On minimax approach to nonparametric adaptive control

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Abstract

The subject of the paper is the adaptive stabilization problem for a functional autoregressive model. The uncertainty is represented by a class of functions satisfying the Hölder condition with given degree of smoothness $s > 0$.

We follow the information approach, first used in adaptive control problems by Nemirovsky and Tsypkin in 1984. We present the asymptotically minimax lower bounds for the mean-square error of stabilization and provide a "quasi-optimal" adaptive control algorithm which attains the minimax rate of convergence up to a logarithmic factor.

Keywords: optimal adaptive control, nonparametric uncertainty, minimax estimation.

1 Introduction

In the adaptive control theory, there is a variety of approaches to the design of control algorithms. Until recently, the major effort in this area of research was directed toward the study of the convergence of these algorithms and the stability of the resulting closed-loop systems. Quite a new problem related to providing the algorithms with the "maximal rate of convergence" has been first formulated and solved in [13]. In that paper the lower (information) bound has been provided for the performance of adaptive control algorithms in the case of linear stochastic systems. An important feature of this result is that it has been established under extremely wide assumptions on the control strategy (an analogous bound, however, limited to the case of certainty equivalent controls, has been recently given in [7]). Also an adaptive control algorithm based on the stochastic approximation method for estimation of the unknown parameters was proposed; it was shown that under certain conditions, the lower bound obtained is attainable with this algorithm. That pioneering result allowed to place the adaptive control problem in the general framework of statistical theory.

Later, these results were extended in [5], [11], where the conditions imposed were relaxed, and the formulations of the lower bounds were simplified. Moreover, new optimal adaptive control algorithms based on the stochastic approximation method with averaging were developed in [12].

In the present paper we address the problem of nonparametric adaptive control of a nonlinear autoregressive model. We consider that the model uncertainty is functional (nonparametric), i.e. it is specified by a class of Hölder functions. [1] Some lower information bounds for this problem has been established in [6]. Our goal here is to provide the correspondent upper bound – an optimal adaptive control algorithm for this situation. We show (cf. Theorem 2) that the certainty equivalent control, based on the adaptive nonparametric estimate of the unknown function is "almost optimal" in the sense of the lower bounds of [6] up to an extra logarithmic factor.

2 The problem of adaptive stabilization

Let us consider a scalar plant described by the nonlinear difference equation

$$y_t = -g(y_{t-1}) + u_{t-1} + e_t, \quad t = 1, 2, \dots, \tag{1}$$

where y_t, u_t, and e_t are the scalar output, control, and unmeasured perturbation (random noise), respectively, at the time instant $t \geq 0$. In what follows it is assumed that $Ee_t = 0$, $0 < Ee_t^2 = \sigma^2 < \infty$, and the function $g: \mathbf{R} \to \mathbf{R}$ belongs to a certain *a priori* specified subclass $\mathcal{F}$ of measurable functions. Furthermore, it is assumed that the disturbances e_t are independent and identically distributed. The initial value y_0 is fixed, and at every time instant $t \geq 0$, the value of y_t is observed without distortion. Therefore, the control u_t, $t \geq 0$, can be taken as a Borel function of the preceding inputs, i.e.,

$$u_t = u_t(y_0, \dots, y_t). \tag{2}$$

Here the right-hand side contains functions of $(t+1)$ variables, i.e., $u_t: \mathbf{R}^{t+1} \to \mathbf{R}$.

Definition 1 *The sequence*

$$U = \{u_t(\cdot) | t \geq 0\} \tag{3}$$

of functions (2) is called a control strategy.

We denote by $\mathcal{U}$ the set of all control strategies. Hence, an arbitrary pair $(g, U) \in \mathcal{F} \times \mathcal{U}$ generates a random process (y_t) with the associated probability measure in the corresponding probability space. The mathematical expectation with respect to this measure is denoted by $E_{g,U}$.

[1]Note that the case of parametric uncertainty and nonlinear autoregressive model has been studied, for instance, in [8].

In the stabilization problem, the goal is to ensure the minimal absolute value of the output y_t, e.g., in the mean-square sense. If the function g was *a priori* known, then using the control

$$u_{t-1} = g(y_{t-1}), \quad t \geq 1,$$

would lead to $y_t \equiv e_t$ and $E_{g,U} y_t^2 \equiv \sigma^2$. Since $E_{g,U} y_t^2 \geq \sigma^2$ for an arbitrary control strategy, an "ideal output" in the problem under study is $y_t = e_t$, $t \geq 1$. Therefore, for an arbitrary control strategy U, the variable

$$\delta_t = y_t - e_t = u_{t-1} - g(y_{t-1})$$

represents the *error of stabilization*, and the problem of adaptive stabilization can be formulated as follows: Find a control strategy $U^* = U^*(\mathcal{F}) \in \mathcal{U}$ such that the controlled process (1) satisfies the condition

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} E_{g,U^*(\mathcal{F})} \left(y_t - \xi_t\right)^2 = 0 \quad \forall\, g \in \mathcal{F}. \tag{4}$$

Definition 2 *The strategy $U^*(\mathcal{F})$ satisfying condition (4) is called an adaptive stabilization strategy with respect to the class $\mathcal{F}$.*

Obviously, if such a strategy $U^*(\mathcal{F})$ exists, it is not unique. We denote by $\mathcal{U}^* = \mathcal{U}^*(\mathcal{F})$ the set of all adaptive stabilization strategies with respect to $\mathcal{F}$.

