Synthesis of Three Term Controllers

with Frequency Domain Data

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- Present Paper - extend these results to the case where only data is available

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PID controller design: CT case

Let the PID controller be of the form

$$C(s) = \frac{K_i + K_p s + K_d s^2}{s(1+sT)}, \qquad T > 0$$

and

$$P(j\omega) = |P(j\omega)|e^{j\phi(\omega)} = P_r(\omega) + jP_i(\omega).$$

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- Using the result above, the subset of the PID gains that satisfy the several given performance requirements.

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- Fix $K_p = K_p^*$, solve

$$K_p^* = -\frac{P_r(\omega) + \omega T P_i(\omega)}{|P(j\omega)|^2} = -\frac{\cos\phi(\omega) + \omega T \sin\phi(\omega)}{|P(j\omega)|}$$

  and let $\omega_1 < \omega_2 < \cdots < \omega_{l-1}$ denote the distinct frequencies which are solutions of the equation above.

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$$\left\{ i_0 - 2i_1 + 2i_2 + \cdots + (-1)^{i-1} 2 i_{l-1} + (-1)^l i_l \right\} \cdot (-1)^{l-1} j = n - m + 2z_r + 2$$

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  - For $n - m$ odd:

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- For the fixed $K_p = K_p^*$ chosen in Step 1, solve for the stabilizing (K_i, K_d) from:

$$\left[K_i - K_d\omega_t^2 + \frac{\omega_t \sin\phi(\omega_t) - \omega_t^2 T \cos\phi(\omega_t)}{|P(j\omega_t)|}\right] i_t > 0$$

for $t = 0, 1, \cdots$.

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for $t = 0, 1, \cdots$.

- Repeat the previous three steps by updating K_p over prescribed ranges.

Lemma 1

Relative Degree and Net Phase Change:

- A. In the Bode magnitude plot of the LTI system $P(j\omega)$, the high frequency slope is $-(n-m)20\text{dB}/\text{decade}$ where $n-m$ is the relative degree of the plant $P(s)$.

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- B. The net change of phase of $P(j\omega), \omega \in [0, \infty)$, denoted $\Delta_0^\infty(\phi)$ is:

$$\Delta_0^\infty(\phi) = -\left[(n-m) - 2(p_r - z_r)\right]\frac{\pi}{2}$$

where p_r and z_r are numbers of RHP poles and zeros of $P(s)$, respectively.

Proof of Lemma 1

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- Let p_l and z_l are number of LHP poles and zeros of $P(s)$. Then the net phase change of $P(j\omega)$ for $\omega \in [0, \infty)$ is

$$\begin{aligned}
\Delta_0^\infty(\phi) &= \left[(z_l - z_r) - (p_l - p_r)\right]\frac{\pi}{2} \\
&= \left[(m - z_r - z_r) - (n - p_r - p_r)\right]\frac{\pi}{2} \\
&= -\left[(n - m) - 2(p_r - z_r)\right]\frac{\pi}{2}.
\end{aligned}$$

Consider an n^{th} degree real polynomial

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\begin{aligned}
a(s) &= a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0, \\
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$$\begin{aligned} i_0 &= \operatorname{sgn}\left[a_R^{p_0}(0)\right], \\ i_k &= \operatorname{sgn}\left[a_R(\omega_k)\right], \quad k = 1, \cdots, l \\ j &= \operatorname{sgn}[a_I(\infty)] \end{aligned}$$

where p_0 the multiplicity of $\omega_0 = 0$ as a zero of $a_I(\omega) = 0$.

Lemma

Real Signature Formula

- For $\deg[a(s)]$ even,

$$\sigma(a) = \left\{ i_0 - 2i_1 + \cdots + (-1)^{l-1} 2 i_{l-1} + (-1)^l i_l \right\} \cdot (-1)^{l-1} \cdot j.$$

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Complex Signature Formula If $a(s)$ is a complex polynomial, let $\omega_0 < \omega_1 < \omega_2 < \cdots < \omega_{l-1}$ denote real, distinct, finite zeros of $a_I(\omega) = 0$ of odd multiplicities and let $\omega_0 = -\infty$, $\omega_l = +\infty$.

- If $\deg[a(s)]$ is even and a_n is real or $\deg[a(s)]$ is odd and a_n is imaginary,

$$\sigma(a) = \frac{1}{2}\left\{(-1)^{l-1} i_0 + 2\sum_{r=1}^{l-1}(-1)^{l-1-r} i_r - i_l\right\} \cdot j.$$

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- If $\deg[a(s)]$ is even and a_n is not purely real or $\deg[a(s)]$ is odd and a_n is not purely imaginary, then

$$\sigma(a) = \frac{1}{2}\left\{2\sum_{r=1}^{l-1}(-1)^{l-1-r} i_r\right\} \cdot j.$$

Sketch of Proof of Procedure

Let us consider the plant and PID controller pair of the form:

$$
\begin{aligned}
P(s) &= \frac{N(s)}{D(s)} = \frac{N_e(s^2) + sN_o(s^2)}{D_e(s^2) + sD_o(s^2)} \\
C(s) &= \frac{K_i + K_p s + K_d s^2}{s(1+sT)}, \quad T > 0
\end{aligned}
$$

where $\deg[D(s)] = n$ and $\deg[N(s)] = m$. Then

$$
\begin{aligned}
\delta(s) &= s(1+sT)D(s) + \left(K_i + K_p s + K_d s^2\right) N(s) \\
\Pi(s) &= \delta(s)N(-s) = s(1+sT)D(s)N(-s) \\
&\quad + \left(K_i + K_p s + K_d s^2\right) N(s)N(-s)
\end{aligned}
$$

with $\deg\Pi(s) = n + m + 2$.

Proof Continue ...

Then $\delta(s)$ is *Hurwitz* if and only if the signature of $\Pi(s)$ (number of LHP roots - number of RHP roots) is:

$$\sigma(\Pi) = n + 2 + z_r - z_l.$$

Equivalently,

$$\begin{aligned} \sigma(\Pi) &= n + 2 + z_r - (m - z_r) = n + 2 + 2z_r - m \\ &= \underbrace{n - m}_{\text{relative degree of plant}} + 2z_r + 2. \end{aligned}$$

Proof Continue ...

$$\begin{aligned}\Pi(j\omega) &= j\omega(1+j\omega T)D(j\omega) + \left(K_i + j\omega K_p - \omega^2 K_d\right) N(j\omega)N(-j\omega)\\ &= R(\omega) + jI(\omega) = R\left(\omega, K_i, K_d\right) + jI\left(\omega, K_p\right)\end{aligned}$$

K_p appears only in $I(\omega)$ and K_i, K_d only in $R(\omega)$. Let

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denote the real, positive, distinct, finite zeros of $I(\omega, K_p^*) = 0$, of odd multiplicities, define $\omega_l = \infty$ and

$$\begin{aligned}&I_0 = \text{sgn}\left[R^{p_0}(0)\right], \quad I_k = \text{sgn}\left[R(\omega_k)\right], \quad k = 1, 2, \cdots, l\\ &j = \text{sgn}[I(\infty)].\end{aligned}$$

Proof Continue ...

The set of PID stabilizing controllers can be found as follows:

- Fix $K = K_p^*$ and set $I(\omega, K_p^*) = 0$. Let $i_t \in \{+1, 0, -1\}$ and $j \in \{+1, -1\}$.

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Proof Continue ...

