CONTROL IN A COLLISION AVOIDANCE PROBLEM FOR AN AIRCRAFT[1]

S. I. Kumkov*, S. G. Pyatko**

*Institute of Mathematics and Mechanics, Ural Branch of the Russian Academy of Sciences, Ekaterinburg, kumkov@imm.uran.ru

**Academy of Civil Aviation, St.-Petersburg

New algorithms for solution of a collision avoidance problem (resolution of a conflict situation) between manoeuvering and non-manoeuvering aircrafts are elaborated. Motion of two aircrafts in the horizontal plane is described by a kinematic equation system, velocities of the aircrafts are constant in time. A manoeuvering aircraft is controlled by variation of the angular velocity of its course. Value of the angular velocity is subject to symmetrical geometric constraints.

A collision (or a conflict situation) is solved if the minimal (with respect to the time of their motions) distance between the aircrafts at the instant of the worst (most close) their encounter is provided equal or larger than the given safe distance tolerance. The object of manoeuvering aircraft under a collision avoidance (or a conflict situation resolution) is minimization of necessary lateral deviation from its original trace under condition of providing the safe distance tolerance.

The whole structure of the resolving manoeuver is prescribed by Technological Rules on the aircraft traffic managing. The manoeuver is comprised of the minimal (in time) S-wise avoidance manoeuver that drives the aircraft from its original trajectory, the line-wise segment for safe flight-by, and the minimal (in time) S-wise returning manoeuver that drives the aircraft back onto its original trajectory.

The cases of uncooperative control are considered: only one of aircrafts from the conflicting pair is prescribed for manoeuvering, the second aircraft does not manoeuver and keeps its course during the encounter and resolution of this conflict situation. In both cases of determination of the manoeuvering aircraft, an avoidance collision problem is formulated not as a differential game but as an optimization problem of optimal control.

Four variants of solving a concrete conflict situation are considered: two variants when one aircraft prescribed for manoeuver implements the avoidance manoeuver (changes its course) in the clock-wise direction and, respectively, in the counter clock-wise direction, and two variants when the other aircraft is prescribed for manoeuver and implements the avoidance manoeuver (changes its course) in the clock-wise direction and, respectively, the counter clock-wise direction.

For all possible variants of manoeuvres a constructive algorithm of the solvability condition for resolving a conflict situation under consideration is derived. For various variants of conflict situations numerical simulation of algorithms of the solvability conditions and optimal manoeuvres construction is fulfilled.

Key words: Aircraft, equation of motion in the horizontal plane, kinematics, collision, conflict situation, solvability condition, manoeuvres, optimal manoeuver, problem of minimization, iteration algorithm.

1 INTRODUCTION

Problem of elaboration of practically meaning schemes and manoeuvres for collision avoidance and resolution of conflict situations between aircrafts is extremely important for providing safe air traffic. In the sequel, for simplification of description we shall mean under a *collision* or *conflict situation* a situation when the minimal distance between the aircrafts at the instant of the worst their encounter is smaller than the given safe distance tolerance. So, we shall use one equivalent term the *conflict situation.* As a result of necessary manoeuver, a conflict situation under consideration is solved if the minimal distance between the aircrafts at the instant of the worst their encounter is provided not smaller than the given safe distance tolerance.

Consideration of existing practical formulations of such problems shows that analysis of a conflict situation and manoeuvres recommended for its resolution are usually rather approximate by virtue both of multi-criterion character of such problems and technical reasons [3–5]. More sophisticated algorithms for analysis and construction of manoeuvres [6] can have complicated logical structure. Strict mathematical formulations for problems of analysis and resolution of conflict situations [9–12] can lead to constructions that are unacceptable from the point of view of technical constraints and Technological Rules for air traffic management [7].

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For example, in [9] such a problem is formulated without a constraint on the lateral deviation of the manoeuvering ship from its original trace. As a result, the lateral deviations that have been realized ([9], Figs.1–4) can become unacceptably large. Formulations, in which the constraint on values of the lateral deviations and constraint on the minimality of the resultant time of being the manoeuvering aircraft outside its trace are absent, can also lead to unacceptably large values of the lateral deviation (see, for example, [10], Fig. 6.6). In some cases [11–12], formulations of conflict solution problems can even demand changing the structure of traces and the air traffic space.

For guaranteed resolution of a conflict situation such algorithms are needed that take directly into account demands of the air traffic management and constructively (i.e., by their building) provide the safe distance tolerance.

In the present work controllable motion of two conflicting aircrafts is considered. The aircrafts move in the horizontal plane along their original traces. Motion of each aircraft is described by a nonlinear ordinary differential equation system of the third order. Two phase coordinates give the aircraft position in the horizontal plane, the third phase coordinate is the course describing the direction of the velocity vector. The velocities of aircrafts are assumed to be constant and their values to be known.

Cases with presence of conflict situations are investigated. Under this, the encounter of the aircrafts is forecasted and analyzed. According to the Technological Rules for air traffic management [7], only one aircraft from the conflicting pair is determined for manoeuvering. The other aircraft continues motion and keeps its course during the resolution of the conflict situation. Thus, the cases of uncooperative control are considered.

In both cases of determination of the manoeuvering aircraft an avoidance collision problem is formulated not as a differential game but as an optimization problem of optimal control, in which the objective of manoeuvering aircraft under conflict situation resolution is minimization of necessary lateral deviation from its original trace under condition of providing the safe distance tolerance.

The whole structure of the resolving manoeuver is prescribed. It is comprised of the minimal (in time) S-wise avoidance manoeuver that drives off the aircraft from its original trajectory, the line-wise segment for safe flight-by, and the minimal (in time) S-wise returning manoeuver that drives back the aircraft onto its original trajectory.

The demand of the minimal lateral deviation of the manoeuvering aircraft from its trace and the demand of the most fast return onto the trace lead to a manoeuver onto a terminal set of special form.

In its essence, this terminal set is a special region of the phase states (of the aircraft under control) such that if the aircraft gets into this set then the conflict situation is resolved and can be resolved in non-unique way. From possible manoeuvres that get the aircraft into this set the optimal one is chosen, which satisfies the additional technological demand of the minimality of the resultant time of being the manoeuvering aircraft outside its trace.

Application of the manoeuvres with the given structure and introduction of the mentioned terminal set fix the scheme of the worst encounter in the resolving manoeuvres. On the basis of the forecast results, this allows to construct the optimal resolving manoeuver and calculate its parameters by simple unified formulas.

For a conflict situation under consideration, existence of the resolving manoeuver of the following four possible variants is justified: two variants when one aircraft prescribed for manoeuver implements the avoidance manoeuver (changes its course) in the clock-wise direction and, respectively, in the counterclock-wise direction, and two variants when the other aircraft is prescribed for manoeuver and implements the avoidance manoeuver (changes its course) in the clock-wise direction and, respectively, the counterclock-wise direction. Variants of manoeuvres, for which the condition of resolution is satisfied, are compared each other by values of their parameters: by the values of necessary minimal lateral deviations and the resultant time of being the manoeuvering aircraft outside its trace. Other practical criteria can also be taken into account.

The elaborated algorithms and corresponding software allow to detect conflict situation and calculate necessary control for its resolution in the real-time mode. These algorithms are designed for application to advanced systems of air traffic management.

The present work is devoted to description and analysis of the basic aspects of the suggested approach in a model formulation, i.e., without disturbances and uncertainties in the input information.

2 PRELIMINARY CONSIDERATION

2.1. Technological demands to a manoeuver resolving a conflict situation

The Technological Rules [7] for operators of the air traffic managing system content the following demands to the resolving manoeuver.

I. Forecast and detection of a conflict situation must be carried out in advance in time and with a reserve in distance sufficient for implementation of the resolving manoeuver.

II. To reduce a workload on an operator and computational complex of the air traffic managing system, resolution of a conflict situation must be realized, as a rule, by a manoeuver of one aircraft only. The second aircraft from the conflicting pair continues motion, keeps its course and does not participate in resolution of this conflict situation.

III. A resolving manoeuver is comprised of the following standard elements: an S-wise avoidance manoeuver, which drives off the aircraft from its original trajectory, and this manoeuver is implemented by control with extremal values of opposite signs on the first and second parts of its time length; a line-wise segment for safe flight-by with keeping its course unchanged; and an S-wise returning manoeuver that drives back the aircraft onto its original trajectory, and this manoeuver is implemented by control with extremal values of opposite signs on the first and second parts of its time length.

IV. To minimize a time workload on an operator for resolution of a concrete conflict situation and to decrease a workload on the crew of the manoeuvering aircraft, the length of the avoidance manoeuver must be minimal in time.