3 Lower information bounds

Let $\mathcal{F} = \mathcal{F}(s, L)$ be a class of bounded functions $g:\ \mathbf{R} \to \mathbf{R}$ satisfying the Hölder condition with constant $L < \infty$ and having degree of smoothness $s > 0$, i.e., there exists the derivative $g^{(k)}$, $k = \max\{i \in \mathbf{N}:\ i < s\}$, such that

$$[g]_s = \sup_{x \neq y} \frac{|g^{(k)}(x) - g^{(k)}(y)|}{|x - y|^{s-k}} \leq L.$$

Assumption 1. We suppose that the function $g(\cdot)$ belongs to $\mathcal{F}(s, L)$ for some $s > 0$, $0 < L < \infty$.

We now introduce the assumptions on the random disturbance e_t.

Assumption 2. (e_t), $t = 1, 2, \ldots$ is a sequence of independent and identically distributed random variables, $Ee_1 = 0$. Further, it is assumed that the random variable e_1 possesses bounded, continuously differentiable probability density function $q(\cdot)$, $\sup |q'(x)| = \|q'\|_\infty < \infty$, and, furthermore,

$$J_4(q) = \int \left(\frac{q'(x)}{q(x)}\right)^4 q(x)dx < \infty.$$

Therefore, there exists the Fisher information

$$J(q) = \int \left(\frac{q'(x)}{q(x)} \right)^2 q(x) dx \le (J_4(q))^{1/2} < \infty.$$

Assumption 3. Consider the functional

$$b(s, L, q) = \left(\frac{L}{(J(q))^s} \right)^{2/(2s+1)} \int (q(x))^{1/(2s+1)} \, dx. \tag{5}$$

We assume that $b(s, L, q)$ is finite for the given density q.

Theorem 1 (Theorem 2 of [6]) *Suppose that Assumptions 1–3 hold true; the observations y_t are generated by Eq. (1) with arbitrary control strategy $U \in \mathcal{U}$ (3). Then there is $C(s) > 0$ such that*

$$\liminf_{n \to \infty} n^{2s/(2s+1)} \sup_{g \in \mathcal{F}(s,L)} n^{-1} \sum_{t=1}^{n} E_{g,U} \left(y_t - e_t \right)^2 \ge C(s) b(s, L, q), \tag{6}$$

where $b(s, L, q)$ is defined in (5).

The bound above looks somewhat complicated, it involves averaging over time. However, this inequality determines the lower bound for *arbitrary* control strategies (cf. the discussion in [13]). We now turn to the problem of attainability of the lower bounds established.

4 Upper bound: adaptive control algorithm

We now describe an adaptive control algorithm. The underlying idea is very simple and has been first realized in [13] in the linear parametric setting: we provide a minimax estimate $\widehat{g}_n(x)$ of the function g, then plug it into the control

$$u_n = \widehat{g}_n(y_n).$$

An optimal adaptive control algorithm of this type has been discussed, for instance, in [6]. Note that the asymptotic properties of the process (y_n) generated by the closed-loop system are defined by the properties of the estimate being constructed. It is clear that if we aim to construct an optimal on the class $\mathcal{F}(s, L)$ estimate $\widehat{g}_n(x)$ the properties of this estimate would depend on the parameters L and s of the class. Furthermore, if the classical minimax framework is concerned, the estimation algorithm itself heavily depends explicitly on those parameters. So the following question arises: how to design an *adaptive estimation algorithm* which only uses the observations and which delivers the estimate $\widehat{g}_n$ of not worse quality than the parameter-dependent estimation, which uses the knowledge of the parameter (s, L) of the class.

In the white-noise framework of nonparametric estimation the answer to this question is given, for instance, in [9], [2] and [1]. In those papers a variety of estimates $\widehat{g}_n$ are proposed, such that the ratio of the estimate risk $R(\widehat{g}_n, \mathcal{F})$ on $\mathcal{F}$ and the minimax risk $R^*(\widehat{g}_n, \mathcal{F})$ remains finite as $n \to \infty$. Our problem is quite different. In order to devise the adaptive estimator to be used in the adaptive control algorithm, we have to confront two difficulties:

1. the regression inputs (y_t) are distributed on $\mathbf{R}$ with the density which vanishes at some points;
2. the observations are dependent.

Here we provide a simple adaptive algorithm adopted for our problem. However, there is a price to pay for the adaptivity – an extra logarithmic factor in the rate of convergence. In what follows we suppose that the variance σ^2 of the noise is known *a priori*. If it is not the case, this parameter can be easily estimated with sufficient accuracy.

4.1 Adaptive estimator

Let us consider the sequence $h_0 = c_0 \frac{\ln m}{m}$, $h_{i_{\max}} = c_1 \ln m$ and $h_i = 2h_{i-1}$ for $i = 1, ..., i_{\max}$. Note that $i_{\max} \le C \ln n$ for some $C < \infty$.