- For each string satisfying the signature formula, the conditions for stability are:

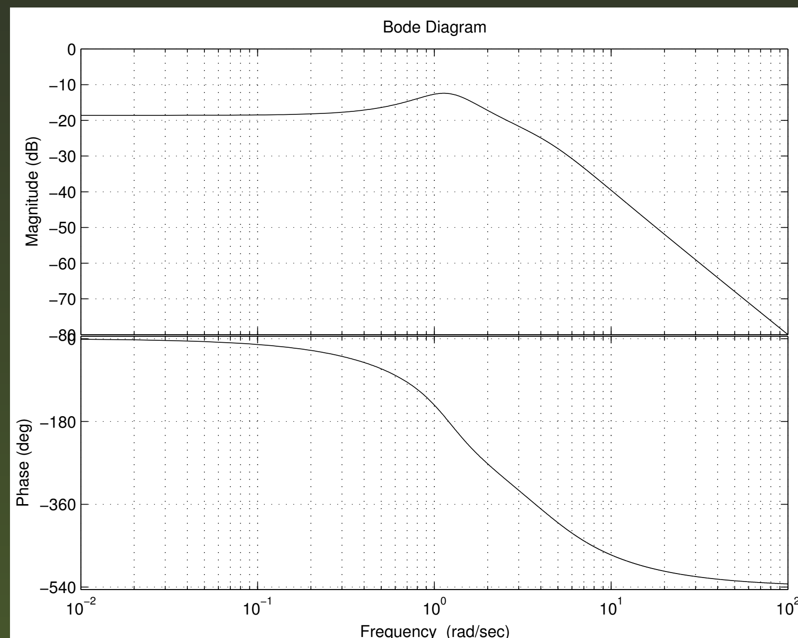
$$\operatorname{sgn}\left[R(\omega_t, K_i, K_d)\right] i_t > 0, \quad \text{for} \;\; t = 0, 1, 2, \cdots,$$

- condition stated in the theorem are equivalent to the above eq.

End of Proof

Example

Consider the *stable* plant



$$-6\frac{\pi}{2} = -\left(\underbrace{(n-m)}_{=2} - 2(\underbrace{p_r}_{=0} - z_r)\right)\frac{\pi}{2} \Rightarrow z_r = 2$$

- The required signature for stability can now be determined and is

$$\sigma(\Pi) = (n-m) + 2z_r + 2 = (2) + 2(2) + 2 = 8.$$

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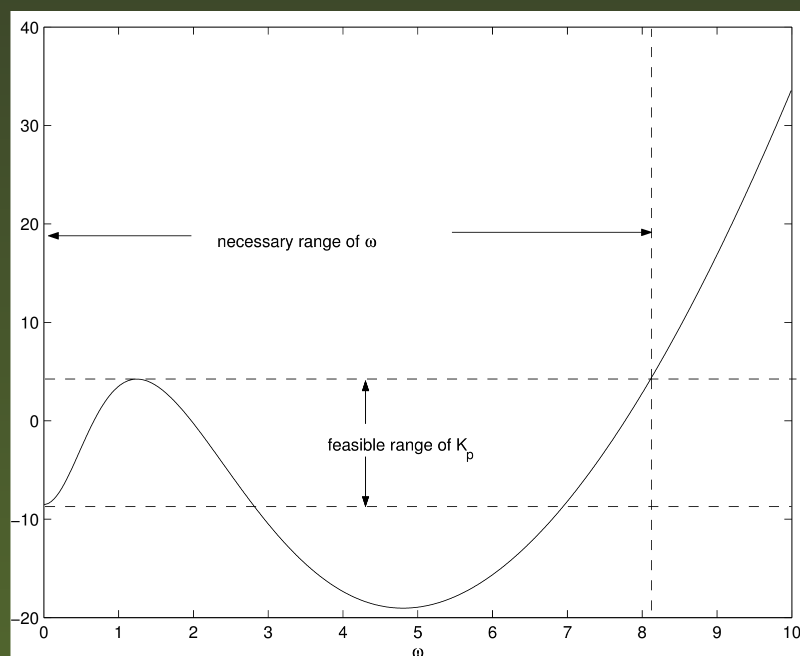
- In other words $l \geq 4$.

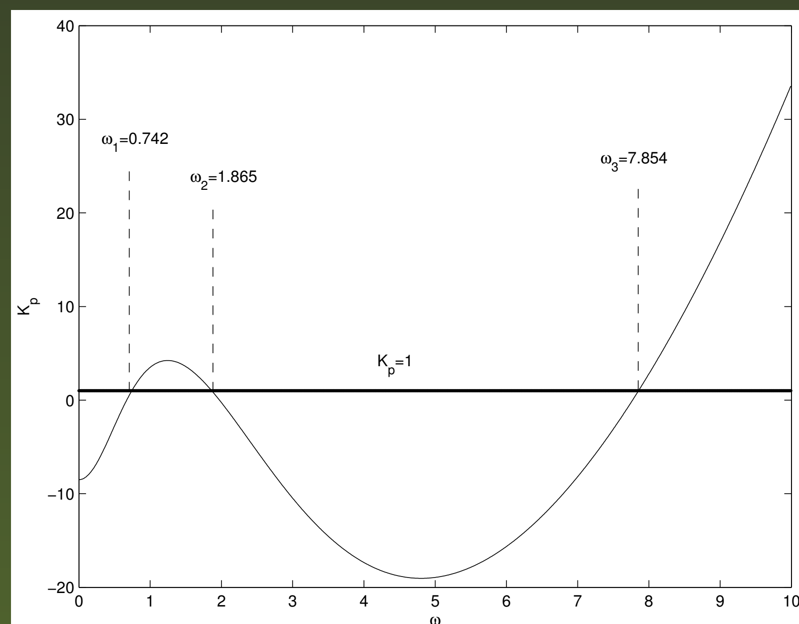
From the figure it is easy to see that K_p^* has at most three positive frequencies as solutions and therefore we have

