

A Dynamic Programming Approach to the Impulse Control Synthesis Problem

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The Bang-Bang Control

$$\dot{x} = Ax + bu$$

$$|u| \leq \mu$$

Time-optimal control:

$$x(0) = x_0 \quad x(\theta) = 0 \quad \theta \rightarrow \min$$

$$u_0(t) = \mu \cdot \text{sign } h_0(t)$$

The Impulse Control

$$dx = Ax dt + b dU(t)$$

$$\frac{dU}{dt} = u \text{ — generalized derivative}$$

$$\text{Var } U \leq \mu$$

Time-optimal control:

$$x(0) = x_0 \quad x(\theta) = 0 \quad \theta \rightarrow \min$$

$$u_0(t) = \sum_{i=1}^k \alpha_i \delta(t - t_i) \quad \sum_{i=1}^k |\alpha_i| \leq \mu$$

$$t_0 \leq t_1 < t_2 < \dots < t_k \leq \theta$$

The Problem

Problem 1. Minimize

$$J(U(\cdot)) = \text{Var}_{[t_0, t_1]} U(\cdot) + \varphi(x(t_1 + 0)) \rightarrow \inf,$$

over $U(\cdot) \in BV([t_0, t_1]; \mathbb{R}^m)$ subject to

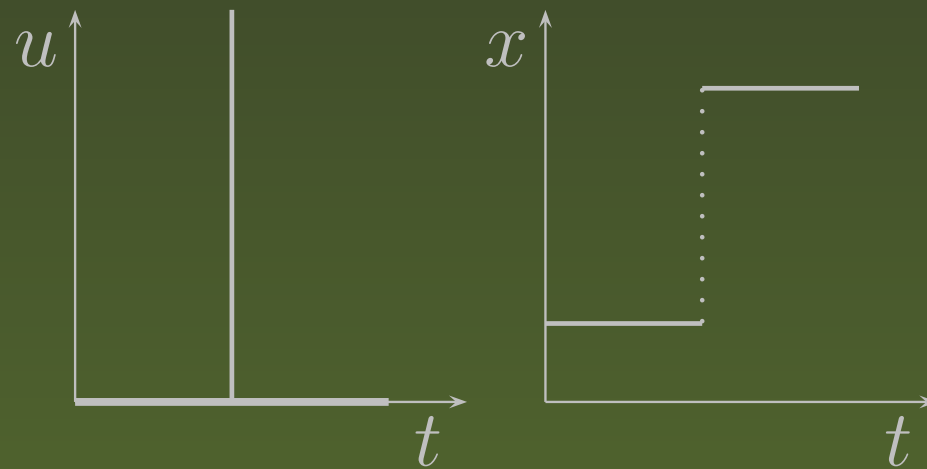
$$dx(t) = A(t)x(t) dt + B(t) dU(t), \quad t \in [t_0, t_1],$$

$$x(t_0 - 0) = x_0.$$

Important particular case:

$\varphi(x) = \delta(x \mid \{x_1\})$: steer from (t_0, x_0) to (t_1, x_1) .

The Ideal Scheme



The Value Function

The minimum of $J(U(\cdot))$ with *fixed* initial position $x(t_0 - 0) = x_0$ is called the *value function*:

$$V(t_0, x_0) = V(t_0, x_0; t_1, \varphi(\cdot)).$$

Calculating the Value Function

Decompose the problem into two parts:

- find optimal $x_1 = x(t_1 + 0)$
- find optimal $U(\cdot)$ to steer from (t_0, x_0) to (t_1, x_1)

$$V(t_0, x_0) = \inf_{x_1 \in \mathbb{R}^n} \left\{ \varphi(x_1) + \sup_{p \in \mathbb{R}^n} \frac{\langle p, x_1 - X(t_1, t_0)x_0 \rangle}{\|B'(\cdot)X'(t_1, \cdot)p\|_{C[t_0, t_1]}} \right\}.$$

$X(t, \tau)$ is the solution to

$$\partial X(t, \tau) / \partial t = A(t)X, \quad X(\tau, \tau) = I.$$

Calculating the Value Function II

$$X(t_1, t_0)x_0 + \int_{t_0}^{t_1} X(t_1, t)B(t) dU(t) = x_1$$

$$\begin{aligned} \langle p, x_1 - X(t_1, t_0)x_0 \rangle &= \int_{t_0}^{t_1} \langle B'(t)X'(t_1, t)p, dU(t) \rangle \leq \\ &\leq \left[\max_{t \in [t_0, t_1]} B'(t)X'(t_1, t)p \right] \cdot \text{Var}_{[t_0, t_1]} U(\cdot) \end{aligned}$$