We start with the construction of the family of *locally linear estimates* of the function $g(x)$, parameterized with the bandwidths $h_i = h_i(x)$. Let

$$\begin{aligned} \theta &= (\theta^0, \theta^1)^T \in \mathbf{R}^2 \\ z_t &= (1, y_{t-1} - x)^T. \end{aligned}$$

Let $h(i, x)$ be the interval of the size $2h_i$, centered at x:

$$h(i, x) = \{y : |y - x| \le h_i\}.$$

We introduce the indicator function $1\{\cdot\}$ and define

$$\widehat{\theta}_{2m}(i, x) = \arg\min_{\theta} \sum_{t=m+1}^{2m} \left(u_{t-1} - y_t - z_t^T \theta\right)^2 1\{y_{t-1} \in h(i, x)\}. \tag{7}$$

We put $\widehat{g}2m(i, x) = \widehat{\theta}^0_{2m}(i, x)$, which is the first component of the coefficient vector $\widehat{\theta}_{2m}(i, x)$ of the best (in the least-square sense) linear approximation, computed for the outputs belonging to the segment $[x - h_i, x + h_i]$. We refer to this estimate as associated with the bandwidth h_i.

Let for some $\kappa > 0$

$$i_0 = \min\left(i : \sum_{t=m+1}^{2m} 1\{y_{t-1} \in h(i, x)\} \ge \kappa \ln m\right). \tag{8}$$

For $i_0 \le i \le i_l$ we define the matrix

$$S_{2m}(i, x) = \left(\sum_{t=m+1}^{2m} z_t z_t^T 1\{y_{t-1} \in h(i, x)\}\right)^{-1}$$

and set for some $\nu > 0$

$$s_{2m}(i,x) = \nu\sqrt{\sigma^2 S_{2m}^{(1,1)}(i,x)\ln m}.$$

where $S_{2m}^{(1,1)}(i,x)$ is the first diagonal entry of $S_{2m}(i,x)$.

The adaptive estimate $\widehat{g}_{2m}(x) = \widehat{g}_{2m}(\hat{i},x)$ is defined as follows: we call some $i_0 \le i \le i_{\max}$ acceptable if all the segments

$$B_i \equiv [\widehat{g}_{2m}(j,x) - s_{2m}(j,x),\, \widehat{g}_{2m}(j,x) + s_{2m}(j,x)]\,, \quad j = i_0, ..., i$$

have a common point (it is evident that an acceptable point exists, i.e. $i = i_0$). Finally, we put $\hat{i}$ the largest of all acceptable i and set

$$\widehat{g}_{2m}(x) = \Pi_R\left(\widehat{g}_{2m}(x)\right), \tag{9}$$

where $\Pi_R = \min(|x|, R)\mathrm{sign}(x)$ is the projector on the segment $[-R, R]$. $\widehat{g}_{2m}(x) = \widehat{g}_{2m}(\hat{i},x)$.

4.2 Control algorithm

Consider the sequence of time instants τ_i, $i = 1, 2, ...,$ $\tau_0 = T > 0$ (an integer constant), $\tau_i = 2\tau_{i-1}$. We denote

$$\tau(n) = \max_i(\tau_i, \tau_i \le n)$$

and set for some $R > 0$ large enough

$$u_n = \begin{cases} 0 & \text{for} \quad n < 2T, \\ \widehat{g}_{\tau(n)}(y_n) & \text{for} \quad n \ge 2T, \end{cases},$$

where $\widehat{g}_{\tau(n)}(\cdot)$ is defined in (9).

Comments: an important feature of the presented adaptive control algorithm (when compared to that in the parametric case) is that the estimate $\widehat{g}$ is not updated each time the new observation y_t arrive. Rather, the observations are processed in a "batch". This is done mainly for technical reasons: indeed, since for $\tau_j \le t < 2\tau_j$ the control actions $u_t = \widehat{g}_{\tau_j}(y_t)$ do only depend on $y_0, ..., y_{\tau_j}$ and y_t, the controlled process $(y_t)_{\tau_j \le t < 2\tau_j}$ constitutes a homogeneous Markov chain with nice mixing properties. The main disadvantage of this approach is that only a part of the observation sample available at the instant t is used to compute the adaptive control. However, as we are not interested in the exact constant in the rate of convergence, the loss of precision due to this fact is not important. Consider the following

Assumption 4. The random disturbances (e_t), $t = 1, 2, ...$ are independent and identically distributed Gaussian random variables, $Ee_1 = 0$, $Ee_1^2 = \sigma^2$.

Theorem 2 *Let Assumptions 1 and 4 hold. Suppose that σ^2 and some upper bound R of $\|g\|_\infty$ are known* a priori. *Then the parameters c_1, ν and κ of Algorithm 1 can be chosen such that for $0 < s \le 2$ and bounded L*

$$\left(\frac{n}{\log n}\right)^{\frac{2s}{2s+1}} \sup_{g\in\mathcal{F}(s,L)} \frac{1}{n}\sum_{t=1}^{n} E_g(y_t^2 - \sigma^2) \le C(s)(L\sigma^{3s})^{\frac{2}{2s+1}}(1 + o(1)). \tag{10}$$

Here $C(s) < \infty$ does not depend on n, σ^2 or L.