V. To minimize a time workload on an operator for resolution of a concrete conflict situation and to decrease a workload on the crew of the manoeuvering aircraft, the length of the returning manoeuver must be also minimal in time.

VI. To except arising secondary (stipulated) conflict situations with other aircrafts, the lateral deviation of the manoeuvering aircraft from its original trace must be of the minimal necessary value.

VII. The lateral acceleration of an aircraft during the resolving manoeuver must not exceed the given tolerance.

VIII. The resultant time length of the whole manoeuver resolving the conflict situation from the beginning and up to the return must be minimal, i.e., the Rules demand the minimal resultant time of being the aircraft outside its original trace.

Further, the demands considered are formalized and directly or implicitly used in formulation of the problem of elaboration of the optimal resolving manoeuver.

2.2. Description of aircrafts motion

Motion of each aircraft in the horizontal plane (Fig. 1) of the standard normal coordinate system [8] is described by the following ordinary differential equation system:

$$\begin{aligned} \dot{x} &= V\text{Cos}\psi, \\ \dot{z} &= V\text{Sin}\psi, \\ \dot{\psi} &= \omega_{\max} u, \\ \omega_{\max} &= k/V. \end{aligned} \tag{1}$$

Here, x is the ordinate of the aircraft position along the axis OX in the standard coordinate system, and this axis is oriented to the North; z is the abscissa of the aircraft position along the axis OZ in the standard coordinate system, and this axis is oriented to the East; ψ is the aircraft course, the angle is counted clock-wise from the axis OX; $\omega_{\max}$ is the maximal (in modulus) admissible value of the angular velocity of the course changing; $k > 0$ is the maximal admissible value of the lateral acceleration of the aircraft; $V > 0$ is the aircraft velocity in absence of disturbances; u is the aircraft control. The original trace of the aircraft is marked with the thick dashes; Ψ_{Tr} is the trace angle; the thick arrow is the aircraft velocity vector; directions of the angles counting are shown with narrow arrows. Equation system (1) is used for implementation of standard navigational computations [3–7]. The following assumptions

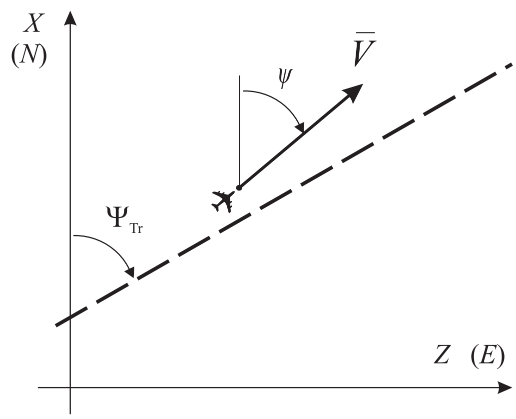

Figure 1: Motion of an aircraft; the standard normal coordinate system

are made about the parameters of system (1). Values V_1 and V_2 of the aircraft velocities are constant in time.

$$V_1 = \text{Const}, \quad V_2 = \text{Const}. \tag{2}$$

Controls of each aircraft, if being prescribed as manoeuvering, is constrained in modulus

$$|u_1| \leq 1 \quad \text{or} \quad |u_2| \leq 1. \tag{3}$$

To simplify investigations and computations, a special version of the standard normal coordinate system is used that is centered by the direction of motion and the trace of the non-manoeuvering aircraft (Fig. 2). For example, let us choose the aircraft moving (conditionally) to the North with the velocity V_1 (Fig. 2). Further for simplicity of description, its parameters and phase variables will be always marked by index "1", and its trace angle $\Psi_1 \equiv 0$ (i.e., conditionally, this aircraft moves to the North along the axis OX). The other aircraft crossing the trace of the aircraft 1, its parameters and phase variables will be always marked by index "2"; the trace of the aircraft 2 is marked in this coordinate system with dashed line. Under this, the angle Ψ_{Tr} of intersection of the aircrafts traces directly coincides with the angle Ψ_2 of the trace of the aircraft 2, and this angle is counted in the standard way in the clock-wise direction from the axis OX.

In Figure 2, the initial positions $x_{1\text{o}}, z_{1\text{o}}$ and $x_{2\text{o}}, z_{2\text{o}}$ of the aircrafts at the conditional initial instant $t_0 = 0$ are also shown. Positions of the aircrafts at the instant T_{p} of the worst their

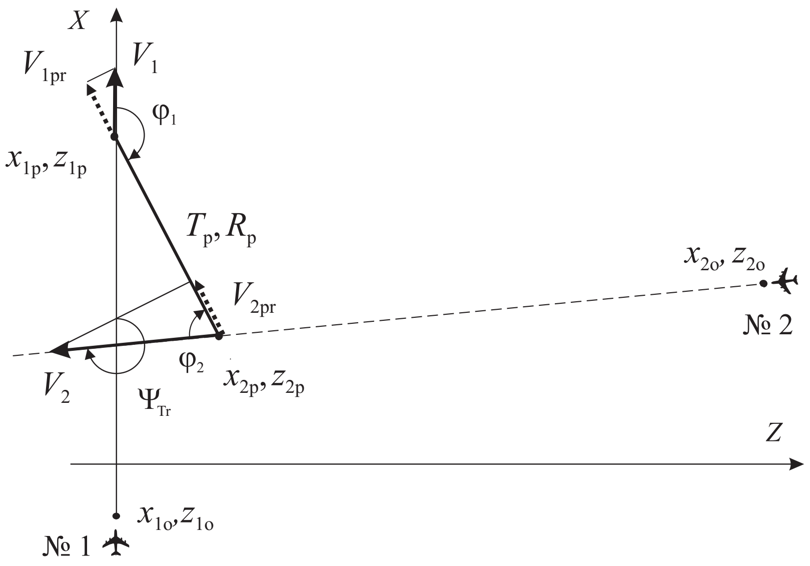

Figure 2: Motion of a pair of aircrafts in the coordinate system centered by the direction of motion and the trace of the non-manoeuvering aircraft 1

encounter are $x_{1\text{p}}, z_{1\text{p}}$ and $x_{2\text{p}}, z_{2\text{p}}$. The thick solid line marks the minimal distance R_{p} between the aircrafts. The dotted arrows illustrate projections $V_{1\text{pr}}$, $V_{2\text{pr}}$ of the aircraft velocities onto the line of sight at this instant.

At the instant of the worst their encounter the time derivative of the distance between them is equal to zero, i.e., $\dot{R} = V_{1\text{pr}} - V_{2\text{pr}} = 0$. The auxiliary angles are shown: φ_1 is the vision angle from the aircraft 1 to the aircraft 2, and φ_2 is the vision angle from the aircraft 2 to the aircraft 1.

The forecast motion of the aircraft 1 along its original trace is described by the finite relations:

$$x_1(t) = x_{1\text{o}} + V_1 t, \quad z_1(t) \equiv 0, \quad \Psi_1 \equiv 0. \tag{4}$$

The forecast motion of the aircraft 2 along its original trace is described by the finite relations:

$$x_2(t) = x_{2\text{o}} + tV_2\text{Cos}\Psi_{\text{Tr}}, \quad z_2(t) = z_{2\text{o}} + tV_2\text{Sin}\Psi_{\text{Tr}}, \quad \Psi_{\text{Tr}} = \Psi_{2\text{o}} \equiv \text{Const}. \tag{5}$$

In (4) and (5) $\Psi_{1\text{o}}$ and $\Psi_{2\text{o}}$ are the directions of traces; Ψ_{Tr} is the angle of intersection of the traces; $x_{1\text{o}}$, $z_{1\text{o}}$ and $x_{2\text{o}}$, $z_{2\text{o}}$ are the positions of aircrafts at the conditional initial instant $t_0 = 0$.

3 PROBLEM FORMULATION

Let a conflict situation and a resolving manoeuver be considered on some sufficiently large time interval $[t_0, T]$. The main condition for constructing the manoeuver is satisfaction of the demand

$$R_{\min} = R(T_{\min}) = \min_{t\in[t_0,T]} R(t) \geq R_{\text{s}}, \tag{6}$$

where R_{s} is the safe distance tolerance; $T_{\min}$ is the instant of the worst encounter; $R_{\min}$ is the minimal distance at this instant.

The problem is in constructing a control satisfying the condition

$$u^*(t) : \min_{u(t)}\{\Delta B(u(t)) \text{ under } R_{\min} \geq R_{\text{s}}\}, \tag{7}$$

where a control of the manoeuvering aircraft is bounded $|u(t)| \leq 1$ by (3), and $\Delta B(\cdot)$ is the lateral deviation of this aircraft form its trace.

