Star-shaped separability with applications

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The most challenging problem in modern optimization theory is a theory of global optimization. We need verifiable necessary and/or sufficient conditions for broad enough classes of optimization problems. The theory of local optimization is based on calculus and its very sophisticated generalizations. Different tools should be used in a theory of global optimization.

For some classes of problems the approach based on separability can be used.

Separability of two convex sets is one of the fundamental facts of convex analysis that can be considered as a geometrical form of Hanh-Banach theorem. Some attempts to extended the notion of separability for star-shaped sets were undertaken by A. Shveidel and later on by Rubinov.

Let $U \subset \mathbb{R}^n$. The set

$$\operatorname{kern} U = \{u \in U : u + \lambda(x - u) \in U$$

$$\text{for all } x \in U \quad \text{and} \quad \lambda \in [0,1]\}$$

is called the kernel of U.

A set U is called star-shaped if $\operatorname{kern} U \neq \emptyset$.

Theorem

$x \in \operatorname{int} \operatorname{kern} U \iff$

there is family $(U_t)_{t \in T}$ of convex sets and $\varepsilon > 0$ such that $U_t \supset B(x, \varepsilon)$ for all $t \in T$ and

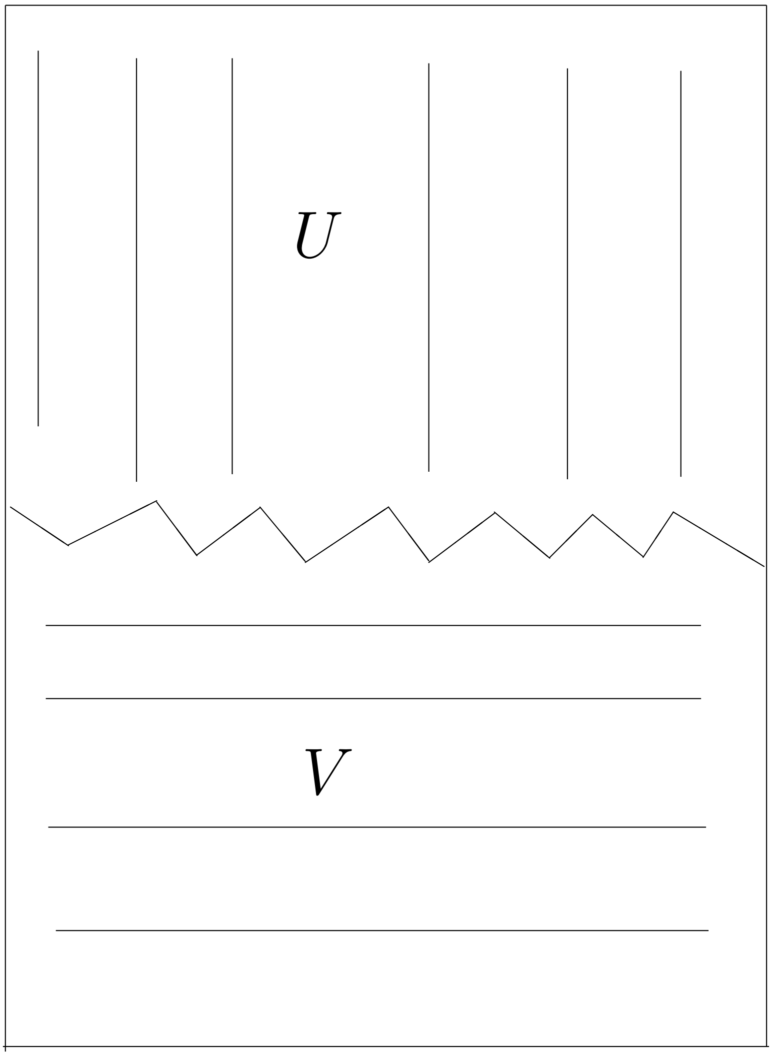
$$U = \bigcup_{t \in T} U_t.$$