$$i_0 - 2i_1 + 2i_2 - 2i_3 + i_4 = 8.$$



Fix $K_p = 1$ and compute the set of ω's that satisfies

$$-\frac{\cos\phi(\omega) + \omega T \sin\phi(\omega)}{|P(j\omega)|} = 1.$$



This leads to the requirement

$$i_0 - 2i_1 + 2i_2 - 2i_3 = 7$$

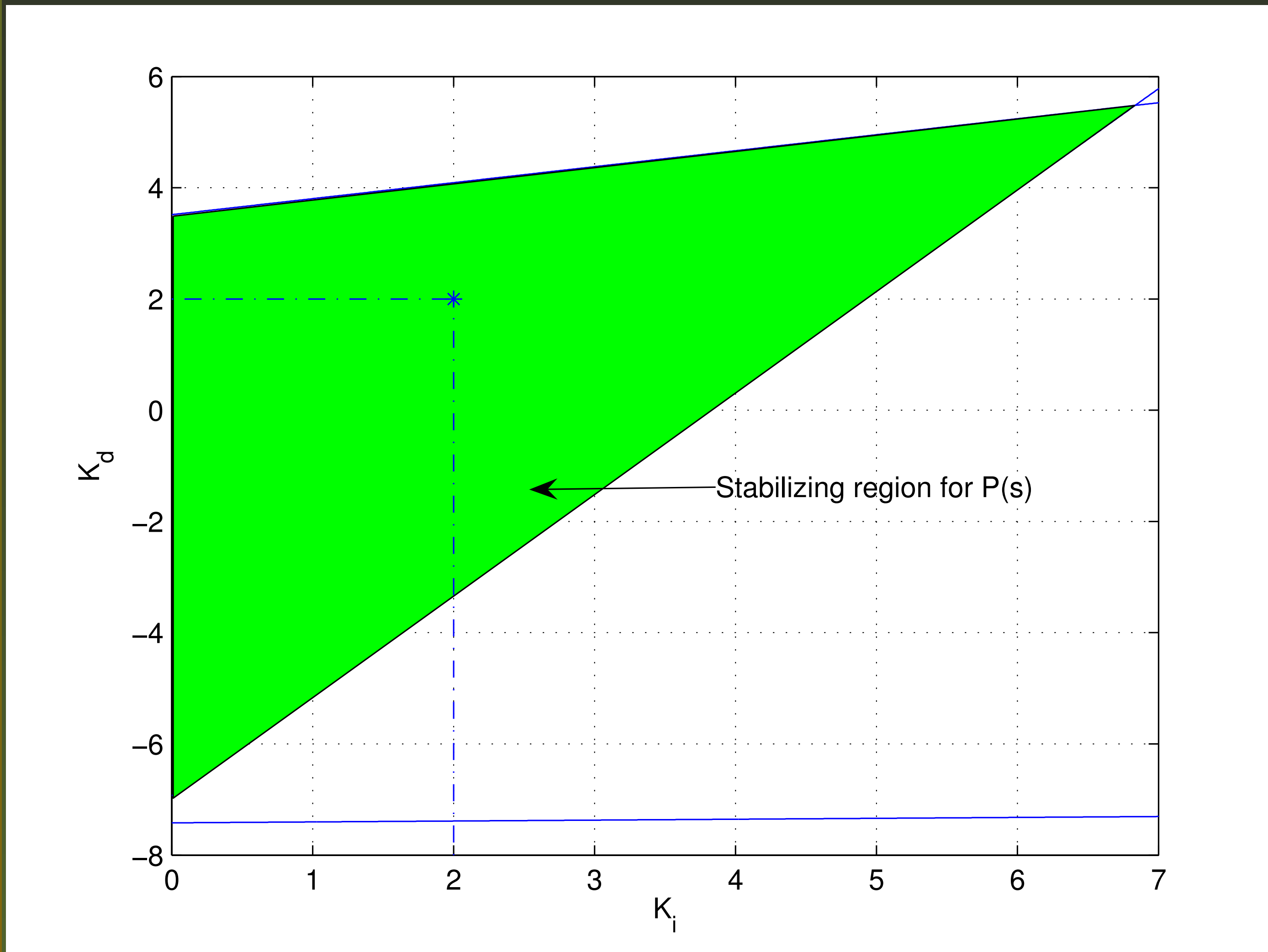
giving the feasible string

$$\mathcal{F} = \{i_0, i_1, i_2, i_3\} = \{1, \ -1, \ 1, \ -1\}.$$

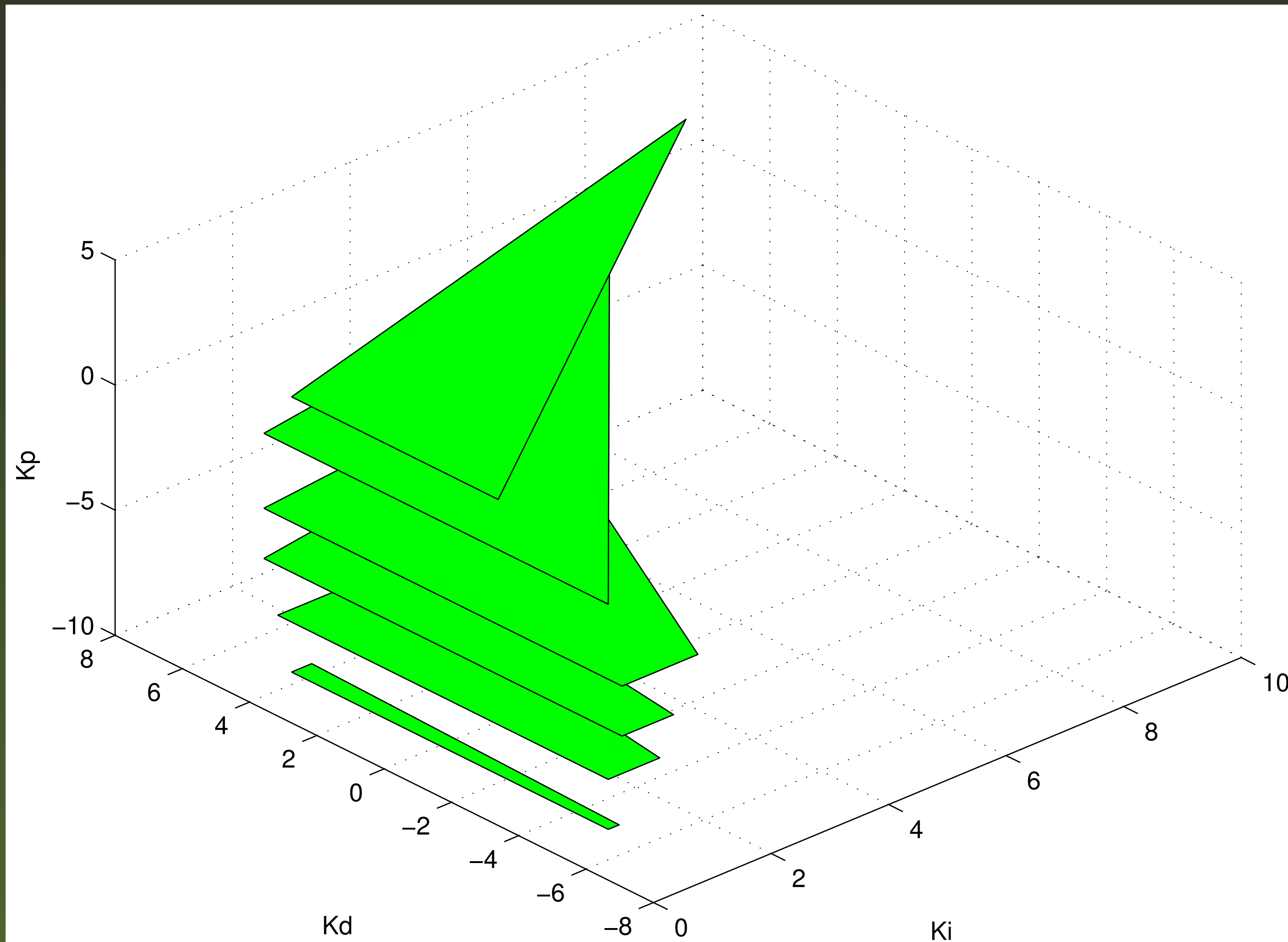
Thus, we have the following set of linear inequalities for stability:

$$\begin{aligned} 0.0138K_i &> 0 \\ -0.1390 + 0.0364K_i - 0.0201K_d &< 0 \\ 0.2791 + 0.0229K_i - 0.0797K_d &> 0 \\ -0.1349 + 0.0003K_i - 0.0182K_d &< 0 \end{aligned}$$

Stabilizing PID Set for $K_p = 1$



Entire Stabilizing PID Set



Performance Specifications

Many performance attainment problems can be cast as stabilization of families of real and complex plants. For example,

- The problem of achieving a gain margin is equivalent to stabilizing the family of *real* plants

$$\mathcal{P}^c(s) = \{KP(s) : K \in [K_{\min}, K_{\max}]\}\,.$$

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- The problem of achieving prescribed phase margin θ_m is equivalent to stabilizing the family of *complex* plants

$$\mathcal{P}^c(s) = \left\{e^{-j\theta}P(s) : \theta \in [0, \theta_m]\right\}.$$

- The problem of achieving an H_∞ norm specification on the sensitivity function $S(s)$, that is, $\|W(s)S(s)\|_\infty < \gamma$ is equivalent to stabilizing the family of *complex* plants

$$\mathcal{P}^c(s) = \left\{ \left[\frac{1}{1 + \frac{1}{\gamma} e^{j\theta} W(s)} \right] P(s) : \theta \in [0, 2\pi] \right\}.$$

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- The problem of achieving an H_∞ norm specification on the complementary sensitivity function $T(s)$, that is, $W(s)T(s)\|_\infty < \gamma$ is equivalent to stabilizing the family of *complex* plants

$$\mathcal{P}^c(s) = \left\{ P(s) \left[1 + \frac{1}{\gamma} e^{j\theta} W(s) \right] : \theta \in [0, 2\pi] \right\}.$$

Assumption

The only information available to the designer is:

- Knowledge of the frequency response magnitude and phase, equivalently, $P^c(j\omega), \omega \in (-\infty, +\infty)$.
- Knowledge of the number of RHP poles, p_r.