Solution of problem (7) becomes complicated when one tries to take into account the technological demands mentioned in Section 2. Under this, the problem is multi-criteria and can become ill posed, and its solution can be unsatisfactory in practical application. From this, necessity of direct taking into account the mentioned technological demands in formulation of the problem is actual.

Demand I is formalized by the assumption that the initial distance $R(t_0)$ between the conflicting aircrafts exceeds with reserve the safe distance tolerance $R_{\rm s}$, and a computed instant $T_{\rm bm}$ for beginning the avoidance manoeuver is smaller of some limit instant $T^*_{\rm bm}$:

$$R(t_0) \gg R_{\rm s}, \quad t_0 \le T_{\rm bm} \le T^*_{\rm bm}. \tag{8}$$

Note that this simple (from the engineering point of view) condition leads to serious difficulties under strict mathematical formulation of problem of the manoeuvering aircraft control since the instant $T_{\rm bm}$ of the maneuver beginning becomes a free unknown parameter.

Demands II, III, IV, and V are taken directly into account in the following way.

1) By the Technological Rules [7], the manoeuvering is preferably implemented by only one aircraft.

2) Structure of the mentioned S-wise avoidance manoeuver is fixed: there are no linear segments in it; on the first half of the manoeuver time length the control has one sign, but on the second half the control sign is opposite; the minimality (in time) of the manoeuver is provided by the fact that the control takes only extremal admissible values. For example,

$$T_{\rm trm} = T_{\rm bm} + \tau^*, \quad u(t) = \begin{cases} +1, & T_{\rm bm} \le \ t \ < T_{\rm bm} + \tau^*/2, \\ -1, & T_{\rm bm} + \tau^*/2 \le \ t \ < T_{\rm trm}. \end{cases} \tag{9}$$

Here, $T_{\rm bm}$ is the instant of the beginning of the avoidance manoeuver; $T_{\rm trm}$ is the instant of termination of the avoidance manoeuver; τ^* is the time length of the manoeuver.

Remark 3.1. *Note one principally important fact that will be used in the sequel. Under the fixed structure of the avoidance manoeuver and the control* (9)*, the course angle ψ of the manoeuvering aircraft motion at the instant $T_{\rm trm}$ of its termination is equal to the initial course of the aircraft and coincides with the trace angle $\Psi_{\rm Tr}$.*

3) In the time interval $\tau_{\rm lt}$ of the mentioned line-wise segment for safe flight-by after the encounter, the prescribed control is equal to zero, and the aircraft moves keeping the same course $\Psi_{\rm Tr}$

$$T_{\rm lt} = T_{\rm trm} + \tau_{\rm lt}, \quad u(t) \equiv 0, \quad T_{\rm trm} \le t \le T_{\rm lt}, \tag{10}$$

where $T_{\rm lt}$ is the instant of termination of the safe flight-by interval.

4) Structure of the S-wise returning manoeuver is also fixed: there are no linear segments in it; on the first half of the manoeuver time length the control has one sign, but on the second half the control sign is opposite; the minimality (in time) of the manoeuver is provided by the fact that the control takes only extremal admissible values. For example,

$$T_{\rm rt} = T_{\rm lt} + \tau^*, \quad u(t) = \begin{cases} -1, & T_{\rm lt} \le \ t \ < T_{\rm lt} + \tau^*/2, \\ +1, & T_{\rm lt} + \tau^*/2 \le \ t \ < T_{\rm rt}. \end{cases} \tag{11}$$

Here, $T_{\rm rt}$ is the instant of termination of the returning manoeuver.

Remark 3.2. *Thus, the given S-wise manoeuvres do not contain line segments between comprising half-arcs of circles of the minimal radii R_{1m} or R_{2m} of turning that are stipulated by the constraints on the maximal admissible values of the lateral accelerations k_1 or k_2 and the maximal admissible values of angular velocities ω_{1max} or ω_{2max} of turning the velocity vectors* *in system* (1). *This directly provides satisfaction of* Demand VII.

Demand VI of minimality of the necessary lateral deviation is satisfied by the described structure of the avoidance manoeuver, in which the special condition of equality of the course ψ to the trace direction angle $\Psi_{\rm Tr}$ of the manoeuvering aircraft is satisfied at the following character instants:

$$\psi(t_0) = \psi(T_{\rm bm}) = \psi(T_{\rm trm}) = \psi(T_{\rm lt}) = \psi(T_{\rm rt}) \equiv \Psi_{\rm Tr}, \tag{12}$$

and by the fact that by construction the necessary lateral deviation is achieved just at the instant of termination of the optimal avoiding manoeuver.

Demand VIII of minimal resultant time length of the whole resolving manoeuver is also satisfied by the described structure of the avoidance manoeuver and by coincidence of the instant $T_{\rm trm}$ of its termination with the instant $T_{\rm min}$ of the worst encounter of the aircrafts.

Remark 3.3. *Note two important facts. Firstly, the given structure of the whole resolving manoeuver and satisfaction of* Demands V,VI, and VII *fix the angular orientation* (*the angles* φ_1 *and* φ_2, Fig. 2) *of the line of sight at the instant of the worst encounter, but shift of the line of sight* (*in parallel to itself, i.e., with the same orientation*) *along the trace of the unmanoeuvreing aircraft remains unconstrained. It significantly simplifies the problem of finding the necessary deviation* (*of the minimal value*) *of the manoeuvering aircraft from its original trace and allows to derive a constructive condition for existence of the resolving manoeuver in conflict situation under investigation. Secondly, satisfaction of* Demands II–VIII *evidently leads to narrowing the class of admissible controls and, in the turn, it can lead to narrowing the types of conflict situations to be solved and to possible decreasing the provided values of the safe distance tolerances.*

Consider a special terminal set $\widetilde{\mathbf{M}}$, which is comprised of positions (x, z, ψ) of the manoeuvering aircraft at the instants $T = T_{\rm trm}$ of termination of possible avoidance manoeuvres with the course $\Psi_{\rm Tr}$, and, additionally, these positions have such a property that during further direct–linear motion of the aircraft with the mentioned course (i.e., under zero value of its control) the minimal distance $R_{\rm min}$ between aircrafts at the instant $T_{\rm min}$ of their worst encounter is guaranteed to be *not smaller that the given safe distance tolerance* $R_{\rm min} > R_{\rm s}$. In the three-dimensional phase space of system (1) the terminal set $\widetilde{\mathbf{M}}$ is a plane

$$\widetilde{\mathbf{M}} = \{x(T_{\rm trm}), z(T_{\rm trm}), \psi(T_{\rm trm}) \equiv \Psi_{\rm Tr}\} \tag{13}$$

parallel to the plane $x \times z$.

In the elaborated approach to construction of the resolving manoeuvres, the projection $\widetilde{M}$ of the terminal set $\widetilde{\mathbf{M}}$ into the plane $x \times z$ is used. The view of the set $\widetilde{M}$, its frontiers $L_{\rm s}$, M, and trajectories of the resolving manoeuvres are considered in the next section.

The avoidance manoeuvres, which satisfy the condition on the course value $\psi(T_{\rm trm}) = \Psi_{\rm Tr}$, begin at points $(x_{\rm bm}, z_{\rm bm})$ on the trace axis at various instants $T_{\rm bm}$. From all admissible resolving manoeuvres one is chosen that gets the aircraft at a special point (x^*, z^*) of the terminal set $\widetilde{M}$ which is placed at the minimal lateral distance from the trace of the manoeuvering aircraft. Such a manoeuver provides (by its construction): satisfaction of condition (6) for resolving the conflict situation, the minimal value of necessary lateral deviation of the avoidance manoeuver, and the minimal time length of this manoeuver.

Since the structure of the line-wise segment for safe flight-by after the encounter and the structure of the returning manoeuver guarantee satisfaction of condition (6), the problem of control of the manoeuvering aircraft is formulated as follows:

Problem 1. ***Construct a control, which gets the manoeuvering aircraft at the point*** $(x^*, z^*) \in \widetilde{M}$ ***of projection of the terminal set*** (13).