Comment: note that in the case of Gaussian disturbances (e_t) the quantity $b(s, L, q)$, defined in (5), is

$$b(s, L, q) = C(s))(L\sigma^{3s})^{\frac{2}{2s+1}}.$$

So the upper bound (10) differs from the lower bound (6) by a logarithmic factor

$$(\log n)^{\frac{2s}{2s+1}}.$$

5 Proof of Theorem 2

First remark that, by construction, the estimate $\widehat{g}(x)$ is frozen as a function of x for $\tau_j < t \leq \tau_{j+1}$. As

$$y_t = (\widehat{g}_{\tau(t)}(y_{t-1}) - g(y_{t-1})) + e_t,$$

this implies that (y_t) constitutes for $\tau_j < t \leq \tau_{j+1}$ a homogeneous Markov chain; we denote by P_j the transition probability, of this chain and by E_{g,τ_j} (P_{g,τ_j}) the expectation (the probability) with respect to the distribution of this chain. It can be easily seen that the chain is Doeblin recurrent and possesses(cf Case (b), p. 197 of [3]) a unique invariant measure π_j. Further, it is exponentially stable, i.e. there are $C < \infty$ and $\lambda > 0$ which do not depend on j such that for any measurable set $A \subseteq \mathbf{R}$

$$|P_j^n(A) - \pi(A)| \leq C \exp(-\lambda n). \tag{11}$$

We now introduce some useful notations. Let $x \in \mathbf{R}$ and $m = \tau_{j+1} - \tau_j$. We define $i^* = i_j^*(x)$ as follows:

$$i^* = \min\left(i : \; L^2 h_i^{2s} > \frac{\sigma^2 \log m}{m\pi_j(h(i,x))}\right). \tag{12}$$

As we shall see, this is the value of the bandwidth which provides an approximate equilibrium of bias and variance of the estimate $\widehat{g}_{\tau_{j+1}}$ at x. We also set for some C, to be chosen later,

$$i_1 = \min\left(i : \; \pi_j(h(i,x)) \geq \frac{C \log m}{m}\right)$$

and denote

$$r^*_{\tau_{j+1}}(x) = L^2 h_{i^*}^{2s} + \frac{\sigma^2 \log m}{m\pi_j(h(i^*,x))}. \tag{13}$$

The proof of the theorem is based on the following proposition which describes the point-wise properties of the adaptive estimate $\widehat{g}_{\tau_{j+1}}(x)$:

Proposition 1 *Let $x \in \mathbf{R}$ be such that $i^* \geq i_1$. Then for any $\alpha > 0$ there are $C(\alpha)$, $C < \infty$ such that*

$$P_{g,\tau_j}\left(|\widehat{g}_{\tau_{j+1}}(x) - g(x)| \geq C(\alpha)\sqrt{r^*_{\tau_{j+1}}(x)}\right) \leq C m^{-\alpha}.$$

Let us first show how the result of Theorem 2 follows from the proposition. We set

$$R_{\tau_{j+1},\pi_j}(\widehat{g},g) = \int E_{g,\tau_j}(\widehat{g}_{\tau_{j+1}}(x) - g(x))^2 \pi_j(dx)$$

and

$$R_{\tau_{j+1}}(\widehat{g},g) = E_g R_{\tau_{j+1},\pi_j}(\widehat{g},g).$$

We start with the following technical lemmas:

Lemma 1 *The invariant measure of the Markov chain $(y_t)_{\tau_j<t\le\tau_{j+1}}$ possesses a Lipschitz continuous density q_j, i.e. there is $C<\infty$ such that*

$$|q_j(x) - q_j(y)| \le C\sigma^{-2}|x-y|, \quad \text{for any } \; x,\, y \in \mathbf{R}.$$

Furthermore, if q_e is the distribution density of e_1,

$$E_g(q_j(x) - q_e(x))^2 \le C\sigma^{-2} R_{\tau_j}(\widehat{g},g).$$

Proof: The results of the lemma are direct consequences of the expression

$$q_j(x) = \int q_j(z) q_e(x - (\widehat{g}_{\tau_j}(z) - g(z)))dz.$$

∎

Let us denote $h^*(x)$ the optimal bandwidth h_{i^*} at x.

Lemma 2 *Let for some $x \in \mathbf{R}$*

$$\frac{\pi_j(h(i^*,x))}{h^*(x)} \ge C(\log m)^{-(2s+1)}. \tag{14}$$

Then

$$h^*(x)\left(\frac{\pi_j(h(i^*,x))}{h^*(x)}\right)^{\frac{1}{2s+1}} \le C\int_x^{x+h^*(x)} q_j^{\frac{1}{2s+1}}(y)dy \tag{15}$$

Proof: Due to the inequality (14) we get from the definition (12) of $h^*(x)$ the following bound:

$$h^*(x) \le C\left(\frac{\sigma^2(\log m)^{\frac{2s}{2s+1}}}{mL^2}\right)^{\frac{1}{2s+1}} = o\left(\frac{\pi_j(h(i^*,x))}{h^*(x)}\right).$$