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  - Fix $K_p = K_p^*$ and solve

$$K_p^* = -\frac{\cos\phi(\omega) + \omega T \sin\phi(\omega)}{|P^c(j\omega)|}$$

  and let $\omega_1 < \omega_2 < \cdots < \omega_{l-1}$ denote the distinct frequencies which are solutions of the above.

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- For the fixed $K_p = K_p^*$ chosen in Step 1, solve for the stabilizing (K_i, K_d) from:

$$\left[K_i - K_d\omega_t^2 + \frac{\omega_t \sin\phi(\omega_t) - \omega_t^2 T \cos\phi(\omega_t)}{|P^c(j\omega)|}\right] i_t > 0$$

for $t = 0, 1, \cdots$.

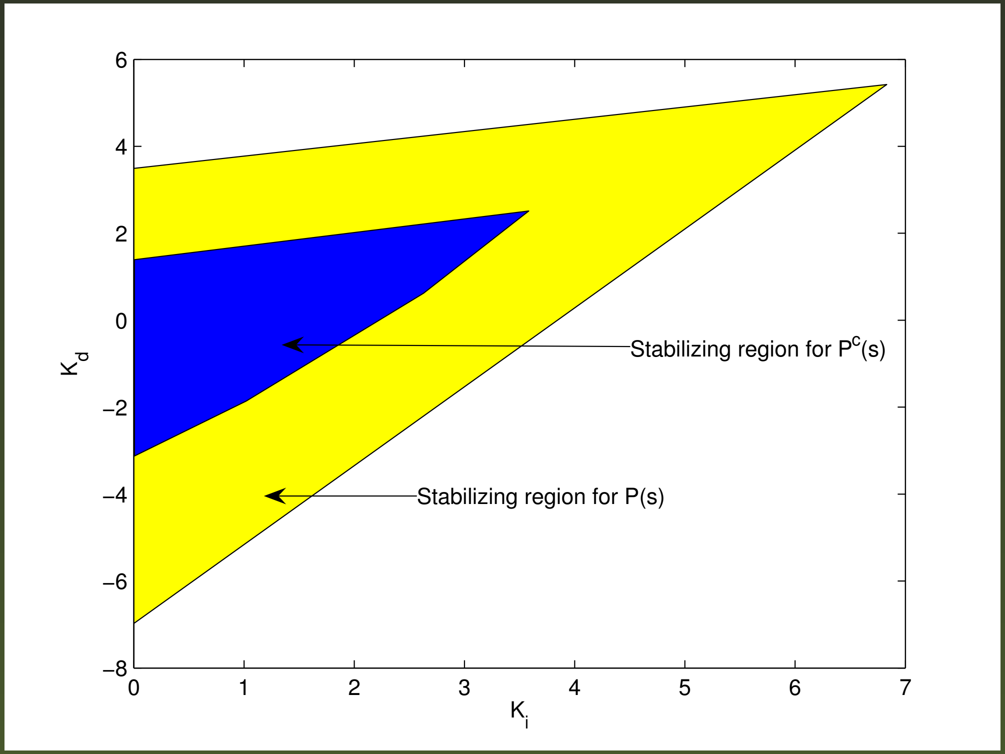
Performance Example

Taking the same frequency domain data set $\mathbf{P}(j\omega)$ used in the previous example, we consider the problem of achieving an H_∞ norm specification on the complementary sensitivity function $T(s)$, that is,

$$\|W(s)T(s)\|_\infty < 1 \quad \text{where } W(s) = \frac{s+0.1}{s+1}.$$

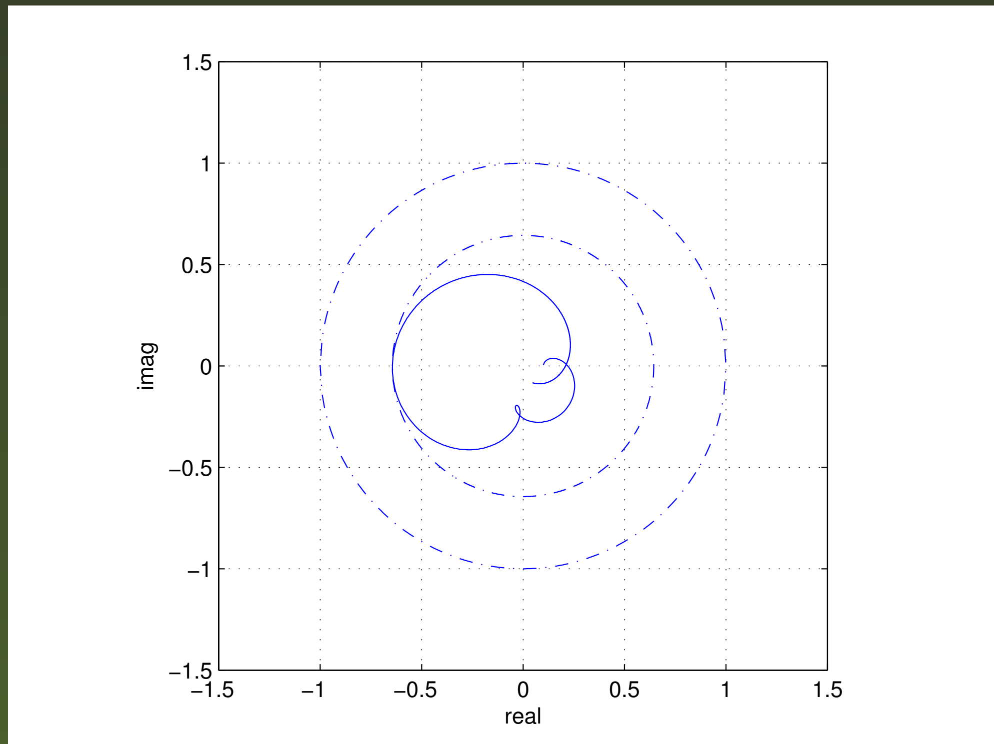
By solving the complex stabilization problem, we have the stabilizing PID controller parameter region that satisfies the given H_∞ norm specification.

The complete set of Stabilizing PID gains for H_∞ specification when $K_p = 1$



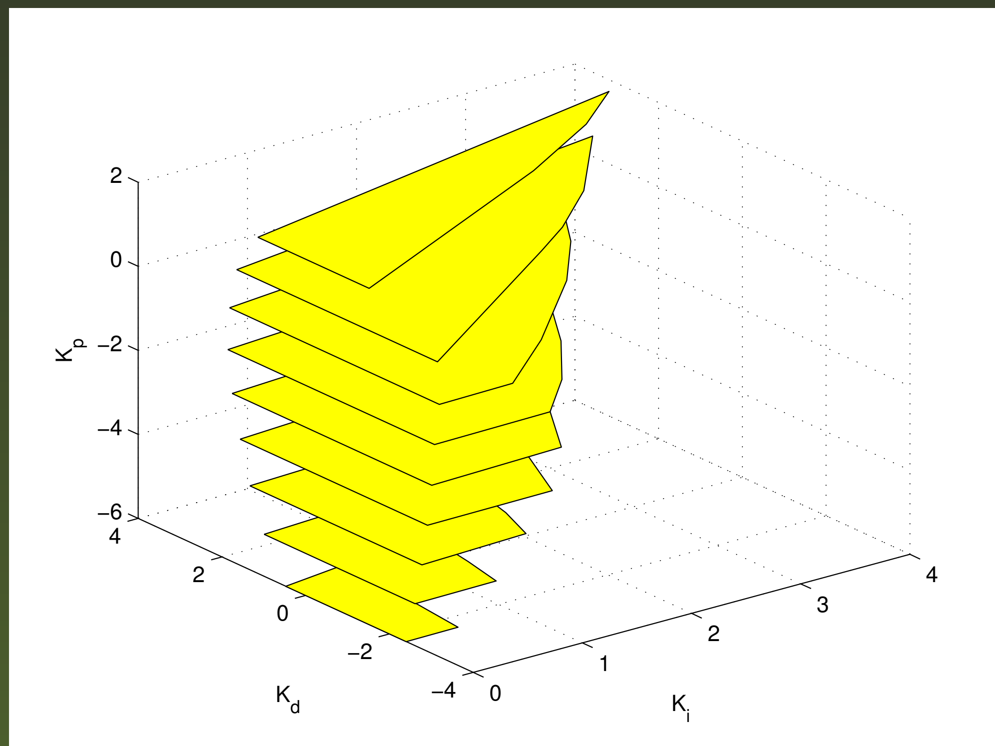
Nyquist plot of $W(s)T(s)$

By selecting a point, we verify that the point selected satisfied the given H_∞ specification.



