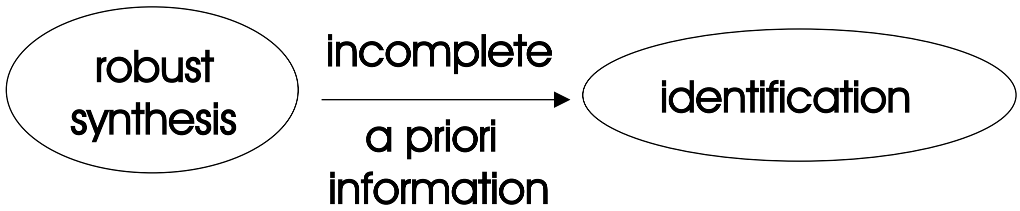
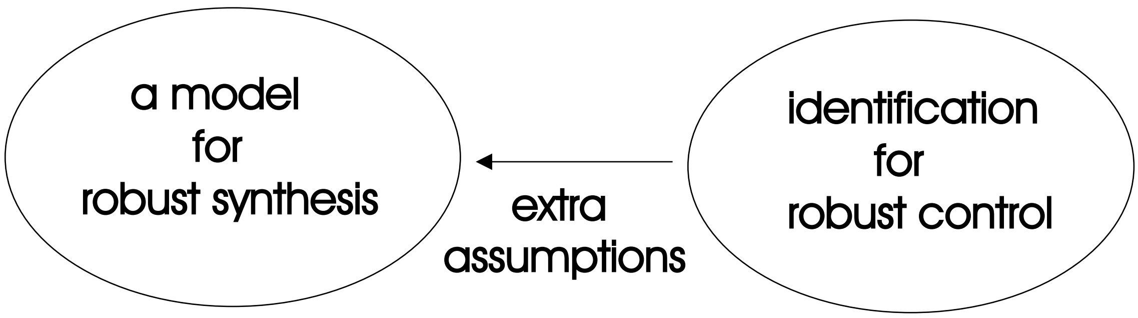
Victor Sokolov
Department of Mathematics, Komi Science Center

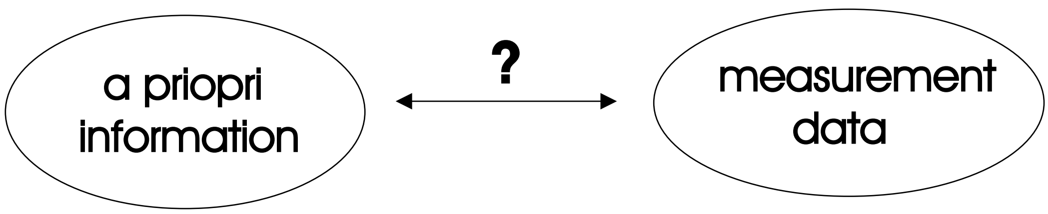
SPECIAL LINEAR AND QUADRATIC FRACTIONAL PROGRAMMING PROBLEMS ARISING IN OPTIMAL ROBUST SYNTHESIS IN ℓ_1



Dominating approach :



model validation



Hierarchy of problems depending on a priori information

Problems	Nominal model	Uncertainty and disturbance description
robust synthesis	given	complete
errors quantification (and subsequent robust synthesis)	given	incomplete (unknown errors bounds)
classical identification of nominal model	unknown	largely irrespective
identification for robust control	unknown	incomplete

Identification criterion :

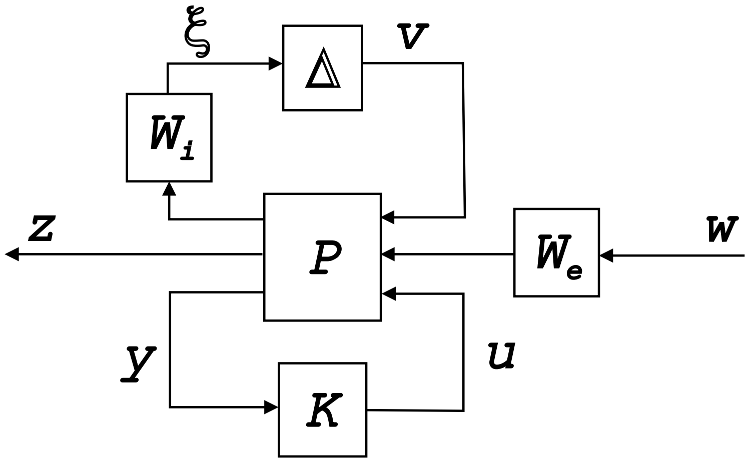
classical - in terms of errors bounds

the most coherent - the control criterion

(Kurzhansky A.B., 1970s: control-oriented state estimation and control on finite time interval)

