A.17}$$

We obtain (A.13) by the inverse replacement (A.16). The matrix (A.14) satisfies the constraints $Q \geq Q_0$ and $\operatorname{tr} Q = \beta$, and its substitution into the left-hand side of inequality (A.13) rearranges it in equality, which proves the lemma.

Corollary A.1. *In the conditions of Lemma A.1, the inequality* (A.13) *assumes for $Q_0 = 0$ the form*

$$\operatorname{tr} AQA^{\mathrm{T}} \leq \sigma^2_{\max}(A)\operatorname{tr} Q, \qquad \forall\, Q \geq 0. \tag{A.18}$$

Corollary A.2. *In the conditions of Lemma A.1, for $\beta > 0$*

$$\max_{\substack{\operatorname{tr} Q \leq \beta \\ Q \geq Q_0}} \frac{\operatorname{tr} AQA^{\mathrm{T}}}{\operatorname{tr} Q} = \frac{\operatorname{tr} AQ_0A^{\mathrm{T}} + (\beta - \operatorname{tr} Q_0)\sigma_{\max}(A)}{\beta}. \tag{A.19}$$

Proof. According to inequality (A.18), the right-hand side of (A.19), which, by virtue of Lemma A.1, is the maximum of $\operatorname{tr} AQA^{\mathrm{T}}/\operatorname{tr} Q$ for all $Q \geq Q_0$ subject to the condition $\operatorname{tr} Q = \beta$, is a monotonically nondecreasing function of the parameter $\beta > 0$ from which (A.19) follows directly, thus proving Corollary A.2.

We denote by $\mathcal{Q}_\infty$ the set of limiting points of all sequences $\{Q_k\}_{k\geq 0}$ generated by the recurrent algorithm (39) subject to the condition $Q_0 \geq 0$, and introduce the functions $g(\cdot):\ R_+^{n\times n} \to R_+$

$$g(Q) = \left(\frac{\operatorname{tr} Q}{\operatorname{tr} AQA^{\mathrm{T}}}\right)^{1/2}, \qquad Q = Q^{\mathrm{T}} \geq 0, \quad Q \neq 0, \tag{A.20}$$

and $Q(\cdot) : [0, \rho_A^{-2}) \to R_+^{n\times n}$ as the solution of the matrix Lyapunov equation

$$Q(\nu) = \nu AQ(\nu)A^{\mathrm{T}} + C, \qquad 0 < \nu < \rho_A^{-2}, \tag{A.21}$$

that is,

$$Q(\nu) = \sum_{k=0}^{\infty} \nu^k A^k C (A^{\mathrm{T}})^k. \tag{A.22}$$

Lemma A.2. *Let* $\sigma_{\max}(A) < 1$ *and the set of all limiting points* $\mathcal{Q}_\infty \subseteq \mathcal{Q}_1$, *where*

$$\mathcal{Q}_1 = \left\{Q = Q^{\mathrm{T}} \mid Q \geq X_1,\ \operatorname{tr} Q \leq \tau_1\right\}, \tag{A.23}$$

and the parameters X_1 *and* τ_1 *satisfy the conditions* $X_1 = X_1^{\mathrm{T}} > 0$ *and* $\operatorname{tr} X_1 \leq \tau_1 \leq +\infty$. *Then,* $\mathcal{Q}_\infty \subseteq \mathcal{Q}_2$, *where*

$$\mathcal{Q}_2 = \left\{Q = Q^{\mathrm{T}} \mid Q \geq X_2,\ \operatorname{tr} Q \leq \tau_2\right\}, \tag{A.24}$$

at that,

$$X_2 = Q(\nu_*), \qquad \tau_2 = \frac{\nu_*}{\nu_* - 1} \operatorname{tr} C, \tag{A.25}$$

$$\nu_* = \inf_{Q\in\mathcal{Q}_1} g(Q) \tag{A.26}$$

and

$$\lambda_{\max}^{-1/2}\left(A^{\mathrm{T}}A\right) \leq \nu_* \leq g(Q^*) = \nu^*. \tag{A.27}$$

Proof. By definition, the set $\mathcal{Q}_\infty$ contains at least one point—namely, the equilibrium point Q^* of the recurrent algorithm (39)—which is the unique root of Eq. (40). Therefore, the inequality $\nu_* \leq g(Q^*) = \nu^*$ providing the right-hand inequality in (A.27) by virtue of Theorem 2 follows from (A.26). The left-hand inequality in (A.27) is a direct consequence of (A.18). Now, we prove (A.24), (A.25).

By iterating Eq. (39) and taking into account that $\nu_k = g(Q_k)$ and, consequently,

$$\liminf_{k\to\infty} \nu_k \geq \nu_*, \tag{A.28}$$

we obtain from the monotonicity of (A.22) in ν that for any limiting point Q_∞ of an arbitrary sequence $\{Q_k\}$ generated by algorithm (A.22) the following inequality is true:

$$Q_\infty \geq Q(\nu_*) = X_2. \tag{A.29}$$

The difference equation

$$\operatorname{tr} Q_{k+1} = \nu_k^{-1} \operatorname{tr} Q_k + \operatorname{tr} C, \quad k = 1, 2, \dots, \tag{A.30}$$

follows now from (39). The condition for asymptotic stability

$$\limsup_{k\to\infty} \nu_k^{-1} \le (\nu_*)^{-1} \le \left(\sigma^2_{\max}(A)\right)^{1/2} < 1 \tag{A.31}$$

is satisfied for it, which makes valid the upper bound

$$\limsup_{k\to\infty} \operatorname{tr} Q_k \le \frac{\operatorname{tr} C}{1 - 1/\nu_*} = \tau_2. \tag{A.32}$$

Consequently, also $\operatorname{tr} Q_\infty \le \tau_2$. The lemma is proved by virtue of arbitrariness of $Q_\infty \subseteq \mathcal{Q}_2$.