Entire set of stabilizing PID gains satisfyinf the H_∞ specification

By sweeping K_p, we have the entire stabilizing PID gains that satisfy the given H_∞ specification as shown in Figure.



First order controller design: CT case

In the unity feedback configuration, let the plant and the controller be

$$P(s) = \frac{N(s)}{D(s)}, \qquad C(s) = \frac{x_1 s + x_2}{s + x_3}$$

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  - $P(j\omega)$ for $\omega \in [0, \infty)$
  - Number of plant RHP poles, p_r.

Characterization of Root Invariant Region

Given the frequency domain data $P(j\omega)$ of the plant, the following two straight lines and one curve given below in the $x_1 - x_2$ plane, for each fixed x_3 completely partitions the first order controller parameter space (x_1, x_2, x_3) such that each and every open region bounded by them consists of parameters that correspond to closed-loop systems with an invariant number of open LHP poles.

$$x_3 + x_2 P(0) = 0$$

$$\begin{cases} x_1(\omega) = \dfrac{1}{|P(j\omega)|^2}\left(\dfrac{P_i(\omega)}{\omega}x_3 - P_r(\omega)\right) \\ x_2(\omega) = -\dfrac{1}{|P(j\omega)|^2}\Big(P_r(\omega)x_3 + \omega P_i(\omega)\Big) \end{cases}$$

$$1 + P(\infty)x_2 = 0.$$

Proof

Consider the characteristic polynomial

$$\begin{aligned}\Pi(j\omega) &= (j\omega + x_3)\left(D_e(-\omega^2) + j\omega D_o(-\omega^2)\right] \\ &\quad +(j\omega x_1 + x_2)\left[N_e(-\omega^2) + j\omega N_o(-\omega^2)\right] \\ &= R(\omega) + j\omega I(\omega)\end{aligned}$$

where

$$\begin{aligned}R(\omega) &= -\omega^2 x_1 N_o(-\omega^2) + x_2 N_e(-\omega^2) + x_3 D_e(-\omega^2) - \omega^2 D_o(-\omega^2) \\ I(\omega) &= x_1 N_e(-\omega^2) + x_2 N_o(-\omega^2) + x_3 D_o(-\omega^2) + D_e(-\omega^2)\end{aligned}$$

Proof continue...

Using the Boundary Crossing Theorem, we have following three conditions.

- (A) Real root crossing condition:

$$\Pi(0) = x_3 D(0) + x_2 N(0) = 0$$

and since $D(0) \neq 0$ from Assumption, equivalently,

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- (B) Complex root crossing condition: From $R(\omega) = 0$ and $I(\omega) = 0$, we have

$$\begin{bmatrix} -\omega^2 N_o(-\omega^2) & N_e(-\omega^2) \\ N_e(-\omega^2) & N_o(-\omega^2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_3 D_e(-\omega^2) + \omega^2 D_o(-\omega^2) \\ -x_3 D_o(-\omega^2) - D_e(-\omega^2) \end{bmatrix}.$$

Proof continue...

Assume $|A(\omega)| \neq 0$ for all $\omega > 0$. Then

$$|A(\omega)| = \omega^2 N_o^2(-\omega^2) + N_e^2(-\omega^2) > 0, \text{ for all } \omega > 0.$$

Simplifying the notations, we write

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{|A(\omega)|} \begin{bmatrix} N_o & -N_e \\ -N_e & -\omega^2 N_o \end{bmatrix} \begin{bmatrix} -x_3 D_e + \omega^2 D_o \\ -x_3 D_o - D_e \end{bmatrix}.$$

Note that

$$P(j\omega) = \underbrace{\frac{N_e D_e + \omega^2 N_o D_o}{D_e^2 + \omega^2 D_o^2}}_{P_r(\omega)} + j \underbrace{\frac{\omega(N_o D_e - N_e D_o)}{D_e^2 + \omega^2 D_o^2}}_{P_i(\omega)}$$

$$|P(j\omega)|^2 = \frac{N_e^2 + \omega^2 N_o^2}{D_e^2 + \omega^2 D_o^2} = \frac{|A(\omega)|}{D_e^2 + \omega^2 D_o^2}$$

Proof continue...

Then we have

$$
\begin{aligned}
x_1(\omega) &= \frac{1}{|P(j\omega)|^2}\left(\frac{N_oD_e - N_eD_o}{D_e^2+\omega^2 D_o^2}x_3 - \frac{N_eD_e+\omega^2 N_oD_o}{D_e^2+\omega^2 D_o^2}\right) \\
&= \frac{1}{|P(j\omega)|^2}\left(\frac{P_i(\omega)}{\omega}x_3 - P_r(\omega)\right) \\
x_2(\omega) &= \frac{1}{|P(j\omega)|^2}\left(-\frac{N_eD_e+\omega^2 N_oD_o}{D_e^2+\omega^2 D_o^2}x_3 - \frac{\omega^2(N_oD_e - N_eD_o}{D_e^2+\omega^2 D_o^2}\right) \\
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&= -\frac{1}{|P(j\omega)|^2}\Big(P_r(\omega)x_3 + \omega P_i(\omega)\Big).
\end{aligned}
$$

- (C) Degree dropping condition:

$$
1 + \frac{n_n}{d_n}x_2 = 1 + P(\infty)x_2 = 0.
$$

Proof continue...

Finally, consider the case when $|A(\omega)| = 0$ for some $\omega \neq 0$.

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it follows that

$$\begin{array}{r} -x_3 D_e(-\omega^2) + \omega^2 D_o(-\omega^2) = 0, \\ -x_3 D_o(-\omega^2) - D_e(-\omega^2) = 0 \end{array} \Rightarrow \omega^2 D_o^2(-\omega^2) + D_e^2(-\omega^2) = 0.$$

Proof continue...

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$$\left\{ \begin{array}{l} N_o(-\omega^2) = N_e(-\omega^2) = 0 \\ D_o(-\omega^2) - D_e(-\omega^2) = 0 \end{array} \right. \quad \text{for some } \omega$$

$$\Downarrow$$

$$D(s), N(s) \text{ has common factor}$$

This is ruled out by Assumption.