Using the Lipschitz property of the density q_j we conclude now that, for instance,

$$\frac{1}{2}q_j(x) \le \frac{\pi_j(h(i^*,x))}{h^*(x)} \le 2q_j(x),$$

what results in the desired bound. ∎

We have the following bounds for the risk $R_{\tau_{j+1},\pi_j}(\widehat{g},g)$

Proposition 2

$$
\begin{aligned}
R_{\tau_{j+1},\pi_j}(\widehat{g},g) &\le CL^{\frac{2}{2s+1}}\left(\frac{\sigma^2\log\tau_j}{\tau_j}\right)^{\frac{2s}{2s+1}}\int q_j^{\frac{1}{2s+1}}(x)dx + O\left(\frac{(\log\tau_j)^{3/2}}{\tau_j}\right);\\
R_{\tau_{j+1}}(\widehat{g},g) &\le CL^{\frac{2}{2s+1}}\left(\frac{\sigma^3\log\tau_j}{\tau_j}\right)^{\frac{2s}{2s+1}} + O\left(\frac{(\log\tau_j)^{3/2}}{\tau_j}\right).
\end{aligned}
$$

Proof: Note first that if $\beta < \frac{1}{2\sigma^2}$,

$$E_g\exp(\beta y_t^2) \le E\exp(\beta(2R+|e_t|)^2) \le C.$$

Thus there exist $C<\infty$ such that

$$\int_{|x|>C(1+\sigma\sqrt{\log m})}\pi_j(dx) \le \frac{1}{m}.$$

We now construct a special partition of the interval $A_m = [-C(1+\sigma\sqrt{\log m}), C(1+\sigma\sqrt{\log m})]$. Let us put $x_0=0$ and $x_i = x_{i-1}+h^*(x_{i-1})$ for $0<i\le i_m^+$ and $x_i = x_{i+1}+h^*(x_{i+1})$ for $i_m^-\le i<0$. Since the invariant measure π_j of the Markov chain possess a bounded density, we conclude from (12) that

$$h^*(x) = O\left((\log m)^{\frac{1}{2s+1}}m^{-\frac{1}{2s+1}}\right),$$

and i_m^- and i_m^+ are at most $O\left((\log m)^{\frac{2s-1}{4s+2}}m^{\frac{1}{2s+1}}\right)$. Recall that $|\widehat{g}_{\tau_{j+1}}(x)-g(x)|\le 2R$, so that

$$
\begin{aligned}
R_{\tau_{j+1},\pi_j}(\widehat{g},g) &= \int E_{g,\tau_j}(\widehat{g}_{\tau_{j+1}}(x)-g(x))^2 q_j(x)dx\\
&\le \sum_{i=i_m^-}^{i^+m-1}\int_{x_i}^{x_{i+1}} E_{g,\tau_j}(\widehat{g}_{\tau_{j+1}}(x)-g(x))^2 q_j(x)dx + \frac{2R}{m}\\
&\le C\sum_{i=i_m^-}^{i^+m-1}\int_{x_i}^{x_{i+1}} r^*_{\tau_{j+1}}(x)q_j(x)dx\\
&\quad +2R\sum_{i=i_m^-}^{i^+m-1}\int_{x_i}^{x_{i+1}} 1\{i^*(x)<i_1(x)\}q_j(x)dx + O\left(\frac{1}{m}\right)\\
&= \delta_m^{(1)}+\delta_m^{(2)}+O\left(m^{-1}\right).
\end{aligned}
$$

Let us estimate first $\delta_m^{(2)}$. From the definition of (12) of i^* and that of i_1 we conclude that $h^*(x)\ge C>0$ for x such that $i^*(x)<i_1(x)$. Thus there are at most $O(\sqrt{\log m})$ non-vanishing terms in the sum which defines $\delta_m^{(2)}$. This provides the bound

$$\delta_m^{(2)} \le C\sqrt{\log m}\frac{\log m}{m} \le C\frac{(\log m)^{3/2}}{m}.$$

Further, it follows from the definitions (12) of $h^*(x)$ and (13) of $r^*_{\tau_{j+1}}(x)$ that there is $C(s) < \infty$ such that if $x \in [x_{i-1}, x_i]$ $h^*(x) \le 2h^*(x_{i-1})$ and $r^*_{\tau_{j+1}}(x) \le C(s) r^*_{\tau_{j+1}}(x_{i-1})$. Thus we can decompose $\delta_m^{(1)}$ as follows:

$$\delta_m^{(1)} \le C \sum_{i=i_m^-}^{i_m^+-1} r^*_{\tau_{j+1}}(x_i)\pi_j(h(i^*, x_i)).$$