LFT model

Robust synthesis in the ℓ_1 setup



Exogenous disturbance : $w \in \ell_\infty$

$$|w(k)|_\infty := \max_i \ |w_i(k)| \ , \qquad \|w\|_\infty := \sup_k \ |w(k)|_\infty \ \leq 1$$

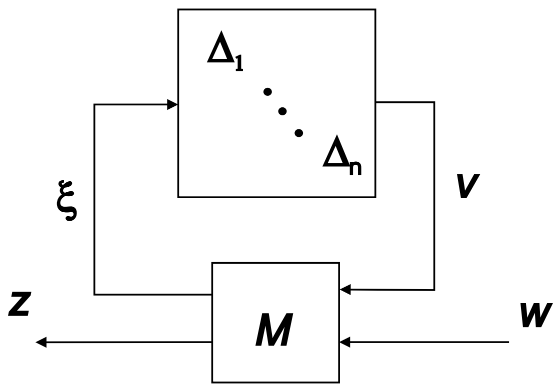
Structured uncertainty :

$$\Delta = \begin{pmatrix} \Delta_1 & & 0 \\ & \ddots & \\ 0 & & \Delta_n \end{pmatrix} : \ell_\infty^p \to \ell_\infty^q$$

$$\|\Delta\| := \sup_{\substack{x \in \ell_\infty^p \\ x \neq 0}} \frac{\|\Delta(x)\|_\infty}{\|x\|_\infty} < 1$$

$P, K, W_e, W_i -$ **LTI systems**

Basic results



$$M = M_1 - M_2 Q M_3\ , \quad Q - \textbf{the Youla parameter}$$

$$J(M) := \sup_{\Delta} \ \sup_{w} \ \|z\|_\infty$$

$$\mathbb{M} = \begin{bmatrix} \|M_{\xi_1 v_1}\|_1 & \cdots & \|M_{\xi_1 v_n}\|_1 & \|M_{\xi_1 w}\|_1 \\ \vdots & & \vdots & \vdots \\ \|M_{\xi_n v_1}\|_1 & \cdots & \|M_{\xi_n v_n}\|_1 & \|M_{\xi_n w}\|_1 \\ \|M_{z v_1}\|_1 & \cdots & \|M_{z v_n}\|_1 & \|M_{z w}\|_1 \end{bmatrix}$$

J.B. Pearson and M. Khammash (IEEE TAC, 1991; S&CL, 1993) :

$$J(\mathbb{M}) \leq 1 \quad \Leftrightarrow \quad \rho(\mathbb{M}) \leq 1 \quad \Leftrightarrow \quad \inf_{D \in \mathbb{D}} \ \inf_{Q} \ \|D^{-1} M(Q) D\|_1\ ,$$

$$\mathbb{D} := \{ \ D \ | \ D = diag(d_1, \cdots, d_{n+1}), \ d_i > 0 \ \}\ .$$

M.H. Khammash, M.V. Salapaka, and T. Vanvoorhis (IEEE TAC, 2001) : relaxations of the nonconvex problem + branch and bound scheme

$$J(\mathbb{M}) \leq \gamma \quad \longrightarrow \quad \left(\frac{C}{\varepsilon} \right)^{n+1}$$

linear programming problems, ε - accuracy

Representation of the control criterion:

$$\mathbb{M} = \begin{bmatrix} \|M_{\xi_1 v_1}\|_1 & \cdots & \|M_{\xi_1 v_n}\|_1 & \|M_{\xi_1 w}\|_1 \\ \vdots & & \vdots & \vdots \\ \|M_{\xi_n v_1}\|_1 & \cdots & \|M_{\xi_n v_n}\|_1 & \|M_{\xi_n w}\|_1 \\ \|M_{z v_1}\|_1 & \cdots & \|M_{z v_n}\|_1 & \|M_{z w}\|_1 \end{bmatrix} =$$

$$\begin{bmatrix} \mathbb{M}_{\xi v} & \mathbb{M}_{\xi w} \\ \mathbb{M}_{zv} & \mathbb{M}_{zw} \end{bmatrix}, \quad \mathbb{M}_{\xi v} - n \times n$$

M.Khammash, IEEE TAC, 1997:

$$J(M) = \mathbb{M}_{zw} + \mathbb{M}_{zv}(I - \mathbb{M}_{\xi v})^{-1}\mathbb{M}_{\xi w}$$

- polynomially fractional (scalar signals).
This representation provides significantly more insight into the way in which various quantities contribute to the worst case performance measure.