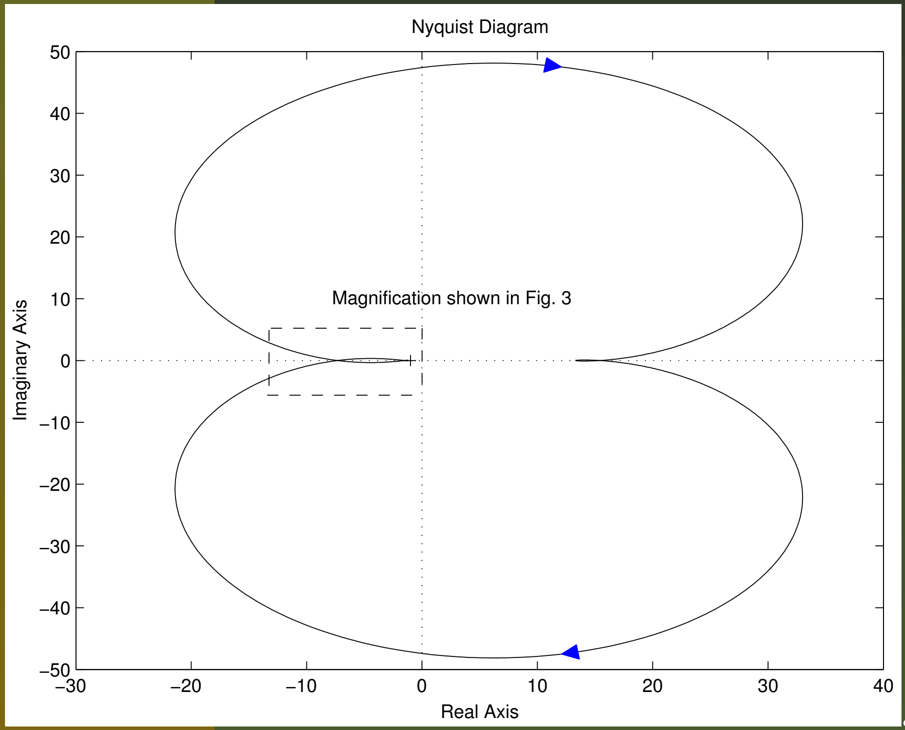
Example

For illustration, we have collected the frequency domain (Nyquist-Bode) data of the stable plant used to illustrate our model-based technique and refer to

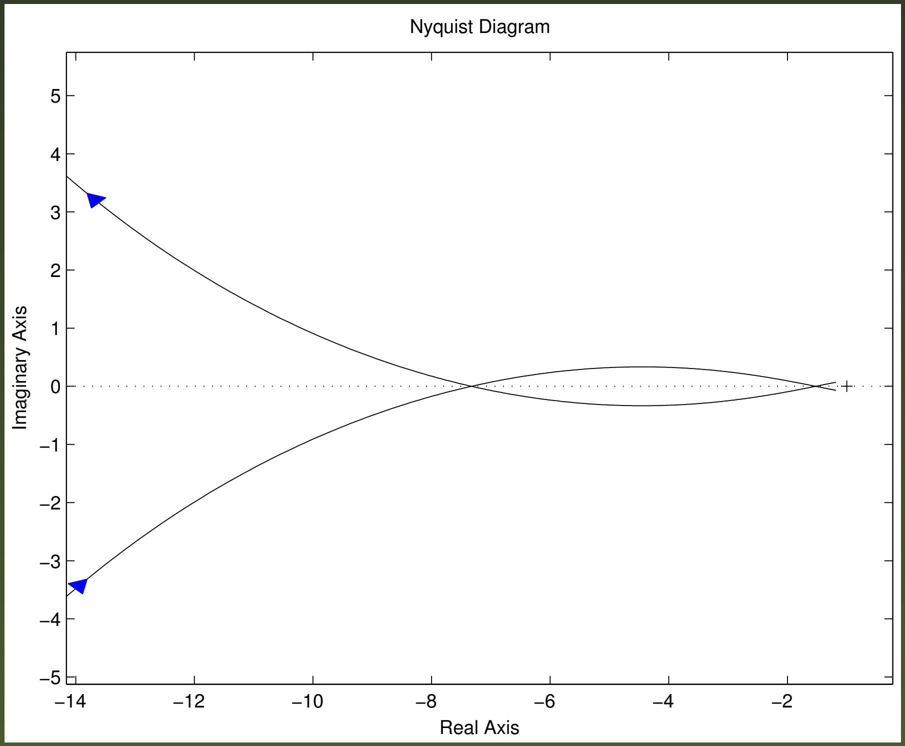
$$\mathbf{P}(j\omega) = \{P(j\omega) : \omega \in (0, 10) \text{ sampled every } 0.01\}.$$

Example continue...

The Nyquist plot of the plant obtained is:



,



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- Then it is easy to see that the straight line

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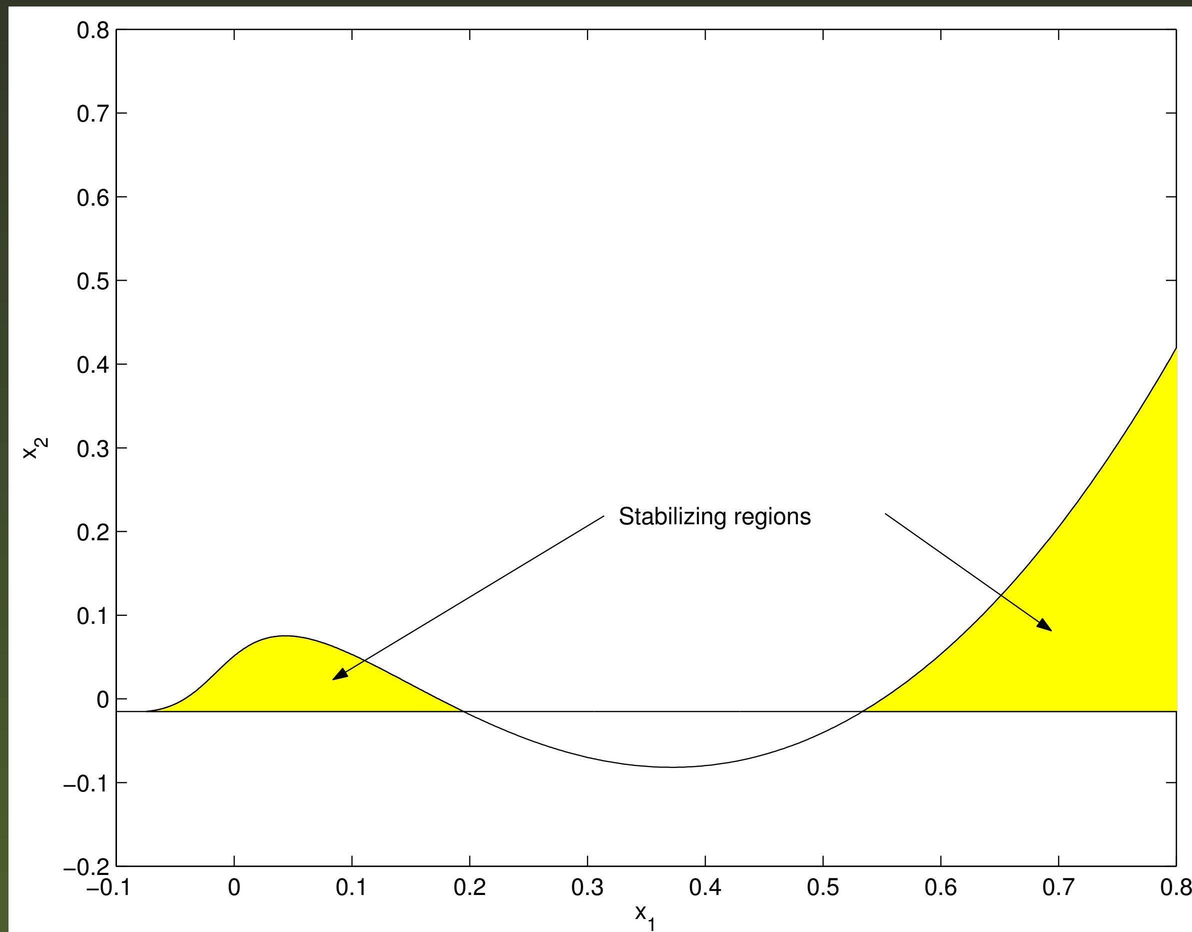
is not applicable.

- After fixing $x_3 = 0.2$, the data points representing the straight line

$$x_3 + x_2 P(0) = 0$$

and the curve $(x_1(\omega), x_2(\omega))$ are depicted.

Stabilizing regions for $x_3 = 0.2$



By testing a point for each root invariant region, we obtained the stabilizing regions which are identical to those in the example used for the model-based technique.