The choice of i^* implies that

$$r^*_{\tau_{j+1}}(x) \le CL^{\frac{2}{2s+1}}(h^*(x))^{\frac{2s}{2s+1}} \left(\frac{\sigma^2 \log m}{m\pi_j(h(i^*, x))}\right)^{\frac{2s}{2s+1}},$$

so that

$$\begin{aligned} \delta_m^{(1)} &\le CL^{\frac{2}{2s+1}} \left(\frac{\sigma^2 \log m}{m}\right)^{\frac{2s}{2s+1}} \sum_{i=i_m^-}^{i_m^+-1} h^*(x_i) \left(\frac{\pi_j(h(i^*, x_i))}{h^*(x_i)}\right)^{\frac{1}{2s+1}} \\ &\le C'L^{\frac{2}{2s+1}} \left(\frac{\sigma^2 \log m}{m}\right)^{\frac{2s}{2s+1}} \left[\sum_{i=i_m^-}^{i_m^+-1} \int_{x_i}^{x_{i+1}} q_j^{\frac{1}{2s+1}}(x)dx + O\left((\log m)^{-1/2}\right)\right] \end{aligned}$$

due to the bound (15). Now we are done: when taking the expectation we get using the second estimate of Lemma 1:

$$\begin{aligned} R_{\tau_{j+1}}(\hat{g}, g) &\le CL^{\frac{2}{2s+1}} \left(\frac{\sigma^2 \log m}{m}\right)^{\frac{2s}{2s+1}} \left(\int q_e^{\frac{1}{2s+1}}(x)dx + R_{\tau_j}(\hat{g}, g)\sqrt{\log m}\right) \\ &\quad + O\left(\frac{(\log m)^{3/2}}{m}\right) \\ &\le C'L^{\frac{2}{2s+1}} \left(\frac{\sigma^2 \log m}{m}\right)^{\frac{2s}{2s+1}} \int q_e^{\frac{1}{2s+1}}(x)dx + O\left(\frac{(\log m)^{3/2}}{m}\right) \\ &\le C''L^{\frac{2}{2s+1}} \left(\frac{\sigma^3 \log m}{m}\right)^{\frac{2s}{2s+1}} + O\left(\frac{(\log m)^{3/2}}{m}\right). \end{aligned}$$

∎

Now return to the proof of the theorem: recall that the estimate $\hat{g}_{\tau(t)}(x)$ depends only on y_ts up to $\tau(t)$. Due to the mixing property (11) we obtain

$$\begin{aligned} E_g\left(\hat{g}_{\tau(t)}(y_{t-1}) - g(y_{t-1})\right)^2 &= E_g\left[E_{g,\tau_j}(\hat{g}_{\tau(t)}(y_{t-1}) - g(y_{t-1}))^2\right] \\ &= R_{\tau(t)}(\hat{g}, g) + C\exp\left(-\lambda(t - (\tau(t) - 1))\right) \\ &= R_{\tau(t)}(\hat{g}, g) + O(\tau^{-1}(t)) \end{aligned}$$

for $t > \tau(t) - \frac{\log \tau(t)}{\lambda}$, and

$$E_g\left(\hat{g}_{\tau(t)}(y_{t-1}) - g(y_{t-1})\right)^2 \le 4R^2 \quad \text{for} \quad \tau(t) < t \le \tau(t) - \frac{\log \tau(j)}{\lambda}.$$

Then

$$\begin{aligned}\sum_{t=\tau_j+1}^{\tau_{j+1}} E_g\left(\widehat{g}_{\tau(t)}(y_{t-1})-g(y_{t-1})\right)^2 &\leq C\log(\tau_j)+\sum_{t=\tau_j+\lambda\log(\tau_j)+1}^{\tau_{j+1}} R_{\tau(t)}(\widehat{g},g)+O(\tau^{-1}(t)) \\ &\leq C(L\sigma^{3s})^{\frac{2}{2s+1}}\tau_j^{\frac{1}{2s+1}}(\log\tau_j)^{\frac{2s}{2s+1}}+O\left((\log\tau_j)^{3/2}\right). \quad (16)\end{aligned}$$

Let us define $j(n)$ by the relation $\tau(n)=\tau_{j(n)}$. We write

$$\begin{aligned}\frac{1}{n}\sum_{t=1}^{n} E_g(y_t-e_t)^2 &= \frac{1}{n}\sum_{t=1}^{2T} E_g(y_t-e_t)^2 \\ &+\frac{1}{n}\sum_{j=1}^{j(n)}\sum_{t=\tau_{j-1}+1}^{\tau_j} E_g\left(\widehat{g}_{\tau(t)}(y_{t-1})-g(y_{t-1})\right)^2 \\ &+\frac{1}{n}\sum_{t=\tau(n)+1}^{n} E_g\left(\widehat{g}_{\tau(t)}(y_{t-1})-g(y_{t-1})\right)^2 \\ &= \delta_1+\delta_2+\delta_3.\end{aligned}$$

For evident reasons $\delta_1 \leq \frac{2TC}{n}$. Note that $\tau(n)\geq n/2$, and

$$\delta_3 \leq CR_n(\widehat{g},g)+O(n^{-1}\log n).$$

On the other hand, since $\tau_j = 2^{j-j(n)}\tau(n)\leq n2^{j-j(n)}$, we obtain from (16):

$$\begin{aligned}\delta_2 &\leq \frac{C(L\sigma^{3s})^{\frac{2}{2s+1}}}{n}\sum_{j=1}^{j(n)}(\log\tau_j)^{\frac{2s}{2s+1}}\tau_j^{\frac{1}{2s+1}}+O((\log tau_j)^{3/2}) \\ &\leq \frac{C(L\sigma^{3s})^{\frac{2}{2s+1}}}{n}(\log n)^{\frac{2s}{2s+1}}\sum_{j=1}^{j(n)}2^{-\frac{j}{2s+1}}n^{\frac{1}{2s+1}}+O((\log n)^{3/2}) \\ &\leq C(L\sigma^{3s})^{\frac{2}{2s+1}}\left(\frac{\log n}{n}\right)^{\frac{2s}{2s+1}}+O\left((\log n)^{5/2}n^{-1}\right)\end{aligned}$$