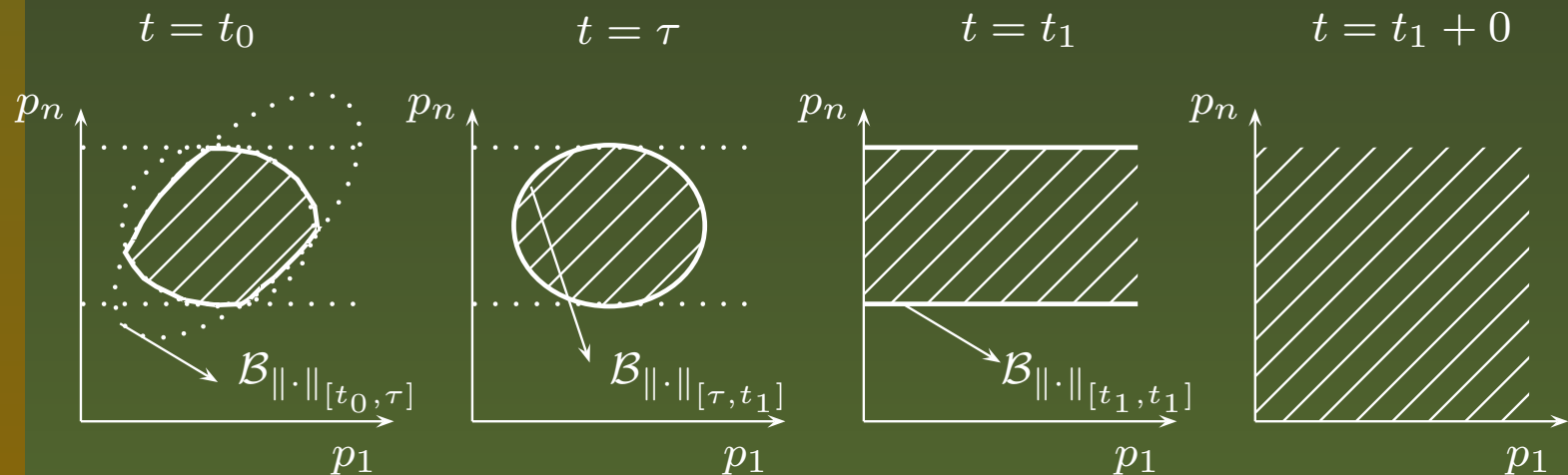
$$\inf_{[t_0, t_1]} \text{Var} U(\cdot) = \sup_{p \in \mathbb{R}^n} \frac{\langle p, x_1 - X(t_1, t_0)x_0 \rangle}{\|B'(\cdot)X'(t_1, \cdot)p\|_{C[t_0, t_1]}}$$

The Conjugate of the Value Function

The value function is convex and its conjugate equals

$$V^*(t_0, p) = \varphi^*(X'(t_0, t_1)p) + \delta\left(X'(t_0, t_1)p \mid \mathcal{B}_{\|\cdot\|_{[t_0, t_1]}}\right)$$

where $\|p\|_{[t_0, t_1]} = \|B'(t)X'(t_1, \cdot)p\|_{C[t_0, t_1]}$.



Calculating the Conjugate

$$V(t_0, x_0) = \inf_{x_1 \in \mathbb{R}^n} \left\{ \varphi(x_1) + \sup_{p \in \mathbb{R}^n} \frac{\langle p, x_1 - X(t_1, t_0)x_0 \rangle}{\|B'(\cdot)X'(t_1, \cdot)p\|_{C[t_0, t_1]}} \right\}.$$

$$V(t_0, x_0) = \inf_{x_1 \in \mathbb{R}^n} \left\{ \varphi(x_1) + \sup_{p \in \mathcal{B}_{\|\cdot\|_{[t_0, t_1]}}} \langle p, x_1 - X(t_1, t_0)x_0 \rangle \right\}.$$

$$V(t_0, x_0) = \sup_{p \in \mathcal{B}_{\|\cdot\|_{[t_0, t_1]}}} \inf_{x_1 \in \mathbb{R}^n} \{ \varphi(x_1) + \langle p, x_1 \rangle - \langle p, X(t_1, t_0)x_0 \rangle \}.$$

$$V(t_0, x_0) = \sup_{p \in \mathcal{B}_{\|\cdot\|_{[t_0, t_1]}}} \{ -\varphi^*(-p) - \langle p, X(t_1, t_0)x_0 \rangle \}.$$