The simplest case of the LFT model structure :
unstructured uncertainty
($n = 1$ - one uncertainty block)

$$\Downarrow$$

$J(M)$ **- quadratic fractional**

A priori information :

P **- nominal plant**
W_e **- weight of exogenous disturbance**
W_i **- weight of inner uncertainty**

Model validation :

$$y(0), \cdots, y(N), u(0), \cdots, u(N) - \textbf{data}, \quad z = y$$

R. Smith (ACC'92): compatibility of linear inequalities;

$$(2\,\mathbf{dim}\,\xi)^N N!$$

linear systems of the order $(2\,\mathbf{dim}\,v + 1)N$**.**

$$M = M(P, K, W_e, W_i)$$

$$\mathcal{M}_N := \textbf{ the set of non-falsified models}$$

Simplest case : $\quad W_e = \delta_e W_e^0 \;, \quad W_i = \delta_i W_i^0 \quad \Rightarrow$ **quadratic inequalities (linear in the case** $\xi = (y, u)'$ **)**

Optimal errors quantification :

Estimation criterion for scaling factors δ_e **and** δ_i

$$\min_{\{\; W_e, W_i \;|\; M \in \mathcal{M}_N\}} \quad J(M)$$

$$G(Q) := \begin{bmatrix} G_{11}(Q) & G_{12}(Q) \\ G_{21}(Q) & G_{22}(Q) \end{bmatrix} = \mathcal{F}_\ell(P, K)$$

$$J(M) = \delta_e \|G_{22} W_e^0\|_1 + \frac{\delta_i \delta_e \|G_{21}\|_1 \|W_i^0 G_{12} W_e^0\|_1}{1 - \delta_i \|W_i^0 G_{11}\|_1}$$

- quadratic fractional in both δ_e, δ_i **and the Youla parameter** Q**.**

$$\Downarrow$$

Optimal robust synthesis with errors quantification

$$\min_Q \quad \min_{\{\; \delta_e, \delta_i \;|\; M \in \mathcal{M}_N\}} \quad J(M)$$

is a difficult task.

Model under coprime factor perturbations

$$(D + \Delta_1 W_y)y = (N + \Delta_2 W_u)u + W_e w \ , \det D(0) \neq 0 \ , \quad z = y$$

D, N, W_y, W_u **- polynomials in the backward shift operator;** D **and** N **- left coprime.**

Structured uncertainty :

$$\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \ , \quad \|\Delta\| \leq 1$$

Unstructured uncertainty :

$$\Delta = [\ \Delta_1 \ \Delta_2 \ ] \ , \quad \|\Delta\| \leq 1$$

$$W_y = \delta_y W_y^0 \ , \quad W_u = \delta_u W_u^0 \ , \quad W_e = \delta_e I$$

The Youla parameterization of stable transfer matrices

$$DX - NY = I$$

$$G_y(Q) = X - NQ \ , \quad G_u(Q) = Y - DQ$$

$$H(\lambda) = \sum_{k=0}^{\infty} H(k)\lambda^k \ , \quad \|H\|_1 = \max_i \sum_j \sum_{k=0}^{\infty} |H_{ij}(k)|$$

structured uncertainty:

$$J(M) = \frac{\delta_e \|G_y(Q)\|_1}{1 - \delta_y \|W_y^0 G_y(Q)\|_1 - \delta_u \|W_u^0 G_u(Q)\|_1}$$

unstructured uncertainty:

$$J(M) = \frac{\delta_e \|G_y(Q)\|_1}{1 - \max \{\delta_y \|W_y^0 G_y(Q)\|_1 \ , \ \delta_u \|W_u^0 G_u(Q)\|_1\}}$$

Classical robust synthesis : $J(M) \leq \gamma \quad \Leftrightarrow$

$$\frac{\delta_e}{\gamma} \|G_y(Q)\|_1 + \delta_y \|W_y^0 G_y(Q)\|_1 + \delta_u \|W_u^0 G_u(Q)\|_1 \leq 1$$

- a linear programming (mixed sensitivity problem)

Model validation

structured uncertainty:

$$|Dy(t) - Nu(t)|_\infty \le \delta_e + \delta_y \max_{s<t} |W_y y(s)|_\infty + \delta_u \max_{s<t} |W_u u(s)|_\infty \ ,$$

$$t = 1, \ldots, N \quad - \ N \text{ linear inequalities}$$

unstructured uncertainty:

$$|Dy(t) - Nu(t)|_\infty \le \delta_e + \max\{\delta_y \max_{s<t} |W_y y(s)|_\infty, \delta_u \max_{s<t} |W_u u(s)|_\infty\}$$

2^N **linear inequalities**

($\delta_y = \delta_u \quad \Rightarrow \quad N$ **inequalities**)