This concludes the proof of the theorem. ∎

5.1 Proof of Proposition 1

To spare the reader some tedious technical details we present the proof of the proposition for the case of the *locally constant* estimator. This estimator is sufficient for the case $s\leq 1$. Further, in this case the minimization problem (7) can be easily solved, and the estimate $\widehat{g}_{2m}(i,x)$ is simply

$$\widehat{g}_{2m}(i,x)=\Pi_R\left((\#h(i,x))^{-1}\sum_{t=m+1}^{2m}(u_{t-1}-y_t)1\{y_{t-1}\in h(i,x)\}\right)$$

and $s_{2m}(i,x) = \nu\sqrt{\frac{\sigma^2 \log m}{\#h(i,x)}}$. Here

$$\#h(i,x) = \sum_{t=m+1}^{2m} 1\{y_{t-1} \in h(i,x)\}$$

stands for the cardinality of the window $h(i,x)$.

Note that

$$\begin{aligned}
\widehat{g}_{2m}(i,x) - g(x) &= (\#h(i,x))^{-1} \sum_{t=m+1}^{2m} (g(y_{t-1}) - g(x)) 1\{y_{t-1} \in h(i,x)\} \\
&\quad + (\#h(i,x))^{-1} \sum_{t=m+1}^{2m} e_t 1\{y_{t-1} \in h(i,x)\} \\
&= b(i,x) + \xi(i,x).
\end{aligned}$$

The assumption on the class $\mathcal{F}(s,L)$ implies that $b(i,x) \leq L h_i^s$. We will need the following technical

Lemma 3

1. *For any $\alpha > 0$ there is $C(\alpha) < \infty$ such that*

$$P_{g,m}\left(|\xi(i,x)| \geq C(\alpha)\sqrt{\frac{\sigma^2 \log m}{\#h(i,x)}},\ i \geq i_0\right) \leq C N^{-\alpha}. \tag{17}$$

2. *There is $\kappa > 0$ such that for any $0 \leq i \leq r$ any α*

$$\begin{aligned}
P_{g,m}&\left(\left|\frac{1\#h(i,x)}{m} - \pi_m(h(i,x))\right| \geq \frac{\pi_m(h(i,x))}{2}\right) \\
&\leq C(\alpha)\left[\exp\left(-\frac{m\pi_m(h(i,x))}{\kappa}\right) + m^{-\alpha}\right].
\end{aligned} \tag{18}$$

Proof: To show the first statement of the lemma it suffices to notice that $\xi(i,x)$ can be rewritten as $\xi(i,x) = \sigma^2 M_m / \langle M_m \rangle$, where M_m is a conditional Gaussian martingale and $\langle M_m \rangle$ its quadratic variation. Since for $i_0 \leq i \leq r$ we have $\sigma^2 c_1 \log m \leq \langle M_m \rangle < \sigma^2 m$, (17) follows immediately from Proposition 2.3 [10].

The bound (18) is a simple φ-mixing analog of the classical Bernstein inequality for independent random variables (cf. [4]). ∎

Now we can continue with the proof of the proposition. Note that there are two possibilities: $\hat{i} \geq i^*$ and $\hat{i} < i^*$.

Case $\hat{i} \geq i^*$. We have immediately

$$\begin{aligned}
|\widehat{g}_{2m}(\hat{i},x)-g(x)| &\leq \Big(|\widehat{g}_{2m}(\hat{i},x)-\widehat{g}_{2m}(i^*,x)|+|\widehat{g}_{2m}(i^*,x)-g(x)|\Big)\,1\{i_0\leq i^*\}\\
&\quad+\Big(|\widehat{g}_{2m}(\hat{i},x)-\widehat{g}_{2m}(i_0,x)|+|\widehat{g}_{2m}(i_0,x)-g(x)|\Big)\,1\{i_0>i^*\}\\
&\leq |\widehat{g}_{2m}(\hat{i},x)-\widehat{g}_{2m}(i^*,x)|1\{i_0\leq i^*\}\\
&\quad+|\widehat{g}_{2m}(i^*,x)-g(x)|1\{i_0\leq i^*\}+4R\,1\{i_0>i^*\}\\
&= \delta_1+\delta_2+\delta_3. \qquad (19)
\end{aligned}$$