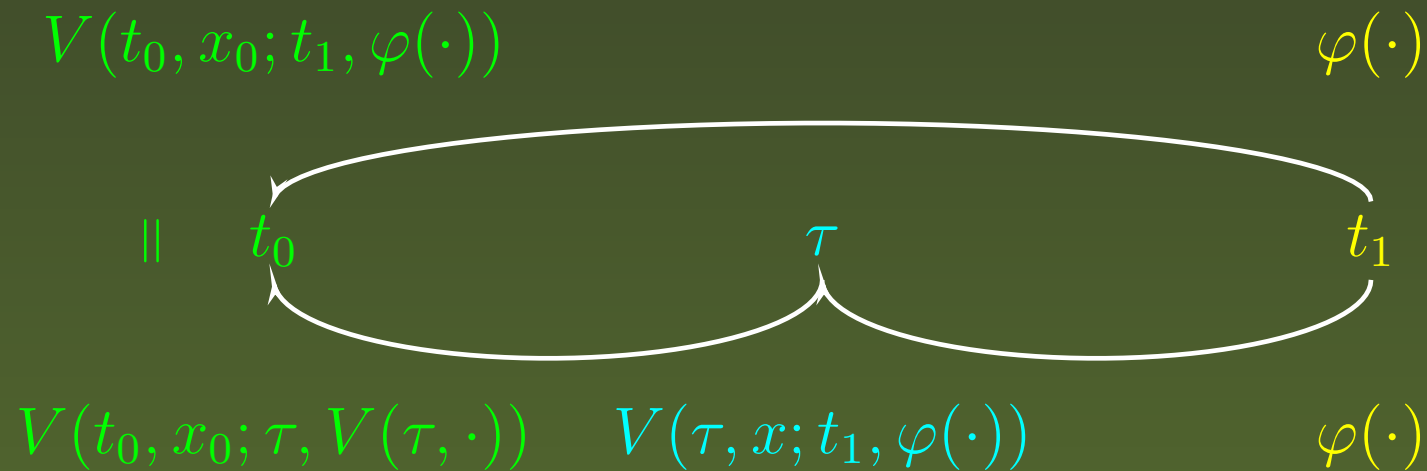
$$V(t_0, x_0) = \sup_{p \in \mathbb{R}^n} \left\{ \varphi^*(p) - \langle p, X(t_1, t_0)x_0 \rangle - \delta\left(p \mid \mathcal{B}_{\|\cdot\|_{[t_0, t_1]}}\right) \right\}.$$

The Principle of Optimality

The value function $V(t, x; t_1, \varphi(\cdot))$ satisfies the *Principle of Optimality*

$$V(t_0, x_0; t_1, \varphi(\cdot)) = V(t_0, x_0; \tau, V(\tau, \cdot; t_1, \varphi(\cdot))),$$

where $\tau \in [t_0, t_1]$.



The Principle of Optimality II

In the general case $V(t_1, x; t_1, \varphi(\cdot)) \leq \varphi(x)$, since

$$V^*(t_1, p) = \varphi^*(p) + \delta(B(t_1)p \mid \mathcal{B}_1).$$

For example, if $\varphi(x) = \delta(x \mid \{x_1\})$ and $B(t_1) = I$, then

$$V(t_1, x; t_1, \varphi(\cdot)) = \|x - x_1\|.$$

However, due to the Principle of Optimality,

$$V(t_1, x; t_1, \varphi(\cdot)) = V(t_1, x; t_1, V(t_1, \cdot; t_1, \varphi(\cdot))).$$

Dynamic Programming Equation

The value function is the viscosity solution to the Hamilton–Jacobi–Bellman equation:

$$\min \{ H_1(t, x, V_t, V_x), H_2(t, x, V_t, V_x) \} = 0,$$

$$V(t_1, x) = V(t_1, x; t_1, \varphi(\cdot)).$$

$$H_1(t, x, \xi_t, \xi_x) = \xi_t + \langle \xi_x, A(t)x \rangle,$$

$$H_2(t, x, \xi_t, \xi_x) = \min_{u \in S_1} \langle \xi_x, B(t)u \rangle + 1 = -\|B^T(t)\xi_x\| + 1.$$