Subsets Achieving Performance Specifications

- Guaranteed Gain Margin Problem:

$$\mathbf{P}_c(j\omega) := \{KP(j\omega) \quad : K \in [1, K^*]\}$$

where $K^* \geq 1$ is the required gain margins. If conditional gain margin is required, it can be treated similarly.

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- Guaranteed Phase Margin Problem:

$$\mathbf{P}_c(j\omega) := \left\{e^{j\theta}P(j\omega) : \theta \in [0, \theta^*]\right\}$$

where $\theta^* \geq 0$ is the required phase margins.

- Guaranteed H_∞ Margin Problem: For the case of the sensitivity function $S(s)$, that is, $\|W(s)S(s)\|_\infty < \gamma$,

$$\mathbf{P}_c(j\omega) := \left\{ P(j\omega)\left[\frac{1}{1+\frac{1}{\gamma}e^{j\theta}W(j\omega)}\right] : \theta \in [0, 2\pi] \right\}.$$

For the case of the complementary sensitivity function $T(s)$, that is, $\|W(s)T(s)\|_\infty < \gamma$,

$$\mathbf{P}_c(j\omega) := \left\{ P(j\omega)\left[1+\frac{1}{\gamma}e^{j\theta}W(j\omega)\right] : \theta \in [0, 2\pi] \right\}.$$

Boundaries of Performance Regions

Note: The family $\mathbf{P}_c(s)$ is in general a complex family. Therefore, for $P_c(s, \theta)$ with a fixed θ, the two straight lines becomes

$$x_3 + x_2 P_r^c(0) = 0$$
$$x_3 + x_2 P_i^c(0) = 0$$

$$1 + x_2 P_r^c(\infty) = 0$$
$$1 + x_2 P_i^c(\infty) = 0$$

where

$$P_c(j\omega) := P_r^c(\omega) + jP_i^c(\omega), \omega \in [-\infty, +\infty].$$

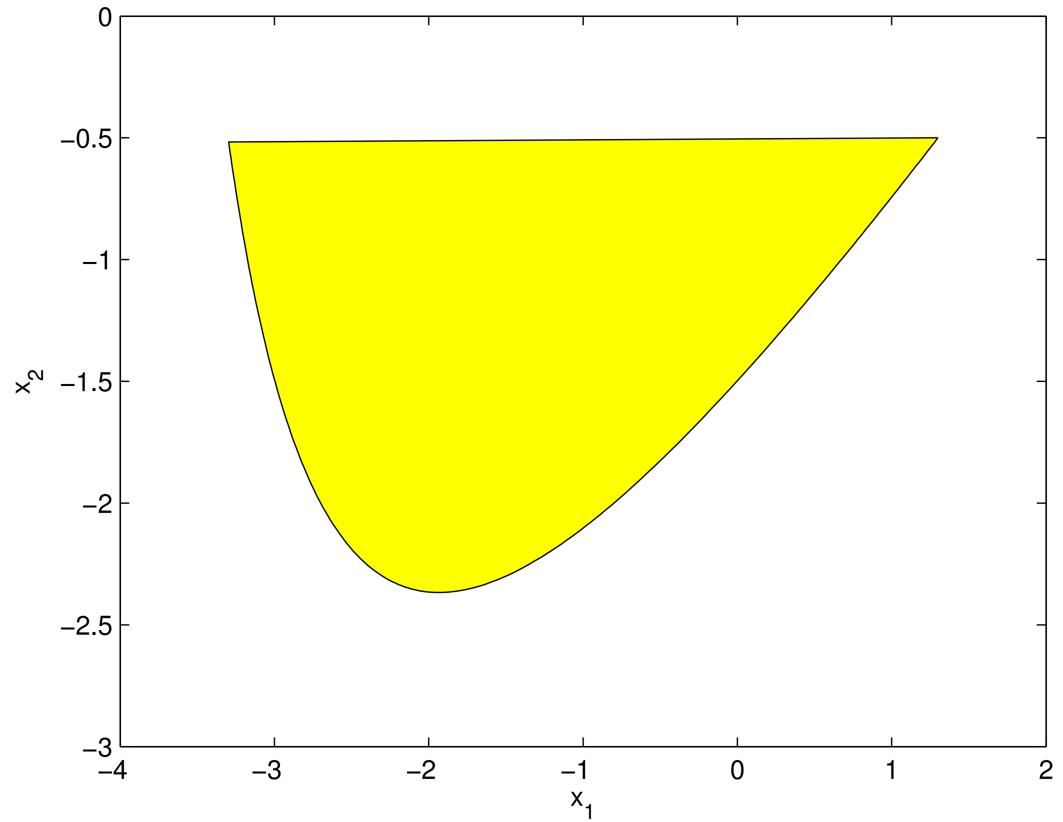
Example

To verify the technique, we consider the example used for illustrating our model-based technique [cdc03]. We collect

$$\mathbf{P}(j\omega) = \{P(j\omega) : \omega \in (-10, 10) \text{ sampled every } 0.01\}.$$

We also have the knowledge that the plant has one RHP pole. We first find the stabilizing region in the controller parameter space. As we did in the previous example, we let $x_3 = 2.5$.

Stabilizing regions for $x_3 = 2.5$



Example continue...

- problem of determining the entire set of first order stabilizing controllers satisfying the required closed-loop performance described by the requirement on the H_∞ norm of the weighted complementary sensitivity function:

$$\|W(j\omega)T(j\omega)\|_\infty < \gamma, \quad \text{for all } \omega$$

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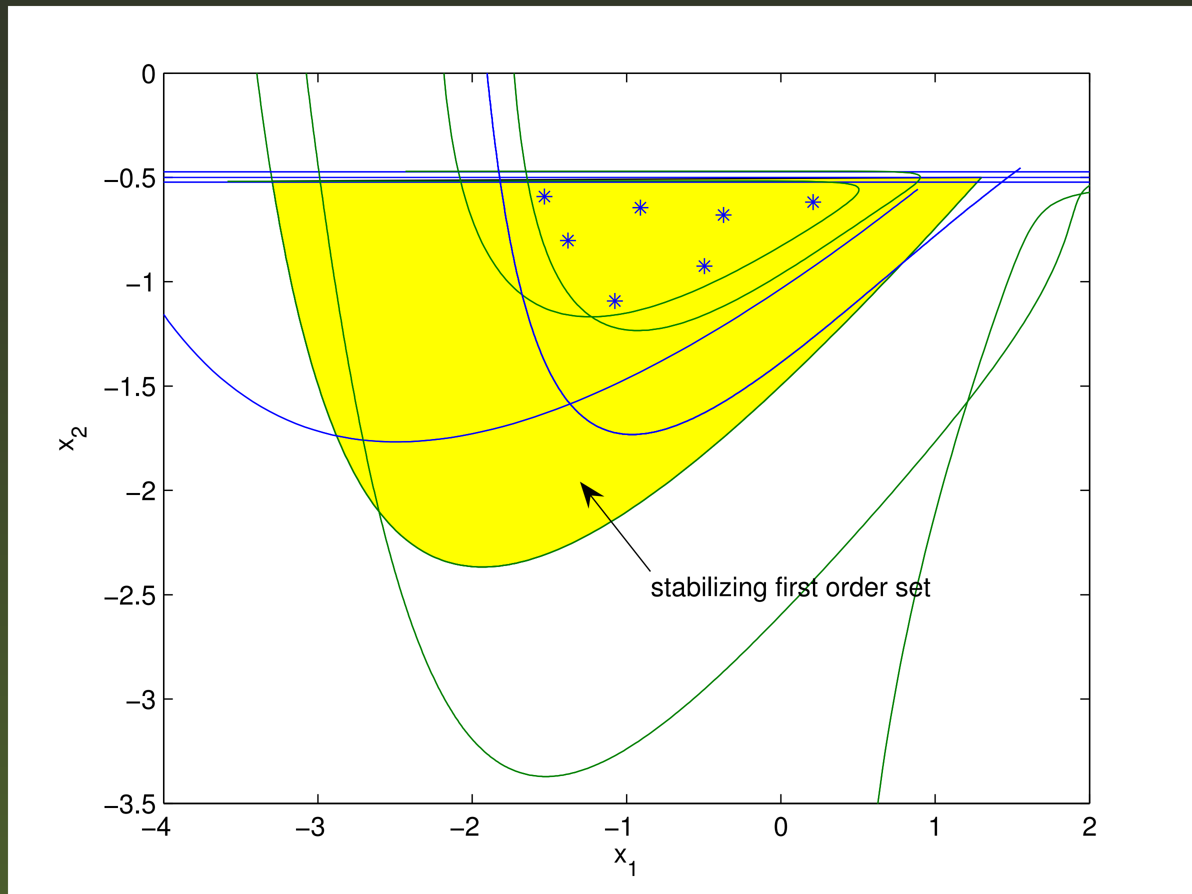
- As shown above, this is equivalent to the problem of simultaneously stabilizing the complex family $P_c(s)$ as well as the original plant $P(s)$.
- In this problem, we let $\gamma = 1$. On the top of the stabilizing region shown stabilizing sets for the complex plant families $\mathbf{P}_c(j\omega, \theta)$ for $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$ are plotted.