4.1. Forecast of the worst encounter

Following [1–6], forecast of a conflict situation at the instant of the worst encounter of the aircrafts is carried out by relations (4) and (5). Thus, it is assumed also that at the initial instant t_0 the aircraft 1 is placed at the axis of its trace and has zero course $\psi_1(t) \equiv 0$ of the velocity vector V_1, and the whole its further motion goes along the axis OX. About the aircraft 2 it is also assumed that its initial position is on the axis of its trace, and the whole its further motion goes along this axis with the constant course $\psi_2(t) \equiv \Psi_{\text{Tr}}$ (Fig. 2). For adopted forecast model of aircraft motion with constant velocities and at constant courses, the worst encounter is detected by zero value of derivation in time of the distance between the aircrafts

$$\begin{aligned} \dot{R} = (x_{2\text{o}} + tV_2\text{Cos}\Psi_{\text{Tr}} - x_{1\text{o}} - V_1 t)(V_2\text{Cos}\Psi_{\text{Tr}} - V_1) + \\ (z_{2\text{o}} + tV_2\text{Sin}\Psi_{\text{Tr}})V_2\text{Sin}\Psi_{\text{Tr}} = 0. \end{aligned} \tag{14}$$

The second derivation in time of the distance is always positive

$$d^2R/dt^2 = (V_2\text{Cos}\Psi_{\text{Tr}} - V_1)^2 + (V_2\text{Sin}\Psi_{\text{Tr}})^2 > 0. \tag{15}$$

Therefore, by (14) the instant of the global minimum of the distance between the aircrafts is found.

Parameters of the worst encounter (in the forecast, Fig. 2) are calculated as follows. The instant T_p of the worst encounter is

$$\begin{aligned} T_\text{p} = -[(x_{2\text{o}} - x_{1\text{o}})(V_2\text{Cos}\Psi_{\text{Tr}} - V_1) + z_{2\text{o}}V_2\text{Sin}\Psi_{\text{Tr}}] \ / \\ [(V_2\text{Cos}\Psi_{\text{Tr}} - V_1)^2 + (V_2\text{Sin}\Psi_{\text{Tr}})^2]. \end{aligned} \tag{16}$$

Positions of the aircrafts at this instant are

$$\begin{aligned} & x_{1\text{p}} = x_{1\text{o}} + V_1T_\text{p}, \quad z_{1\text{p}} \equiv 0, \\ & x_{2\text{p}} = x_{2\text{o}} + V_2T_\text{p}\text{Cos}\Psi_{\text{Tr}}, \quad z_{2\text{p}} = z_{2\text{o}} + V_2T_\text{p}\text{Sin}\Psi_{\text{Tr}}. \end{aligned} \tag{17}$$

The forecast minimal distance R_p is

$$R_\text{p} = \sqrt{(x_{2\text{p}} - x_{1\text{p}})^2 + (z_{2\text{p}} - z_{1\text{p}})^2}. \tag{18}$$

The auxiliary angle φ_1 between the direction of the velocity vector of the aircraft 1 and direction of the line of sight to the aircraft 2 (i.e., clock-wisely from the axis OX) is

$$\varphi_1 = \begin{cases} \pi - \text{Arctg}(z_{2\text{p}}/(x_{1\text{p}} - x_{2\text{p}})), & x_{1\text{p}} > x_{2\text{p}}, \quad z_{2\text{p}} > 0, \\ \pi - \text{Arctg}(z_{2\text{p}}/(x_{1\text{p}} - x_{2\text{p}})), & x_{1\text{p}} > x_{2\text{p}}, \quad z_{2\text{p}} < 0, \\ \text{Arctg}(z_{2\text{p}}/(x_{2\text{p}} - x_{1\text{p}})), & x_{1\text{p}} < x_{2\text{p}}, \quad z_{2\text{p}} > 0, \\ 2\pi + \text{Arctg}(z_{2\text{p}}/(x_{2\text{p}} - x_{1\text{p}})), & x_{1\text{p}} < x_{2\text{p}}, \quad z_{2\text{p}} < 0\,. \end{cases} \tag{19}$$

The auxiliary angle φ_2 between the direction of the velocity vector of the aircraft 2 and direction of the line of sight to the aircraft 1 (i.e., clock-wisely from the vector $\vec{V}_2$) is

$$\varphi_2 = \begin{cases} \pi - (\Psi_{\text{Tr}} - \varphi_1), & \text{if } 0 < \Psi_{\text{Tr}} < \pi, \\ \pi + \Psi_{\text{Tr}} - \varphi_1, & \text{if } \pi < \Psi_{\text{Tr}} < 2\pi\,. \end{cases} \tag{20}$$

On the basis of the angles φ_1 and φ_2, condition (14) of the worst encounter can be rewritten in the following useful equivalent form:

$$V_1|\text{Cos}\varphi_1| = V_2|\text{Cos}\varphi_2|. \tag{21}$$

The intersection point of the traces has the coordinates

$$x_{\mathrm{Tr}} = x_{2\mathrm{o}} - z_{2\mathrm{o}}/\mathrm{tg}\Psi_{\mathrm{Tr}}, \quad z_{\mathrm{Tr}} \equiv 0. \tag{22}$$

Remark 4.1. *Relation* (16) *for calculation of the forecast instant of the worst encounter detects reliably the point of the global minimum of the distance only for line-wise trajectories of the aircrafts and for exactly known parameters of their motions. If there are some uncertainty in description of their motions or disturbances in the input information, a more sophisticated forecast procedure* (*realized step-wisely in time*) *would be necessary.*

4.2. Classification of types situations of the worst encounter

Analysis shows that formulas for calculation of the resolving manoeuvres in various conflict situations and for various pictures of the worst encounter are often similar each other or have the same components, or components in the formulas differ only with signs because of different angles between traces, various directions of the aircraft manoeuver, various their relative positions an directions of velocities, *etc.*

An attempt to construct the unified group of the computational formulas (with internal logical conditions to take into account specific properties of each type of the worst encounter) leads to algorithms of rather complicated structure. With the objective of detailed investigation of specific properties of each type of the worst encounter, the following approach was chosen.

For each type of the worst encounter the complete block of computational formulas is derived. These formulas take into account specifics of each type of encounter. For each type the block is written in a simple form and can be simply programmed (coded) and reliably justified. Types of the worst encounters, schemes of the corresponding resolving manoeuvres and their calculations were investigated in details in [1,2]

Remark 4.2. *Simulation has shown that in numerical calculations it is fruitful for reliable determination of an encounter type to use* (*in analysis of logical conditions*) *some quantitative tolerances onto linear values* (*coordinates, distances, and so on*), *onto angular values* (*course angles, angles of intersection of the traces, angles of orientation of the line of sight*), *and onto values of linear and angular velocities.*

4.3. Computation of parameters of an S-wise avoiding manoeuver and condition of existence for the resolving manoeuver

It is necessary to justify existence of the resolving manoeuver in the following cases.

A. The aircraft 1 is prescribed for manoeuver and implements it in the counterclock-wise direction. (Here and in next variants the other keeps its course.)

B. The aircraft 1 is prescribed for manoeuver and implements it in the clock-wise direction.

C. The aircraft 2 is prescribed for manoeuver and implements it in the counterclock-wise direction.

D. The aircraft 2 is prescribed for manoeuver and implements it in the clock-wise direction.

Consider a way of solving the formulated problem and computational relations on a simple example of a conflict situation shown in Fig. 3, where the aircraft 2 is prescribed for manoeuvering, and the resolving manoeuver is implemented in the counterclock-wise direction.

Example 1. Input numerical data: $x_{1\mathrm{o}} = -20830$ m, $z_{1\mathrm{o}} = 0$ m, $V_1 = 100$ m/sec, $x_{2\mathrm{o}} = 0$ m, $z_{2\mathrm{o}} = 32030$ m, $V_2 = 152$ m/sec, $\Psi_{\mathrm{Tr}} = 1.5\pi$ rad, the maximal value of the lateral acceleration of the aircraft 2 $k_2 = 3.04$ m/sec^2, the maximal value of the course changing $\omega_{\max} = 0.02$ rad/sec, the minimal radius of turn $R_{1\mathrm{m}} = 7600$ m; value of the safe distance tolerance $R_{\mathrm{s}} = 3000$ m. The value $\Psi_{\mathrm{Tr}} = 1.5\pi$ is chosen for demonstrative presentation.