The Control Structure

$$H_1(t, x) = 0 \longleftarrow (t, x) \longrightarrow H_2(t, x) = 0$$

$$\begin{array}{c} \downarrow \\ \text{wait} \\ dU(t) = 0 \end{array}$$

choose jump direction

$$d = -B'(t)V_x$$

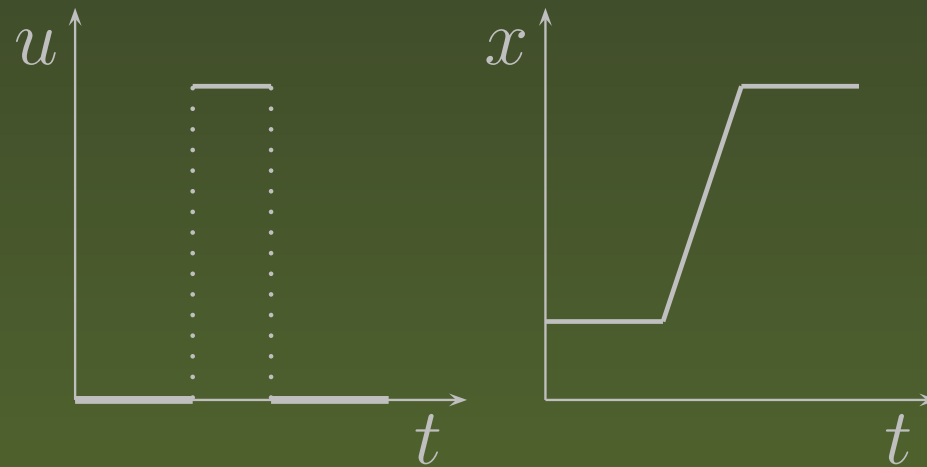
choose jump amplitude

$$\min \alpha \geq 0 : H_1(t, x + \alpha d) = 0$$

jump

$$U(\tau) = \alpha \cdot d \cdot \chi(\tau - t)$$

The Realistic Scheme



The Additional Constraint Approach

Introduce a hard bound on the control, $u(t) \in \mathcal{B}_\mu$, and consider the corresponding problem:

Problem 2. Minimize

$$J_\mu(u(\cdot)) = \int_{t_0}^{t_1} \|u(t)\| dt + \varphi(x(t_1)) \rightarrow \inf,$$

over $u(\cdot) \in L_1([t_0, t_1]; \mathbb{R}^m)$ subject to

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0,$$

$$\|u(t)\| \leq \mu.$$

The Value Function

The minimum of $J_\mu(U(\cdot))$ with *fixed* initial position $x(t_0) = x_0$ is called the *value function*:

$$V_\mu(t_0, x_0) = V_\mu(t_0, x_0; t_1, \varphi(\cdot)).$$

The value function of the Problem 2 is the viscosity solution to the Hamilton–Jacobi–Bellman equation

$$\frac{\partial V_\mu}{\partial t} + \min_{u \in \mu \mathcal{B}_1} \left\{ \left\langle \frac{\partial V_\mu}{\partial x}, A(t)x(t) + B(t)u \right\rangle + \|u\| \right\} = 0,$$

$$V_\mu(t_1, x) = \varphi(x).$$

Calculating the Value Function

The value function of the Problem 2 is

$$V_{\mu}(t, x) = \sup_{p \in \mathbb{R}^n} \left\{ \langle p, X(t_1, t_0)x_0 \rangle - \mu \int_t^{t_1} (\|B'(\tau)X'(t_1, \tau)p\| - 1)_+ d\tau - \varphi^*(p) \right\},$$

and its conjugate in x is

$$V_{\mu}^*(t, p) = \varphi^*(p) + \mu \int_t^{t_1} (\|B'(\tau)X'(t_1, \tau)p\| - 1)_+ d\tau.$$

The Convergence of V_μ

As $\mu \rightarrow \infty$,

$$V_\mu(t, x) \rightarrow V(t, x), \quad V_\mu^*(t, x) \rightarrow V^*(t, x).$$

Under certain assumptions

$$0 \leq V_\mu(t, x) - V(t, x) = O(\mu^{-1}).$$

The HJB equation for Problem 1 may also be derived in the limit from the one for Problem 2, as $\mu \rightarrow \infty$.