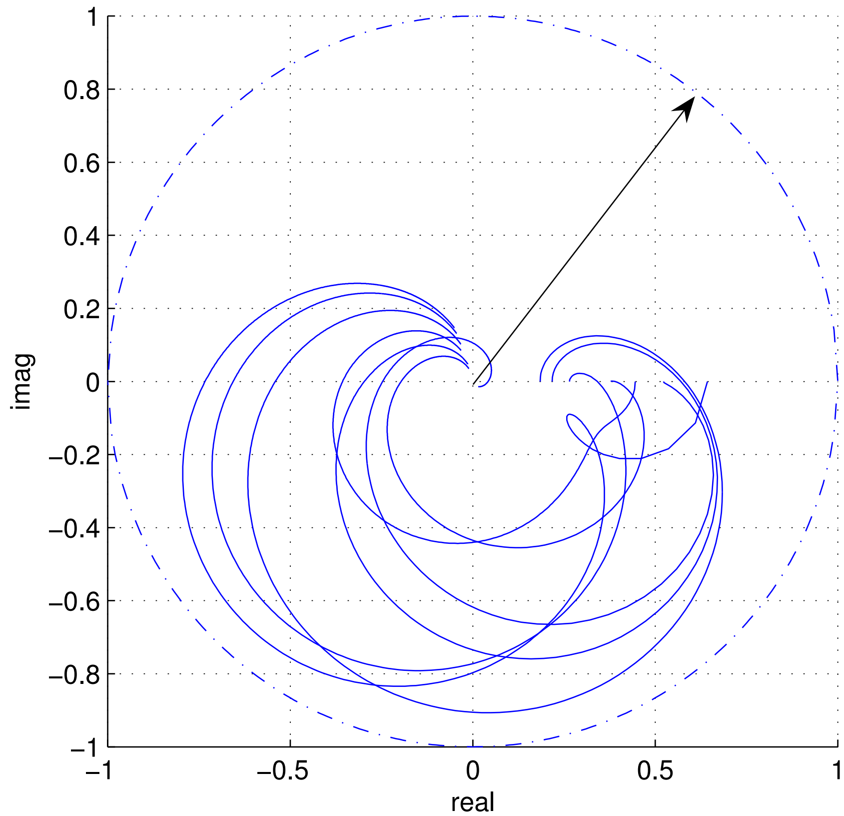
Example continue...

- To verify, a number of points inside the performance region, construct the corresponding controllers, and Nyquist plots of $W(s)T(s)$ have been plotted.

First order controllers satisfying H_∞ performance



Nyquist plots of $W(s)T(s)$ with selected controllers



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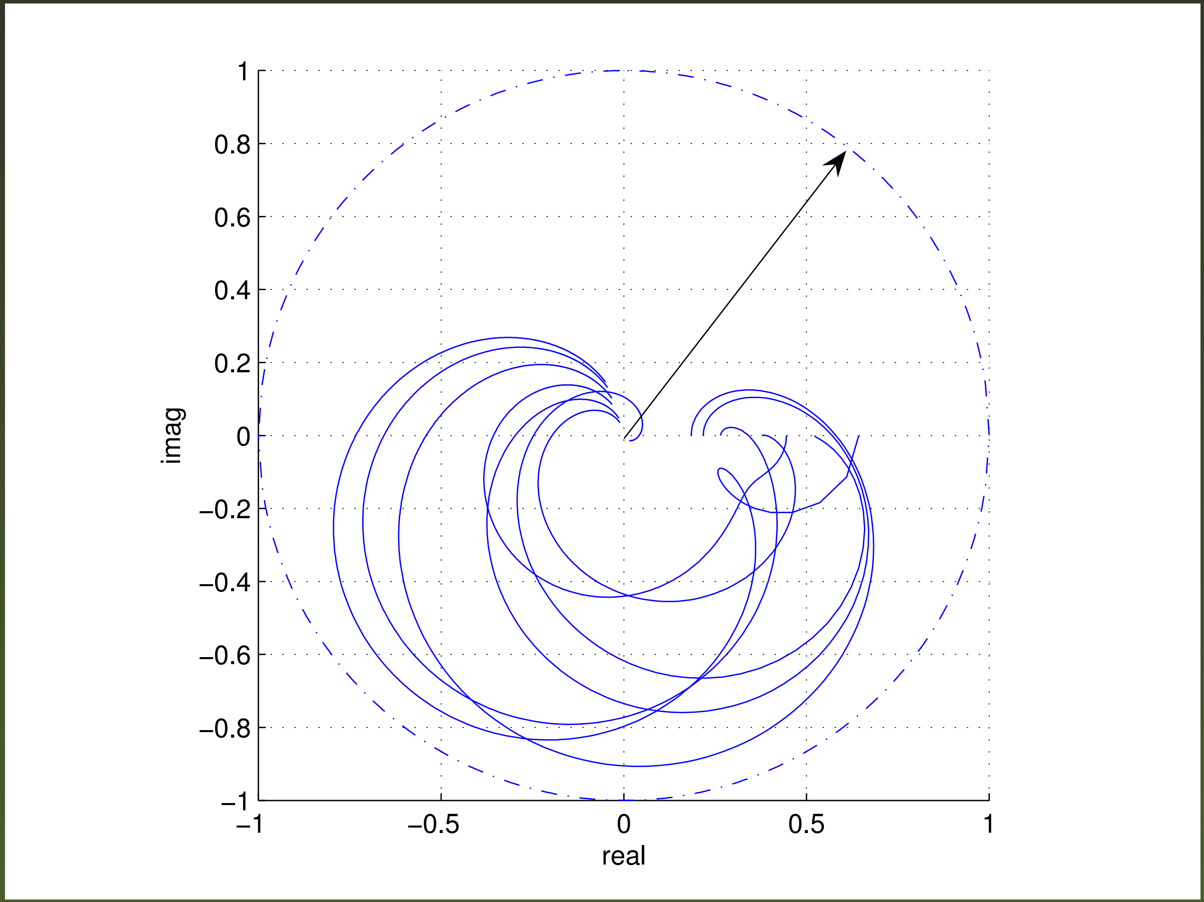

- We observe from the Nyquist plots, every test set satisfies the H_∞ performance requirement.

Concluding Remarks

- We have shown that the complete set of PID and first order stabilizing controllers achieving stability and various meaningful performance specifications can be found from the frequency response of the plant and knowledge of the number of RHP plant poles.

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  - that this calculation can be done by a nested linear programming procedure and
  - that only knowledge of the frequency response and number of RHP poles is sufficient.

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- It is also worth investigating how this procedure can be modified to accommodate incomplete or finite frequency data.
- An important area of research is MIMO PID control and the extension of the results given here to the multi-variable case.

End of Presentation

Thank You