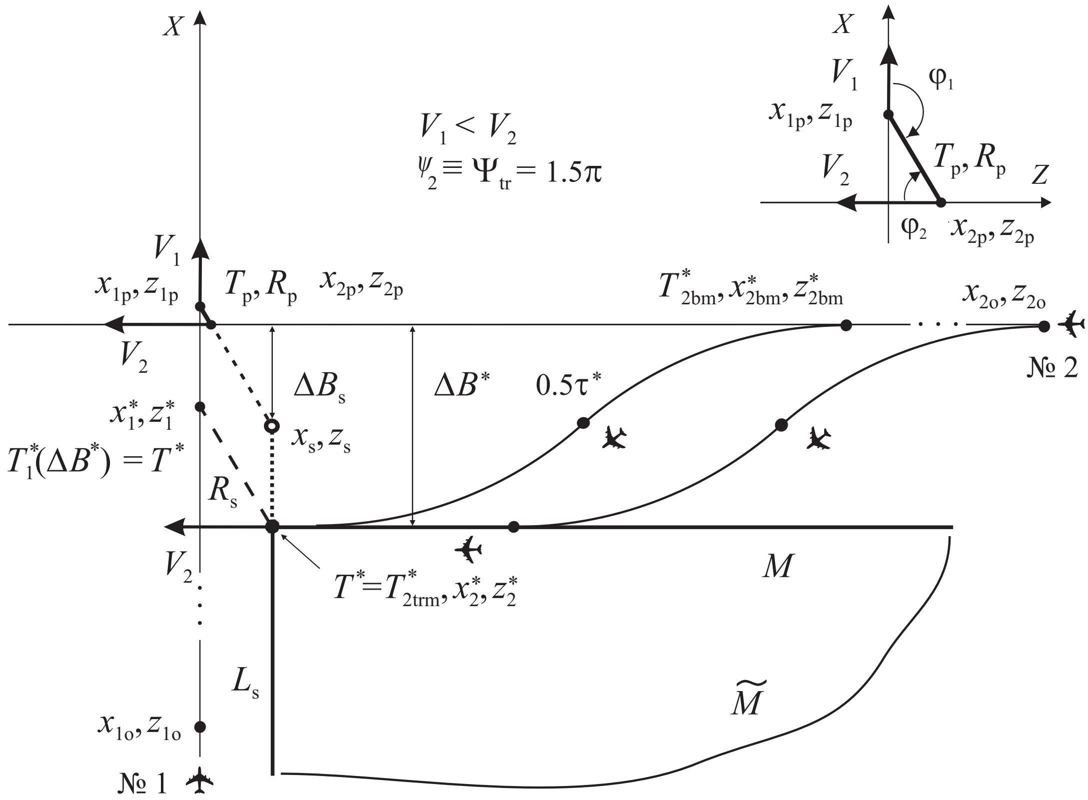

Figure 3: Example of a conflict situation and the resolving manoeuver of given structure and minimal time length

Results of the forecast (Fig. 3, zoomed fragment): $T_p = 209.9$ sec, $R_p = 202.4$ m, $x_{1p} = 169.1$ m, $z_{1p} = 0$ m, $x_{2p} = 0$ m, $z_{2p} = 111.2$ m, the auxiliary angles $\varphi_1 = 2.55$ rad, $\varphi_2 = 0.98$ rad. Thus, at the instant T_p of the worst encounter the aircraft 2 is placed back–right with respect to the aircraft 1.

A conflict situation of this type is detected by the following conditions (in correspondence with Remark 4.2, the tolerance $\Delta_d = 100$ m on values of coordinates in introduced):

$$x_{1p} > x_{2p} + \Delta_d, \quad z_{2p} > \Delta_d, \quad \pi < \Psi_{Tr} < 2\pi. \tag{23}$$

Positions of the aircrafts at the initial instant t_0 are shown by points (x_{1o}, z_{1o}) and (x_{2o}, z_{2o}). The aircraft velocity vectors are marked by thick arrows, and $V_1 < V_2$.

Since the minimal distance R_p (the thick line) between the aircrafts is smaller than the safe distance tolerance R_s, a resolving manoeuver is needed. For manoeuvering the aircraft 2 is prescribed, which is placed back–right w.r.t. the aircraft 1 (in the back half-sphere) at the instant of the worst encounter.

On continuation of the line of sight from the aircraft 1 to the aircraft 2 (this continuation is marked with short thick dashes) an auxiliary point (x_s, z_s) is introduced in such a way that the length of the segment $(x_{1p}, z_{1p}) - (x_s, z_s)$ is equal to the given value of the safe distance tolerance R_s. In the example, for the value $R_s = 3000$ m, the coordinates of the auxiliary point are $(x_s = -2337.1$ m, $z_s) = 1648.8$ m.

Through the point (x_s, z_s) a line L_s is drawn in parallel to the axis of the trace of the non-manoeuvering aircraft 1 (in our case in parallel to the axis OX). Between this axis and the line L_s a line (thick long dashes) parallel to the line of sight $(x_{1p}, z_{1p}) - (x_{2p}, z_{2p})$ is drawn. At the left end of this new position of the line of sight the black circle marks a peculiar position (x^*_1, z^*_1) of the aircraft 1 at such an instant T^*_2, for which the aircraft 2 with the course Ψ_{Tr} is placed just at the right end (x^*_2, z^*_2) of this new line of sight.

If such an instant exists, then the condition for resolving the conflict situation is satisfied since the distance between the points (x_1^*, z_1^*) and (x_2^*, z_2^*) is equal to R_s by construction. Moreover, by virtue of relations (14) and (21) this distance will be the minimal one at the instant T_2^*, and simultaneously, this instant is the instant of the worst encounter. This is so since by virtue of the given structure of the avoidance manoeuver, the aircraft 2 comes at the line L_s with the course Ψ_{Tr} of the velocity V_2.

Note that for each given value of the lateral deviation to be realized by an avoidance manoeuver, its time length, the instant and the position of the beginning on the original trace of the aircraft 2 are defined uniquely.

Under this, it can be seen that if a lateral deviation of some arbitrary value to be realized is given, then the manoeuvring aircraft 2 gets the line L_s at the instant, which does not coincide with the necessary instant. For example, let the manoeuver be realized that gets the aircraft 2 at the point (x_s, z_s). For the shown angle $\Psi_{Tr} = 1.5\pi$ rad of intersection of the traces the corresponding lateral deviation ΔB_s (Fig. 3) of the aircraft 2 from its trace is

$$\Delta B_s = (R_s - R_p)\mathrm{Sin}\varphi_2, \tag{24}$$

and in our case $\Psi_{Tr} = 1.5\pi$ rad, the coordinates of this this point are calculated

$$x_s = x_{2p} - \Delta B_s, \quad z_s = z_{2p} + (R_s - R_p)\mathrm{Cos}\varphi_2. \tag{25}$$

The direct check shows that the aircraft 2 gets the point (x_s, z_s) *with forestalling the instant, at which the aircraft* 1 *gets the left end of the line* $(x_{1p}, z_{1p}) - (x_s, z_s)$, i.e.,

$$T_{2trm}(x_s, z_s) < T_1(x_{1p}, z_{1p}) = T_p. \tag{26}$$

During further motion of the aircrafts, this fact leads to a minimal distance between them that is smaller than the given value of the safe distance tolerance R_s.

If the point $(x_{2trm}, z_{2trm} \equiv z_s)$, at which the aircraft 2 gets the line L_s, is chosen more and more lower of the point (x_s, z_s) by increasing the value of the lateral deviation ΔB, then the mentioned effect of forestalling remains. It is important that under this increasing of the lateral deviation ΔB, the forestalling decreases. Simultaneously, the time length τ of the avoidance manoeuver increases, and the instant $T_{2bm}(\Delta B)$ of the manoeuver beginning decreases. In Fig. 3 this vertical part of the line L_s is marked with thick dots.

For various values of the lateral deviation ΔB the instances $T_{2trm}(\Delta B)$ of termination of the avoidance manoeuver of the aircraft 2 on the line L_s and the instances $T_1(\Delta B)$ of the aircraft 1 achieving the opposite left point of the corresponding lines of sight are shown in Table 1 for numerical data of Example 1. In Figure 4 curves of the dependencies $T_{2trm}(\Delta B)$ (circles, the

Table 1. Solution of the equation for time equlity, $R_s = 3000$ m

ΔB, m	2500	2600	2700	2800	2900	3000
$T_{2trm}(\Delta B)$, sec	202.8	203.1	203.4	203.7	203.9	204.3
$T_1(\Delta B)$, sec	208.2	207.2	206.2	205.2	204.2	203.2

$T^* = 204.0$ sec; $\Delta B^* = 2936.0$ m

solid line) and $T_1(\Delta B)$ (crosses, the line in dashes) are given.