The Control Synthesis

The optimal feedback control strategy for Problem 2 is the minimizer in the HJB equation:

$$\mathcal{U}_\mu^*(t, x) = \begin{cases} \{0\}, & \|\zeta\| < 1; \\ [0, -\mu\zeta], & \|\zeta\| = 1; \\ \left\{ -\mu \frac{\zeta}{\|\zeta\|} \right\}, & \|\zeta\| > 1, \end{cases} \quad \zeta = B'(t) \frac{\partial V_\mu}{\partial x}$$

(in points where $V_\mu(t, x)$ is differentiable in x).

Example: The Problem

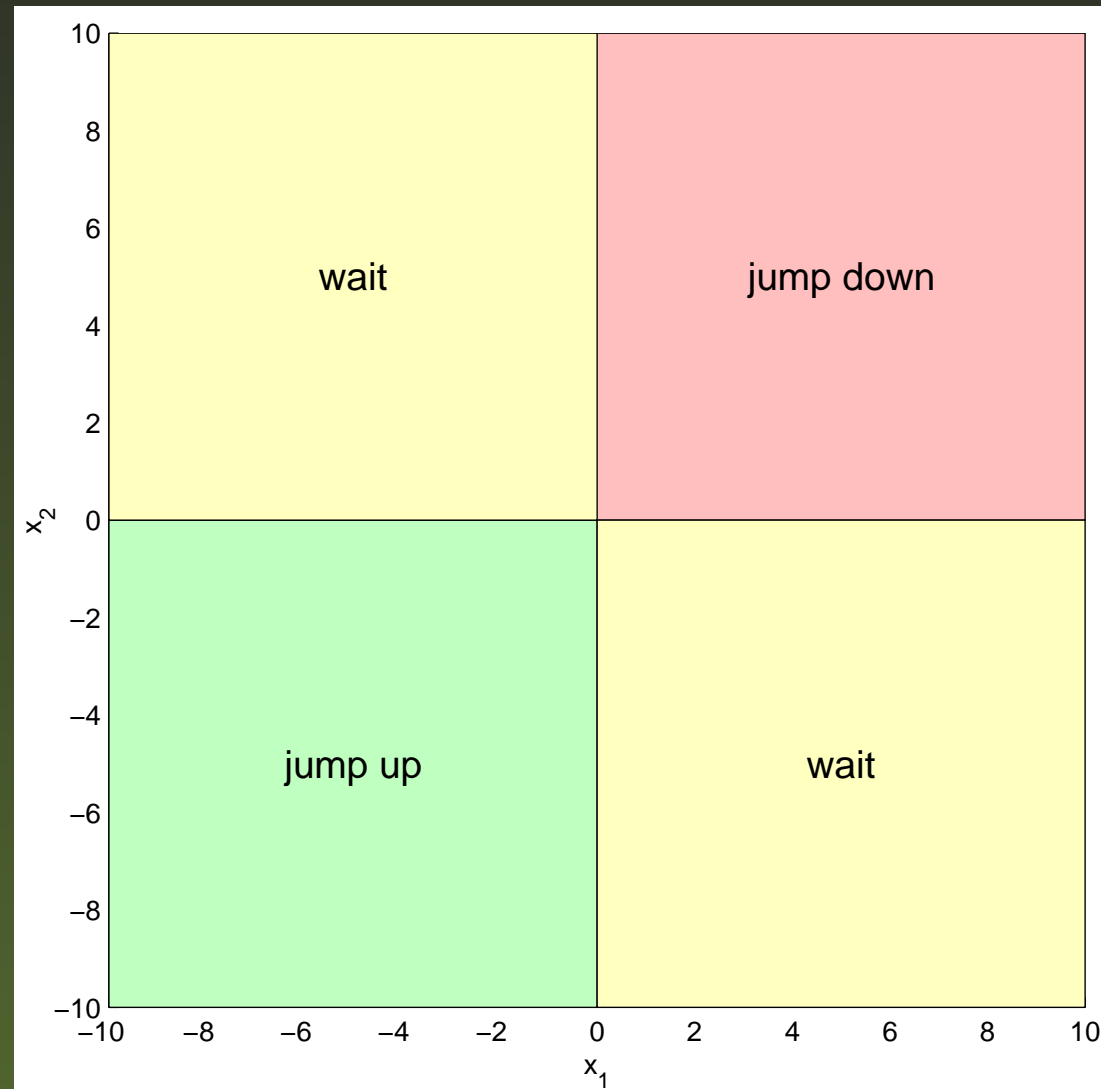
$$\begin{cases} dx_1(t) = x_2(t) dt, \\ dx_2(t) = -x_1(t) dt + dU(t), \end{cases} \quad 0 \leq t \leq \frac{\pi}{2},$$