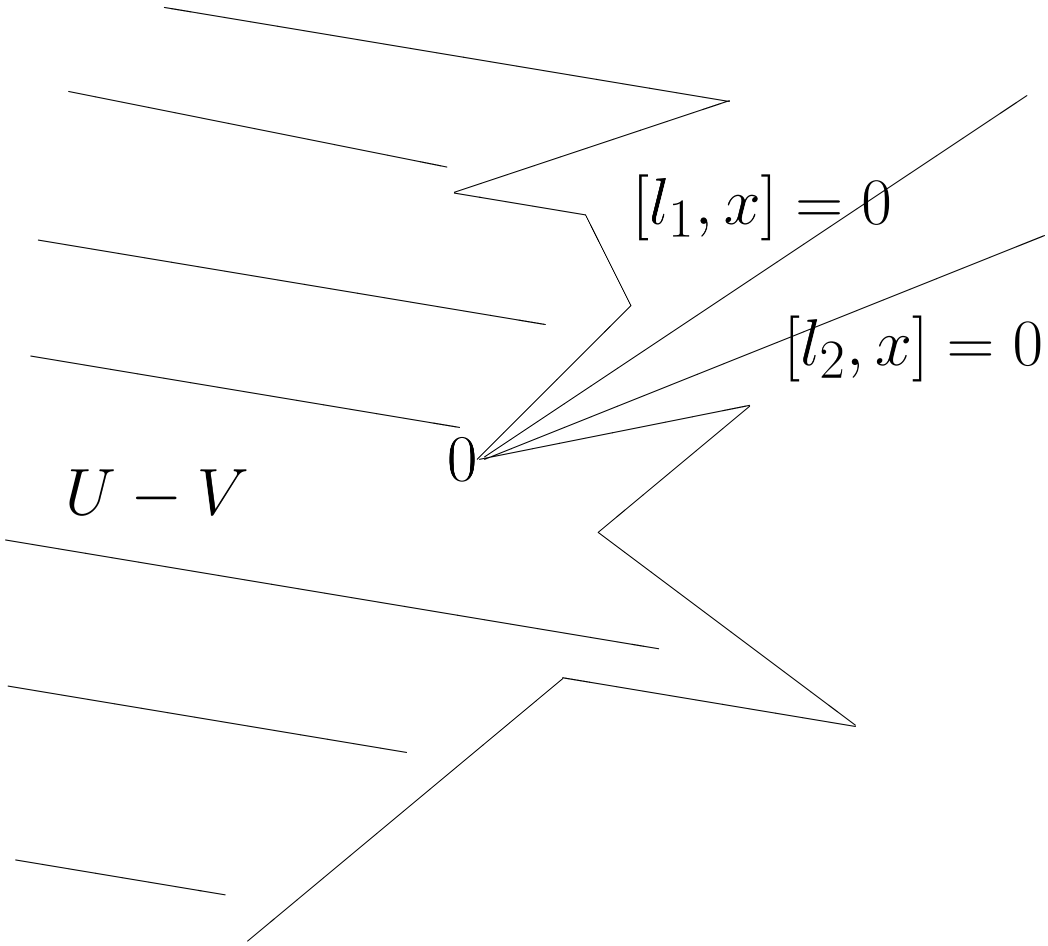
Weak separability of star-shaped sets

Let U and V be subsets of $\mathbb{R}^n$ and $\ell = (l_i)_{i=1,\dots,m}$ be a collection of linearly independent vectors. The sets U, V are said to be weakly separated by vectors $(l_i)_{i=1,\dots,m}$ if for each pair $u \in U, v \in V$ there exists $i \in I$ such that $[l_i, u] \leq [l_i, v]$.

Theorem (Shveidel (1997), Rubinov (2000)) Let U and V be star-shaped sets such that $\operatorname{int}\operatorname{kern} U \neq \emptyset$ and $(\operatorname{int} U) \cap V = \emptyset$. Then U and V are weakly separated.

Geometry:





$[l_1, x] = 0$
$[l_2, x] = 0$
0
$U - V$

The following well-known corollary of Hahn-Banach theorem is a classical result of the approximation theory. Let U be a convex subset of a normed space X and $x \notin U$ and let $\bar{u} \in U$ be best approximation of x by elements of U:

$r := \min\{\|u - x\| : u \in U\} = \|\bar{u} - x\|.$

Then there exists a linear function l such that $l(u) \leq l(\bar{u}) \leq l(v)$ for all $u \in U$ and $v \in B(x, r) = \{y : \|x - y\| \leq r\}$.

Reformulation: $\bar{u}$ is best approximation of x by $U \iff \exists l$:

$0 = (-l, l)(\bar{u}, \bar{u}) = \min\{(-l, l)(u, v) : (u, v) \in U \times B(x, r)\}$.

If U is strictly convex then also

$((u, v) \in U \times V,\ (u, v) \neq (\bar{u}, \bar{u})) \implies (-l, l)(u, v) > 0$.