These curves are calculated by the following way. Auxiliary variables are defined by relations

$$\begin{aligned} &x_{Tr} = x_{2o} - z_{2o}/\mathrm{tg}\Psi_{Tr}, \quad z_{Tr} = 0, \\ &x_s = x_{1p} + R_s\mathrm{Cos}\varphi_1, \quad z_s = R_s\mathrm{Sin}\varphi_1, \\ &\Delta\widetilde{x} = (x_{1p} - x_{Tr})(R_s - R_p)/R_p, \\ &\widetilde{x}_2 = x_s + \Delta\widetilde{x}, \quad \widetilde{z}_2 = z_s. \end{aligned} \tag{27}$$

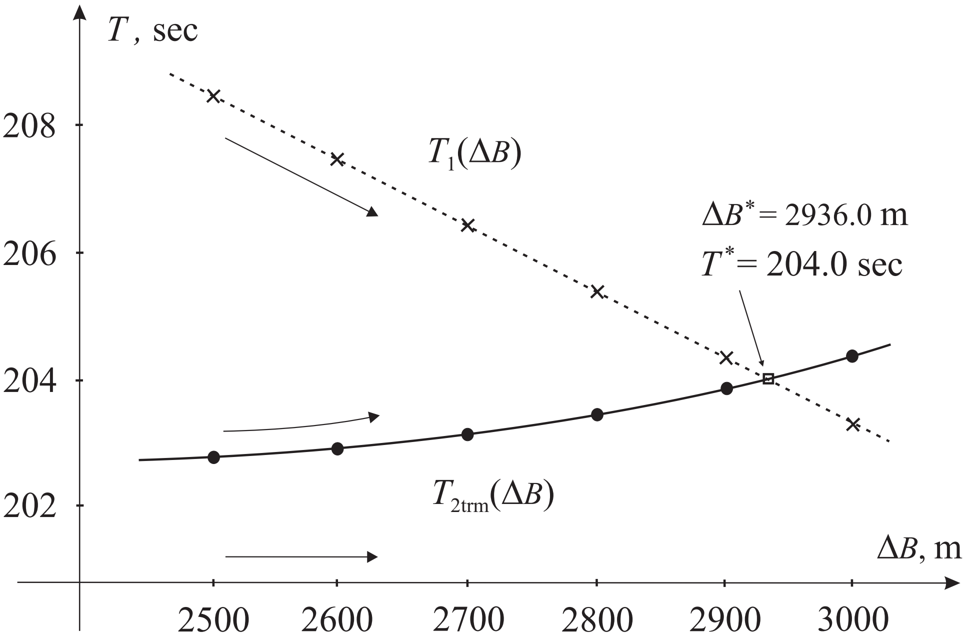

Figure 4: Curves of dependencies $T_1(\Delta B)$ (dashes) and $T_{2\text{trm}}(\Delta B)$ (solid line) *via* the realized lateral deviation

Here, $(x_{\text{Tr}}, z_{\text{Tr}})$ is the point of intersection of the traces; $(x_{\text{s}}, z_{\text{s}})$ is an auxiliary point; $\Delta\widetilde{x}, \widetilde{x}_2, \widetilde{z}_2$ are auxiliary variables.

The instant $T_1(\Delta B)$, at which the aircraft 1 gets the left end of the corresponding line of sight, is found

$$\begin{aligned} &\Delta x = \Delta B/\text{Cos}(3\pi/2 - \Psi_{\text{Tr}}), \quad x_2(\Delta B) = \widetilde{x}_2 - \Delta x, \\ &T_1(\Delta B) = T_{\text{p}} - (x_{\text{s}} - x_2(\Delta B))/V_1. \end{aligned} \tag{28}$$

By the input value of ΔB, the time length τ of the avoidance manoeuver, coordinates $(x_{2\text{bm}}, z_{2\text{bm}})$ of the beginning point on the original trace, the instant $T_{2\text{bm}}(\Delta B)$ of the beginning, and the instant $T_{2\text{trm}}(\Delta B)$ of termination are calculated

$$\begin{aligned} &\text{Cos}\psi_{\text{m}} = 1 - \Delta B/(2R_{2\text{m}}), \quad \text{Sin}\psi_{\text{m}} = \sqrt{1 - (\text{Cos}\psi_{\text{m}})^2}, \\ &\psi_{\text{m}} = \text{Arctg}(\text{Sin}\psi_{\text{m}}/\text{Cos}\psi_{\text{m}}), \quad \tau = 2\psi_{\text{m}}/\omega_{2\max}, \\ &\Delta L_{2\text{m}} = 2R_{2\text{m}}\text{Sin}\psi_{\text{m}}, \\ &\widetilde{\widetilde{z}}_2 = z_{\text{s}} - \Delta B\text{Sin}(3\pi/2 - \Psi_{\text{Tr}}), \\ &z_{2\text{bm}} = \widetilde{\widetilde{z}}_2 + \Delta L_{2\text{m}}\text{Cos}(3\pi/2 - \Psi_{\text{Tr}}), \\ &T_{2\text{bm}} = (z_{2\text{o}} - z_{2\text{bm}})/(V_2|\text{Sin}\Psi_{\text{Tr}}|), \\ &T_{2\text{trm}}(\Delta B) = T_{2\text{bm}} + \tau, \end{aligned} \tag{29}$$

where $R_{2\text{m}} = V_2/\omega_{2\max}$ is the minimal radius of turn of the manoeuvering aircraft 2; for a fixed structure of the S-wise manoeuver the value of the lateral deviation is constrained by the condition $\Delta B \leq 2R_{2\text{m}}$, and variation of the course of this aircraft is constrained by the condition $|\psi_2 - \Psi_{\text{Tr}}| \leq \pi/2$; $\widetilde{\widetilde{z}}_2$ is an auxiliary variable; $(x_{2\text{bm}} \equiv 0, z_{2\text{bm}})$ is a point of the beginning of the aircraft 2 manoeuver (Fig. 3).

By virtue of linear decreasing the value $T_1(\Delta B)$ and monotonic growth of the value $T_{2\text{trm}}(\Delta B)$ (Fig. 4), the points (x_1^*, z_1^*) and (x_2^*, z_2^*) mentioned above (Fig. 3), at which the following equality of times is fulfilled, are obligatory found:

$$T_{2\text{trm}}(\Delta B) = T_1(\Delta B) = T^* = T_{\min}. \tag{30}$$

In Figure 4, the corresponding point of intersection of these dependencies are marked with a small square, the minimal lateral deviation necessary for resolving this conflict situation is ΔB^*, the instant of termination of the resolving manoeuver is $T^* = T^*_{\rm 2trm}(\Delta B^*)$.

In Example 1 under consideration, parameters of the constructed manoeuver, which is optimal both in minimal necessary lateral deviation and in minimal time length under the given structure, are: the minimal lateral deviation $\Delta B^* = 2936.0$ m, the instant of termination $T^* = T^*_{\rm 2trm}(\Delta B^*) = T_1(\Delta B^*) = T_{\rm min} = 204.0$ sec, the coordinates of the aircraft 1 position $x^*_1 = -429.8$ m, $z_{\rm 1p} = 0$ m, the coordinates of the "limit corner" point $x^*_2 = -2936.0$ m, $z_{\rm 2p} = 1648.8$ m, the time length of the avoidance manoeuver $\tau^* = 63.2$ sec, the maximal variation of the aircraft 2 course 0.632 rad (36.2 degrees), the instant of the beginning of the manoeuver $T^*_{\rm 2bm} = 140.8$ sec, and the aircraft 2 coordinates at this instant $x^*_{\rm 2bm} = 0$ m, $z^*_{\rm 2bm} = 10628.3$ m.

When the realized lateral deviation ΔB increases, the time-lag (Table 1 and Fig. 4) of the aircraft 2 reaching the line $L_{\rm s}$ appears. It means that the actual instant of the worst encounter of the aircrafts happens at an internal point of the set $\widetilde{M}$, and this instant is smaller (earlier) than the instant $T_{\rm 2trm}(\Delta B)$ when the aircraft 2 gets the line $L_{\rm s}$. Thus, the actual distance $R_{\rm min}$ of the worst encounter *will be guaranteed larger* than the given value of the safe distance tolerance $R_{\rm s}$. Note that under this, the corresponding lateral deviation of the manoeuvering aircraft will exceed the minimal needed value ΔB^*.

So, in the considered conflict situation the auxiliary line

$$L_{\rm s} = \{x_2 \leq x^*_2, z_2 \equiv z^*_2\} \tag{31}$$

can be used as the left frontier of the terminal set $\widetilde{M}$ (Fig. 3). By construction, this line is a totality of positions and instants with guaranteed providing the safe distance tolerance.

For the input data of Example 1 and taking into account curves of Fig. 4, the equation for the instants $T_1(\Delta B^*)$ and $T_{\rm 2trm}(\Delta B^*)$ equality takes the form

$$\begin{aligned} T_{\rm p} + |x_{\rm s}|/V_1 - \Delta B/V_1 = (2R_{\rm 2m}{\rm ArcCos}(1 - \Delta B/(2R_{\rm 2m})))/V_2 + (z_{\rm 2o} - z_{\rm s})/V_2 - \\ (2R_{\rm 2m}\sqrt{1 - (1 - \Delta B/(2R_{\rm 2m}))^2})/V_2, \quad \Delta B \leq 2R_{\rm 2m}, \end{aligned} \tag{32}$$

where $R_{\rm 2m} = V_2/\omega_{\rm 2max}$ is the minimal radius of turn of the manoeuvering aircraft 2.