$$\text{Var } U(\cdot) \rightarrow \inf, \\ [0, \frac{\pi}{2}]$$

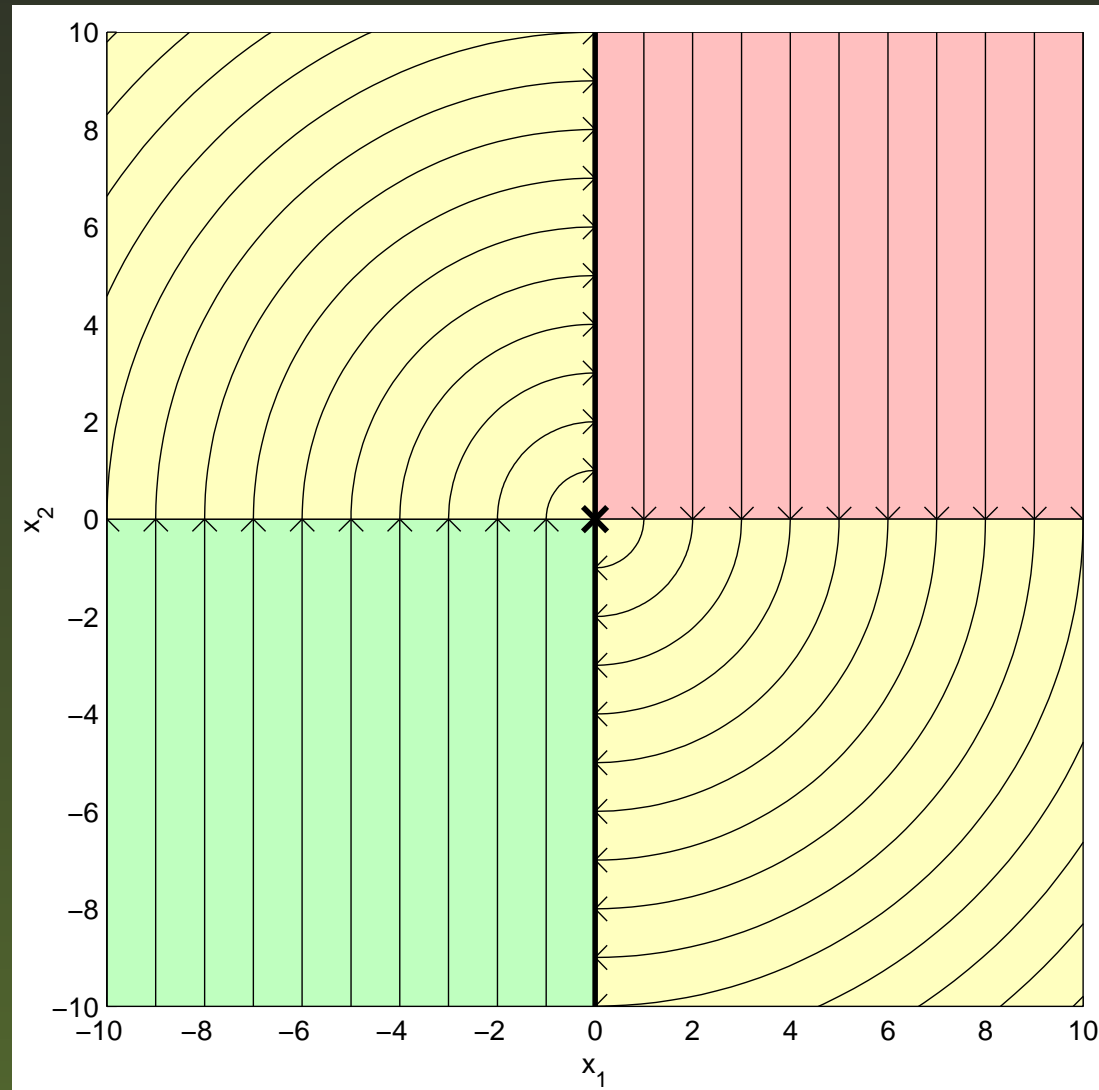
$$x_1(0-0) = x_1^0, \quad x_2(0-0) = x_2^0, \quad x_1(\frac{\pi}{2}+0) = 0, \quad x_2(\frac{\pi}{2}+0) = 0.$$