$\Downarrow$

Optimal errors quantification

$$\min_{\{\ \delta_e, \delta_y, \delta_u,\ |\ M \in \mathcal{M}_N\}} \quad J(M)$$

$$J(M) = \frac{\delta_e \|G_y(Q)\|_1}{1 - \delta_y \|W_y^0 G_y(Q)\|_1 - \delta_u \|W_u^0 G_u(Q)\|_1} \quad -$$

linear fractional programming

Optimal robust synthesis with errors quantification

$$\min_{Q} \min_{\{\ \delta_e, \delta_y, \delta_u,\ |\ M \in \mathcal{M}_N\}} \quad J(M) \quad -$$

a special quadratic fractional programming.

(Polyak B.T. and Halpern M.E, IJAC&SP, 2001: superstability $\Rightarrow$ linear fractional control criterion in terms of controller parameters)

Global optimization in special case

$$W_y = \delta_y I \quad (W_y^0 = I)$$

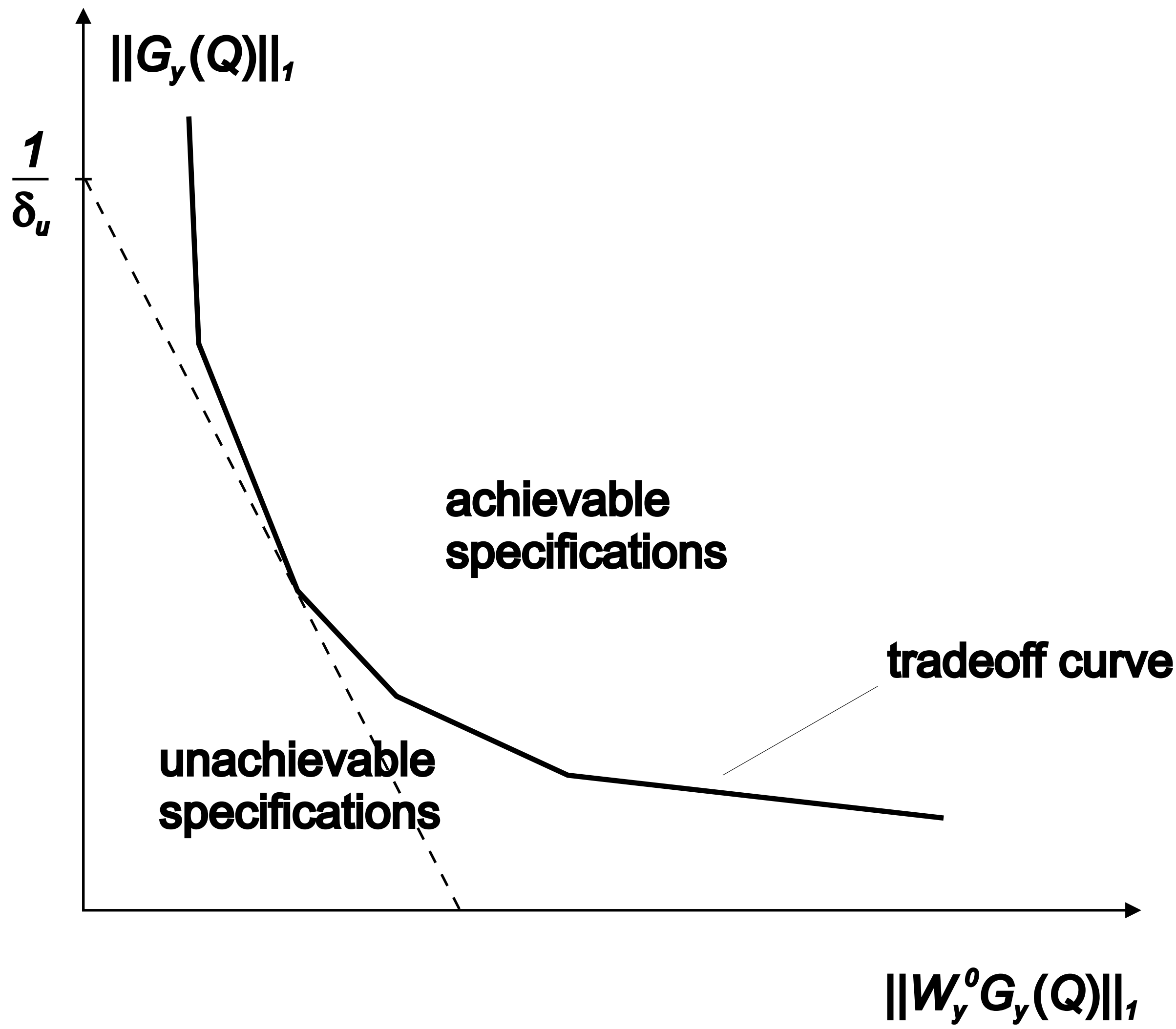
Classical robust synthesis : $J(M) \leq \gamma \quad \Leftrightarrow$

$$(\frac{\delta_e}{\gamma} + \delta_y)\|G_y(Q)\|_1 + \delta_u\|W_u^0 G_u(Q)\|_1 \leq 1$$