In spite of the fact that this equation is transcendental, it can be fast (over small number of steps) solved with necessary accuracy, for example, by the bisection method or the zero-point calculation method. As a result, the minimal value of the necessary lateral deviation of the manoeuvering aircraft 2 from its original trace is found.

All the rest parameters of the manoeuver (in the coordinate system centered by the direction and the trace of the non-manoeuvering aircraft) are calculated as follows:

$$\begin{aligned} &{\rm Cos}\psi^*_{\rm m} = 1 - \Delta B^*/(2R_{\rm 2m}), \quad {\rm Sin}\psi^*_{\rm m} = \sqrt{1 - ({\rm Cos}\psi^*_{\rm m})^2}, \\ &\psi^*_{\rm m} = {\rm Arctg}({\rm Sin}\psi^*_{\rm m}/{\rm Cos}\psi^*_{\rm m}), \quad \tau^* = 2\psi^*_{\rm m}/\omega_{\rm 2max}, \\ &\Delta L^*_{\rm m} = 2R_{\rm 2m}{\rm Sin}\psi^*_{\rm m}, \quad T^*_{\rm bm} = T^* - \tau^*, \\ &x^*_{\rm 2bm} = x_{\rm 2o} + V_2T^*_{\rm bm}{\rm Cos}\Psi_{\rm Tr}, \quad z^*_{\rm 2bm} = z_{\rm 2o} + V_2T^*_{\rm bm}{\rm Sin}\Psi_{\rm Tr}, \\ &x^*_2 = -\Delta B^*, \quad z^*_2 = z_{\rm 2s}, \\ &x^*_1 = x_{\rm 1o} + V_1T^*, \quad z^*_1 = 0, \\ &x^*_{\rm 1bm} = x_{\rm 1o} + V_1T^*_{\rm bm}, \quad z^*_{\rm 1bm} = 0. \end{aligned} \tag{33}$$

Remark 4.3. *Note one interesting fact: when a realized value of the lateral deviation $\Delta B > \Delta B^*$ increases, the dependence $T_{\rm 2trm}(\Delta B)$ converges to a linear, and its inclination coefficient approaches (with the opposite sign) to one of the dependence $T_1(\Delta B)$.*

Remark 4.4. *The considered constructions hold under satisfaction of condition* (8)*, i.e., under sufficient reserve of the initial distance between the conflicting aircrafts and the initial* *time for beginning the S-wise avoidance manoeuver. If the point* $(x_{2\text{trm}}, z_{2\text{trm}} \equiv z_{\text{s}})$ *is shifting down the point* $(x_{\text{s}}, z_{\text{s}})$ *over the line* L_{s} *but satisfaction of equality* (30) *is not achieved, it means that this conflict situation was detected too late and can not be resolved by the standard S-wise manoeuver. Its resolution can be possibly done by the one-side extremal control, similarly to the approach described, for example, in* [9].

Remark 4.5. *The formulas for of computation the values* ΔB^*, T^* *and other parameters of the S-wise avoidance manoeuver have the sense till a reasonable value of variation of the course* $\psi_{2\max}$ *of the manoeuvering aircraft at the point of the control switch. For example,* Fig. 5, *under* $\psi_{2\max} = \pi$ *at the instant of the control switch, the aircraft* 2 *has turned to the direction opposite to the direction of the non-manoeuvering aircraft* 1 *motion. Under this, if the necessary equality* (30) *is not achieved, then the manoeuver can be complemented by a linear segment parallel to the line* L_{s}*. But it is equivalent to changing the structure of the standard avoidance manoeuver and leads to changing the formulas of computation of the manoeuver parameters* τ^*, (x_1^*, z_1^*), (x_2^*, z_2^*), *and* T^**. So, this question lies out of the frames of the present investigation.*

Construct now the upper frontier M of the set $\widetilde{\mathbf{M}}$ and its projection $\widetilde{M}$. Let from points (x_2, z_2) be placed at the right from the point $(x_{2\text{bm}}^*, z_{2\text{bm}}^*)$ on the trace of the aircraft 2 (Figs. 3, 5, and 6) for instants $T_{\text{bm}} < T_{\text{bm}}^*$ the avoidance manoeuvres be started with the parameters ΔB^* τ^*. The points of termination of these manoeuvres at the instants $T_{2\text{trm}}(x_2, z_2)$ by virtue of the condition

$$\psi_2(T_{2\text{trm}}) \equiv \Psi_{\text{Tr}}$$

comprise the line M, which sources from the point (x_2^*, z_2^*) to the right. This line has the following important properties:
– all further linear motions of the aircraft 2 over this line terminate at the point (x_2^*, z_2^*);
– the instants of termination of these motions (Figs. 3, 5, and 6) at the point (x_2^*, z_2^*) are always equal to T^* and, by construction, always coincide with the instant $T_{\min}$ of the worst encounter of the conflicting aircrafts;
– the minimal distance between the aircrafts coincides, by construction, with the given value of the safe distance tolerance R_{s}.

The line M bounds the set $\widetilde{M}$ from above. Under various values of the lateral deviation ΔB the view of the avoidance manoeuver is shown in Figs. 5 and 6. Here, on the trace of the aircraft 2 the position (x_2^{**}, z_2^{**}) and the instant $T_{2\text{bm}}^{**}$ of the manoeuver beginning are shown. This manoeuver realizes the lateral deviation ΔB^{**} equal to two minimal radii $R_{2\text{m}}$ of turn of the manoeuvering aircraft 2. The corresponding trajectory (thin solid curve) is a limit one under retaining the given structure of the S-wise manoeuver. On this trajectory the maximal variation of the course ψ_2 of the manoeuvering aircraft 2 is achieved.

It is seen (Figs. 5 and 6) that manoeuver of the given structure provides (by its possibilities) realization of the lateral deviation not larger than the value $\Delta B \leq \Delta B^{**}$ (see Remark 3.3). Technical analysis [1,2] shows that the value ΔB^{**} can achieve several kilometers for contemporary aircrafts that could be sufficient for resolution of practically all types of conflict situations.

Direct check shows that any S-wise manoeuver of the given structure with the time length $\tau^* \leq \tau \leq \tau^{**}$ and beginning from points (x_2, z_2) located at the right from the point (x_2^{**}, z_2^{**}) on the trace axis of the aircraft 2 and beginning at instants $T_{\text{bm}} < T_{\text{bm}}^{**}$ terminates always inside the set $\widetilde{M}$. This fact under the linear further motion of the aircraft 2 guarantees providing the safe distance tolerance R_{s} when the aircraft gets onto the frontier L_{s} of the mentioned terminal set.

Thus, the limit corner point (x_2^*, z_2^*) and the corresponding avoidance manoeuver satisfy, by construction, all technological demands described in Section 2.

Note one (very important for practice) property of the aircraft 2 manoeuver to the limit