Proof of Theorem 5. As in the proof of Theorem 4, the case of an arbitrary weight matrix $V > 0$ is reduced without loss of generality to the case of the identity matrix $V = I$ which is examined below to simplify calculations.

We use repeatedly Lemma A.2 to construct a monotonically nonincreasing sequence of the sets $\mathcal{Q}(i)$, $i = 1, 2, \dots$, containing the set $\mathcal{Q}_\infty$ of all limiting points of all sequences $\{Q_k\}_{k\ge 0}$ generated by the recurrent algorithm (39) under an arbitrary initial condition $Q_0 \ge 0$. Since $Q_k \ge C$ for all $k \ge 1$, in (A.23) one can take $X_1 = C$, $\tau_1 = +\infty$. For each $i \ge 1$, we assume consecutively, beginning from $i = 1$, that in (A.23) $\mathcal{Q}(i) = \mathcal{Q}_1$ and from (A.24) obtain $\mathcal{Q}(i+1) = \mathcal{Q}_2$; we also denote the corresponding ν_* in (A.26) by $\nu_*(i)$:

$$\nu_*(i) = \inf_{Q\in\mathcal{Q}(i)} g(Q). \tag{A.33}$$

By construction, $\mathcal{Q}(2) \subseteq \mathcal{Q}(1)$, and, consequently, $\nu_*(2) \ge \nu_*(1)$. From (A.25) of ν_*, we obtain consecutively $\mathcal{Q}(i+1) \subseteq \mathcal{Q}(i)$ and $\nu_*(i+1) \ge \nu_*(i)$ for all $i \ge 1$ by virtue of the monotone dependence of X_2 and τ_2. Therefore, by passing to the limit for $i \to \infty$, we obtain the finite limit $\nu_\infty = \lim_{i\to\infty} \nu_*(i) \le \nu^*$, and from (A.24)–(A.26)

$$\mathcal{Q}_\infty \subseteq \bigcap_{i=1}^{\infty} \mathcal{Q}(i) = \{Q = Q^{\mathrm{T}} \mid Q \ge X_\infty, \operatorname{tr} Q \le \tau_\infty\}, \tag{A.34}$$

$$X_\infty = Q(\nu_\infty), \qquad \tau_\infty = \frac{\nu_\infty}{\nu_\infty - 1} \operatorname{tr} C, \tag{A.35}$$

where

$$\nu_\infty = \left(\frac{\tau_\infty}{\operatorname{tr} A X_\infty A^{\mathrm{T}} + (\tau_\infty - \operatorname{tr} X_\infty)\sigma^2_{\max}(A)}\right)^{1/2} \tag{A.36}$$

with regard for (A.33) and Corollary 2 to Lemma A.1. By means of direct substitution one can easily ascertain (with regard for (39), (A.30) and the definition of the equilibrium point Q^*) that the equation system (A.35), (A.36) has solution

$$X_\infty = Q^*, \quad \tau_\infty = \operatorname{tr} Q^*, \quad \nu_\infty = \left(\frac{\operatorname{tr} Q^*}{\operatorname{tr} A Q^* A^{\mathrm{T}}}\right)^{1/2}. \tag{A.37}$$

(We note that no other solution can exist if $\operatorname{tr} X_\infty = \tau_\infty$ because the equation system (A.35), (A.36) becomes equivalent to (40) according to Theorem 2.) Now, it suffices to demonstrate uniqueness of this solution if $\operatorname{tr} X_\infty \le \tau_\infty$ because then the fact that the set of limiting points is a one-point set, that is, $\mathcal{Q}_\infty = \{Q^*\}$, follows from (A.34) and the equality $\operatorname{tr} X_\infty = \tau_\infty$. We note that, as was already observed above, the condition $\operatorname{tr} X_\infty \le \tau_\infty$ is equivalent to the inequality $\nu_\infty \le \nu^*$ by virtue of monotonicity of the functions $\operatorname{tr} Q(\nu)$ and $\tau(\nu)$ and the fact that $\tau(\nu) = \operatorname{tr} Q(\nu)$ only for $\nu = \nu^*$.

By assuming for brevity that $\nu = \nu_\infty$, $\lambda = \sigma^2_{\max}(A)$ and denoting by $F(\nu)$ the function obtained in the right-hand side of (A.36), we eliminate the variables X_∞ and τ_∞ from the system (A.35), (A.36). The resulting equation with one unknown ν after simple transformations taking into account the Lyapunov equation (A.21) can be rearranged equivalently in

$$F^{-2}(\nu) - \nu^{-2} = \left(\lambda - \nu^{-1}\right)\left(1 - \left(1 - \nu^{-1}\right)\frac{\operatorname{tr} Q(\nu)}{\operatorname{tr} C}\right). \quad \text{(A.38)}$$

Therefore, the roots of the equation $F(\nu) = \nu$ can arise only if one of the factors in the right-hand side of (A.38) vanishes. The second factor can be zero only if $\nu = \nu^*$, and the first factor has a unique root $\nu = 1/\lambda$ which by the condition of this theorem does not lie to the left of the point ν^*, which completes the proof.

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