Achievable spec. : $\{\ (\|G_y(Q)\|_1, \|W_u^0 G_u(Q)\|_1)\ |\ Q \in \ell_1\ \}$

Tradeoff curve : $\min_Q\ \lambda\|G_y(Q)\|_1 + (1-\lambda)\|W_u^0 G_u(Q)\|_1,\ \lambda \in [0,1]$

$Q(k) = 0$ **for** $k > K \Rightarrow$ **finite linear programming**



General LFT algorithm

by M.H. Khammash, M.V. Salapaka, and T. Vanvoorhis
(IEEE TAC, 2001) for the problem

$$J(M) \leq \gamma \quad -$$

$$\left(\frac{C}{\varepsilon}\right)^3 \quad \textbf{linear programs}$$

Direct approach for CFP model

Classical optimal robust synthesis : $\min_{Q} \; J(M)$

$$\ln \frac{1}{\varepsilon} \quad \textbf{linear programs}$$

Optimal errors quantification : $\min_{\{\ \delta_e, \delta_y, \delta_u,\ \mid\ M \in \mathcal{M}_N\}} \; J(M)$

a simple linear fractional program

Optimal robust synthesis with errors quantification:

$$\min_{Q} \; \min_{\{\ \delta_e, \delta_y, \delta_u,\ \mid\ M \in \mathcal{M}_N\}} \; J(M)$$

partitioning of the interval $[0, \delta_u^{max}] \quad \Rightarrow$

$$\frac{C}{\varepsilon} \quad \textbf{linear programs over } Q$$

$$\frac{C}{\varepsilon} \quad \textbf{simple linear fractional programs over } \delta_e, \delta_y, \delta_u$$

Identification for robust control : the first order plant

Plant P :

$$y(t) + ay(t-1) = bu(t-1) + \delta_y \Delta_y y(t) + \delta_u \Delta_y u(t) + \delta_e w(t)$$

Controller K :

$$bu(t) = ay(t) + r, \quad r - \textbf{set point}$$

Control criterion :

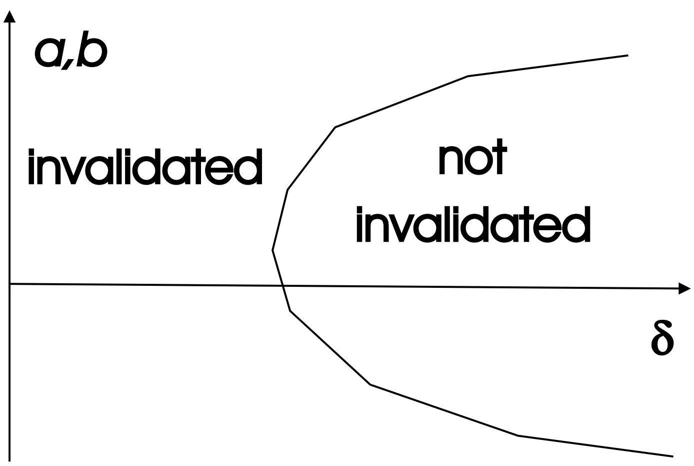
$$J(M) = \sup_w \sup_\Delta \|y - r\|_\infty =$$

$$\frac{|b|(\delta_e + \delta_y r) + (1 + |a|)\delta_u r}{|b|(1 - \delta_y) - \delta_u |a|}$$

- quadratic fractional

Model validation linear constraints : $s = 1, 2, \cdots, N$

$$|y(s) - ay(s-1) - bu(s-1)| \leq \delta_e + \delta_y \max_{k<s} |y(k)| + \delta_u \max_{k<s} |u(k)|$$



Problems	LFT model	CFP model
robust synthesis	nonconvex quadratic fractional	linear fractional
optimal errors quantification	nonconvex quadratic rational	linear fractional
robust synthesis with errors quantification	difficult task	nonconvex quadratic fractional (linear constraints)
identification of nominal model	difficult task	nonconvex quadratic fractional (quadratic constraints)
identification of nominal model with errors quantification	difficult task	nonconvex cubic fractional (quadratic constraints)