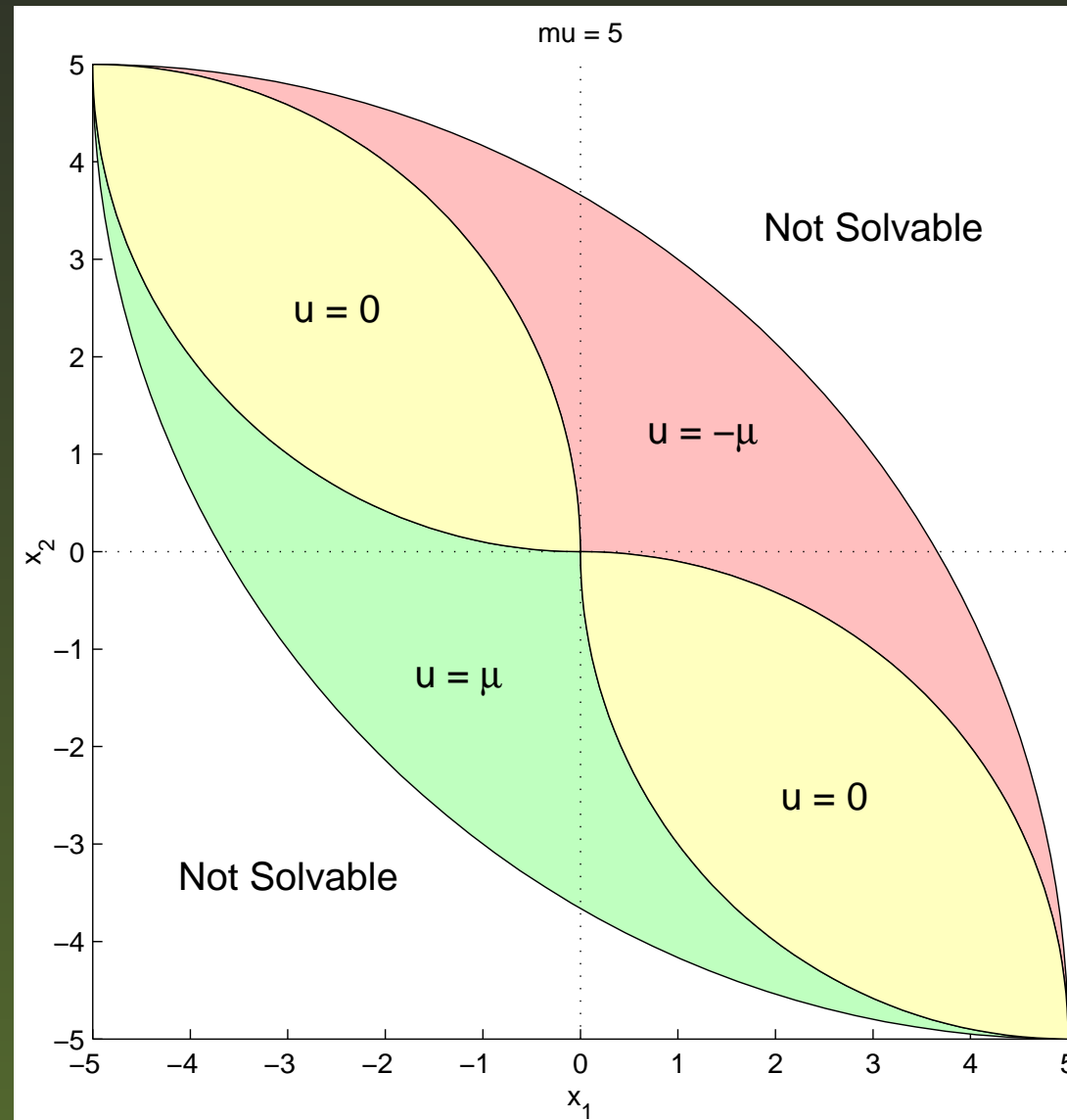
Example: The Ideal Control



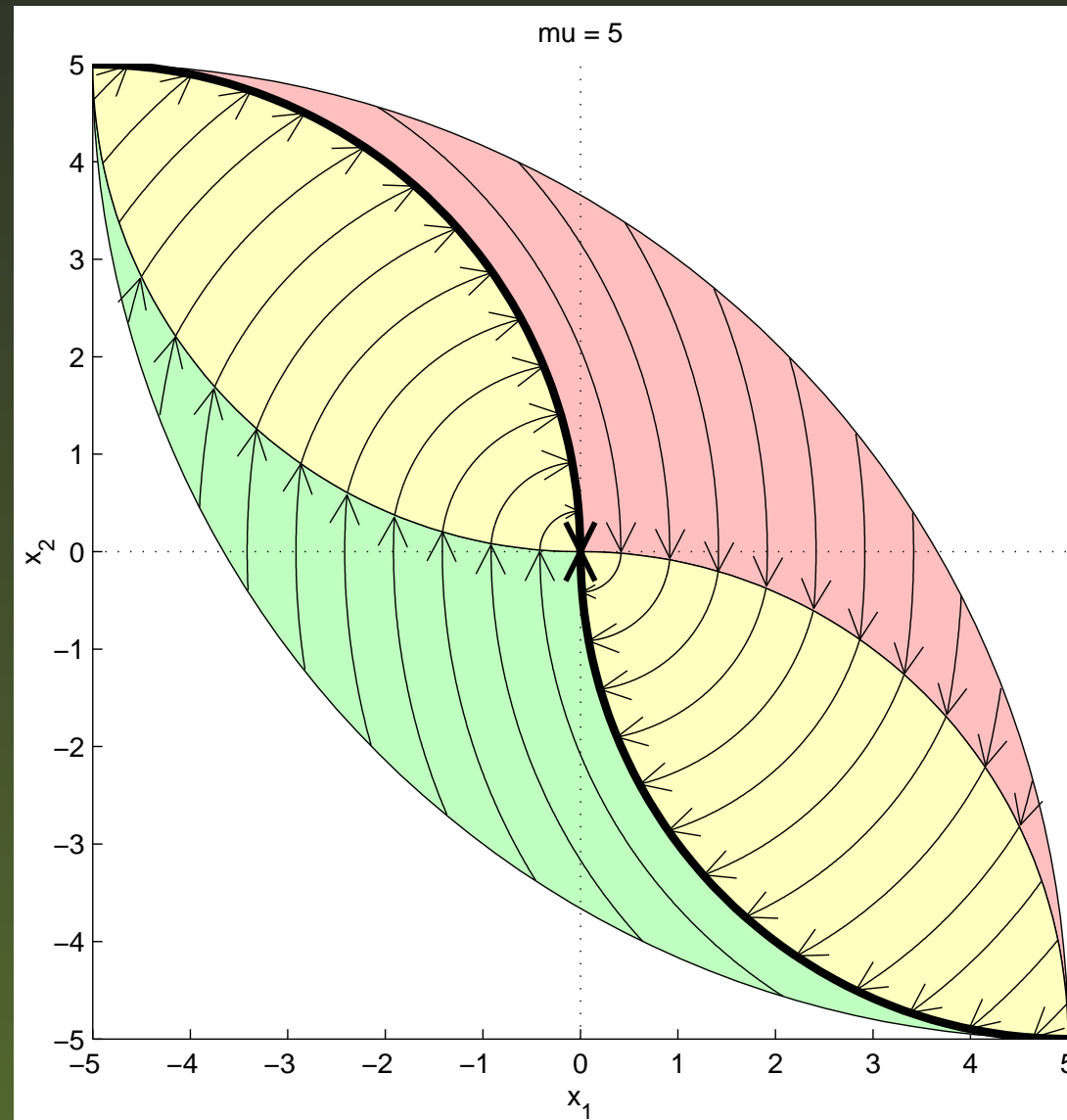
Example: The Ideal Trajectories



Example: The Realistic Control



Example: The Realistic Trajectories



Example: The Convergence