Consider a generalization of this result on a star-shaped sets.

Let $\|\cdot\|$ be a norm in $\mathbb{R}^n$.

Let $U \subset \mathbb{R}^n$ be a star-shaped set and $x \notin U$, let $r = \min\{\|u - x\| : u \in U\}$. Then the intersection $U \cap \{v : \|x - v\| < r\}$ is empty, so the sets U and $\{v : \|x - v\| \leq r\}$ can be weakly separated. We do not need to have exactly a norm in order to prove this result, so we consider a more general situation.

Let $\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be a function such that $\forall\ x, y \in \mathbb{R}^n, \alpha \in [0,1]$:

$\rho(x, y + \alpha(x - y)) \leq \rho(x, y)$.

ρ enjoys this property $\iff$ $\forall r > 0$ the "balls" $B(x,r) = \{y : \rho(x,y) \leq r\}$ are star-shaped with $x \in \operatorname{kern} B(x,r)$.

We need to have star-shaped balls $B(x,r)$ such that

(1) $x \in \operatorname{int} \operatorname{kern} B(x,r)$ for all $r > 0$.

(2) the inequality $\rho(x,y) < r$ holds for interior points of $B(x,r)$.

Definition A function $\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ is called a star-shaped distance if

(i) $\rho(x, x) = 0 \, \forall \, x$ and $\rho(x, y) > 0 \ \forall x \neq y$;

(ii) $\forall \, x \in \mathbb{R}^n$, $r > 0 \, \exists V(x)$:

$\rho(x, y + \alpha(x' - y)) \leq \alpha r + (1 - \alpha)\rho(x, y)$, $y \in \mathbb{R}^n, x' \in V(x), \alpha \in [0, 1]$.

Let $\rho_x(y) = \rho(x, y), \qquad y \in \mathbb{R}^n$.

(iii)ρ_x has no local maxima $\forall \, x \in \mathbb{R}^n$.

(iv) ρ_x is continuous $\forall \, x \in \mathbb{R}^n$.

Proposition Let ρ be a star-shaped distance. Then $B(x,r)$ is star-shaped and $\operatorname{int} B(x,r) = \{v \in \mathbb{R}^n : \rho(x,v) < r\} \,\forall\, x \in \mathbb{R}^n,\ r > 0$.

We now give an example of a star-shaped distance.

Let $(f_t)_{t\in T}$ be a uniformly continuous family of convex functions $f_t : \mathbb{R}^n \to \mathbb{R}_+$ such that $f_t(0) = 0$ and $\inf_{t\in T} f_t(x) > 0$ for $x \neq 0$. Then function $\rho(x,y) = \inf_{t\in T} f_t(x-y)$ is a star-shaped distance in $\mathbb{R}^n$.

Theorem Let ρ be a star-shaped distance on $\mathbb{R}^n$ and $U \subset \mathbb{R}^n$ be a radiant set. Let $x \notin U$, $\bar{u} \in U$ and $r = \rho(x, \bar{u})$. Then

1) If $r = \min_{u \in U} \rho(x, u)$ then there exists m linearly independent vectors $l_1, \ldots, l_m$ such that:

(i) $[l_1, x] = \ldots = [l_m, x] = 1$.

(ii) for each $u \in U$ and $v \in B(x, r)$ with $u \neq v$ there exists an index i such that $[l_i, u] < [l_i, v]$.

2) If there exist m linearly independent vectors l_i such that the condition (ii') below holds then $r := \rho(x, \bar{u}) = \min_{u \in U} \rho(x, u)$. Here

(ii') $U \times B(x, r) = \bigcup_{i=1}^{m} (U \times B(x, r))_i$ where $(U \times B(x, r))_i = \{(u, v) \in U \times B(x, r) : [l_i, u] \leq [l_i, v]\}$.

(Condition (ii') means that for every pair (u, v) with $u \in U$ and $v \in B(x, r)$ there exists i such that $[l_i, u] \leq [l_i, v]$.)

If U is a strictly convex set and $\rho(x, y) = \|x - y\|$ then the classical result follows from Theorem with $m = 1$. A version for arbitrary convex sets can be easily given.

Further research

How to find $l_1, \ldots, l_m$? There are some ideas how to do it for $\rho(x, y) = \inf_{t \in T} f_t(x - y)$.

This approach can be extended for a broad class of global optimization problems with the objective $f(x) = \min_t f_t(x)$, where f_t is a special family of convex functions.