From the bound (18) and the definition of i_0 and i_1 we have for some α large enough:

$$P_{g,m}(\delta_2>0)1\{i_1\leq i^*\}\leq P_{g,m}(i_0>i_1)\leq Cm^{-\alpha}.$$

Further,

$$\begin{aligned}
|\widehat{g}_{2m}(i^*,x)-g(x)|1\{i_0\leq i^*\} &\leq |b(i^*,x)|+|\xi(i^*,x)|1\{i_0\leq i^*\}\\
&\leq Lh^s_{i^*}+|\xi(i^*,x)|1\{i_0\leq i^*\},
\end{aligned}$$

and on the set $\{i_1\leq i^*\}$

$$\begin{aligned}
P_{g,m}\left(\delta_1\geq C\sqrt{r^*_{2m}(x)}\right) &\leq P_{g,m}\left(|\xi(i^*,x)|>C'\sqrt{\frac{\sigma^2\log m}{m\pi_m(h(i^*,x))}}\right)1\left\{\pi_m(h(i^*,x))c\frac{\log m}{m}\right\}\\
&\leq Cm^{-\alpha}
\end{aligned}$$

by Lemma 3. On the other hand,

$$|\widehat{g}_{2m}(\hat{i},x)-\widehat{g}_{2m}(i^*,x)|\leq\nu\sqrt{\frac{\sigma^2\log m}{\#h(i^*,x)}}.$$

Again, from (18) we get

$$\begin{aligned}
&P_{g,m}\left(\delta_2>C\sqrt{r^*_{2m}(x)}\right)1\{i_1\leq i^*\}\\
&\leq P_{g,m}\left(\frac{\#h(i^*,x)}{m}<C'\pi(h(i^*,x))\right)1\left\{\pi_m(h(i^*,x))\geq c\frac{\log m}{m}\right\}\leq Cm^{-\alpha}.
\end{aligned}$$

When summing up the above results we get from (19)

$$P_{g,m}\left(|\widehat{g}_{2m}(x)-g(x)|\geq C(\alpha)\sqrt{r^*_{2m}(x)}\right)1\{i_1\leq i^*\leq\hat{i}\}\leq Cm^{-\alpha}. \qquad (20)$$

Case $\hat{i} < i^*$. Since

$$\begin{aligned} |\widehat{g}_{2m}(i,x) - \widehat{g}_{2m}(r,x)| &\le |\widehat{g}_{2m}(i,x) - g(x)| + |\widehat{g}_{2m}(r,x) - g(x)| \\ &\le |b(i,x)| + |\xi(i,x)| + |b(r,x)| + |\xi(r,x)| \\ &\le Lh_i^s + Lh_r^s + |\xi(i,x)| + |\xi(r,x)|, \end{aligned}$$

we have for $r < i < i^*$,

$$Lh_r^s < Lh_i^s < \sqrt{\frac{\sigma^2 \log m}{m\pi_m(h(i,x))}} < \sqrt{\frac{\sigma^2 \log m}{m\pi_m(h(r,x))}}.$$

This implies the estimate

$$\begin{aligned} &\{|\widehat{g}_{2m}(i,x) - \widehat{g}_{2m}(r,x)| > s_{2m}(r,x)\} \\ &\subseteq \left\{ |\xi(i,x)| > \frac{s_{2m}(r,x)}{2} - \sqrt{\frac{\sigma^2 \log m}{m\pi_m(h(r,x))}} \right\} \\ &\quad \cup \left\{ |\xi(r,x)| > \frac{s_{2m}(r,x)}{2} - \sqrt{\frac{\sigma^2 \log m}{m\pi_m(h(r,x))}} \right\} \end{aligned} \tag{21}$$

Now let us recall the definition (8) of i_0. Let i' be defined as follows:

$$i' = min\left(i : \ \pi_m(h(i,x)) \ge \frac{\kappa}{2} \frac{\log m}{m} \right).$$

Then for κ (the same as in (8)) large enough

$$P_{g,m}(i_0 < i') \le Cm^{-\alpha},$$

and, moreover,

$$\max_{i \ge i'} P_{g,m}\left(s_{2m}(i,x) < \frac{\nu}{4}\sqrt{\frac{\sigma^2 \log m}{m\pi_m(h(r,x))}} \right) \le Cm^{-\alpha}.$$

On the other hand, it can be easily seen that

$$\begin{aligned} \{\hat{i} < i^*\} &= \cup_{i=i_0}^{i^*} \{\exists\, r < i : \ |\widehat{g}_{2m}(i,x) - \widehat{g}_{2m}(r,x)| > s_{2m}(r,x)\} \\ &\subseteq \cup_{i=i_0}^{i^*} \cup_{r=i_0}^{i} \{|\widehat{g}_{2m}(i,x) - \widehat{g}_{2m}(r,x)| > s_{2m}(r,x)\}. \end{aligned}$$

When taking into account (21) we obtain

$$\begin{aligned} P_{g,m}(\hat{i} < i^*) &\le P_{g,m}(i_0 < i') + 2\sum_{i=i'}^{i^*}\sum_{r=i'}^{i} \left[P_{g,m}\left(|\xi(r,x)| > \frac{\nu}{4}\sqrt{\frac{\sigma^2 \log m}{\# h(r,x)}} \right) \right. \\ &\quad \left. + P_{g,m}\left(s_{2m}(i,x) < \frac{\nu}{4}\sqrt{\frac{\sigma^2 \log m}{m\pi(h(r,x))}} \right) \right] \\ &\le C(m^{-\alpha} + (\log m)^2 m^{-\alpha}), \end{aligned} \tag{22}$$

due to (17). When summing up the bounds of (20) and (22) we obtain the statement of the proposition. ∎

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