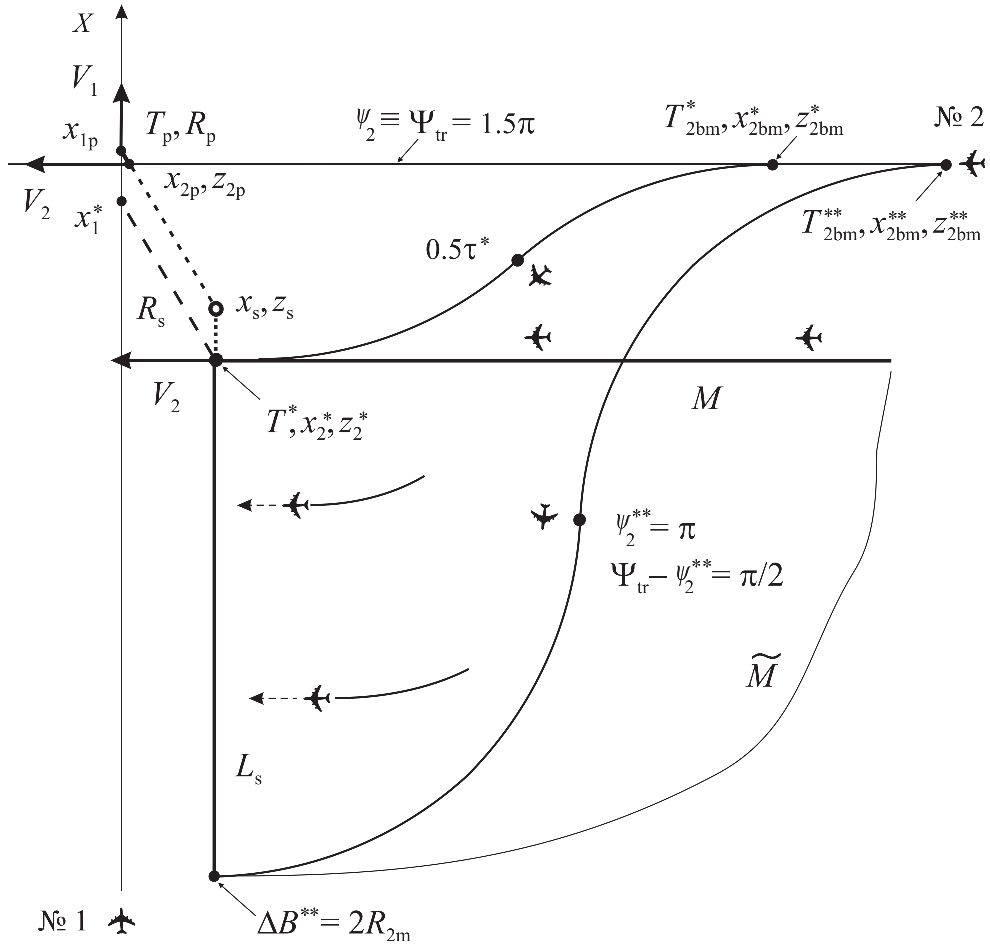

Figure 5: The near frontiers of the projection $\widetilde{M}$ of the terminal set $\widetilde{\mathbf{M}}$

corner point. When the aircraft 2 manoeuvres towards the non-manoeuvering aircraft 1, resolution of the conflict situation happens earlier than the forecast instant T_{p} of the worst their encounter since always the inequality holds $T^* < T_{\text{p}}$.

Now the structure of the *optimal resolving avoidance manoeuver* can be described: the manoeuvering aircraft 2 with the original course Ψ_{Tr} of velocity V_2 is controlled by the standard S-wise manoeuver of the minimal time length τ^* and beginning at the most late instant $T^*_{2\text{bm}}$ from the peculiar point $(x^*_{2\text{bm}}, z^*_{2\text{bm}})$ at its trace axis. The aircraft is brought to the limit point (x^*_2, z^*_2) of the terminal set $\widetilde{M}$ with the course Ψ_{Tr} at the instant $T^* = T^*_{2\text{trm}} = T_1(x^*_1, z^*_1)$, at which the non-manoeuvering aircraft 1 is placed at the opposite end (the point (x^*_1, z^*_1))of the line of sight. The instant $T^*_{2\text{bm}}$ of the manoeuver beginning stands on the minimal time τ^* form the terminal instant T^*.

Figure 6 illustrates the general view of the projection $\widetilde{M}$ into the plane $x \times z$ of the terminal set $\widetilde{\mathbf{M}}$. On the aircraft 2 trace axis peculiar positions of beginning the avoidance manoeuver (corresponding the instants $T^*_{2\text{bm}}$ and $T^{**}_{2\text{bm}}$) are marked. For these positions the admissible time length $\tau^* \leq \widetilde{\tau}(\Delta B) \leq \tau^{**}$ of the manoeuver depends on the admissible value of realized lateral deviation at the instant when the aircraft gets onto the frontier line L_{s}.

The right frontier L_0 (the curve is marked in solid thick) of the set $\widetilde{M}$ is comprised of the terminal points of the S-wise manoeuver from the initial point t_0, $(x_{2\text{o}}, z_{2\text{o}})$ for various values of the realized lateral deviation ΔB. The similar curve (dotted) near the left frontier L_{s}) can be first realized from the initial point $T^{**}_{2\text{bm}}$, $(x^{**}_{2\text{bm}}, z^{**}_{2\text{bm}})$. The lower frontier M^{**} is comprised of the terminal points of the aircraft 2 manoeuvres beginning from its trace axis under the maximal realized value ΔB^{**} of the lateral deviation and linear segments of further motions

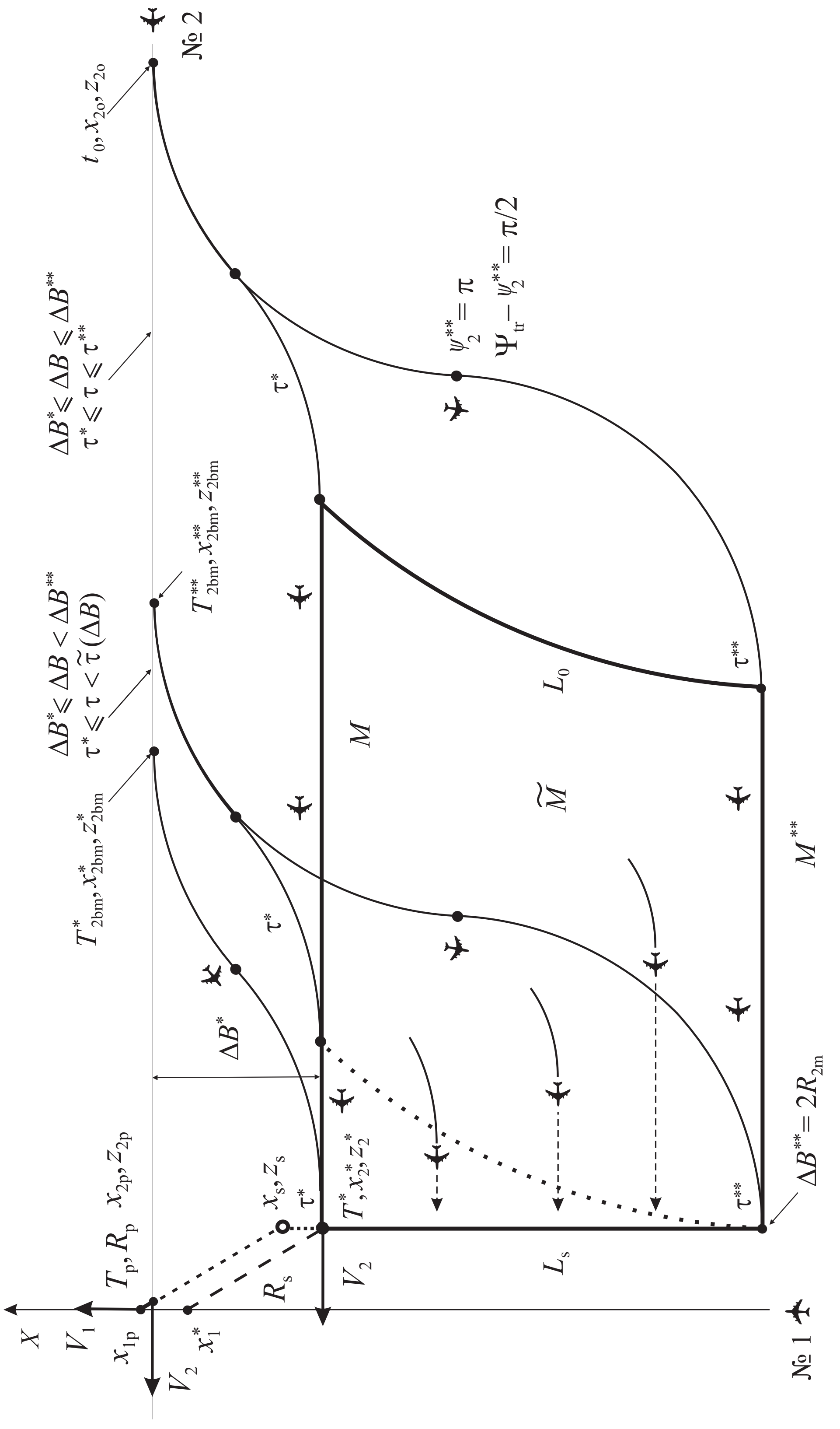

Figure 6: The general view of the projection $\widetilde{M}$ of the terminal set $\widetilde{\mathbf{M}}$

up to the frontier L_s.

Figures 5 and 6 allow to conclude that, indeed, by construction for all points of the set $\widetilde{M}$ under $\psi_{2trm} \equiv \Psi_{Tr}$, the minimal distance between the aircrafts at the instant of the worst encounter is guaranteed to be not smaller than R_s. This minimizes the workload onto an air traffic manager: the conflict situation is obligatory resolved if the manager brings the manoeuvering aircraft at any internal point of the set $\widetilde{M}$. If it is necessary to deviate the craft "*not too far*" from the original trace, then it is sufficient to bring the aircraft to the corner of the set $\widetilde{M}$, i.e., to a region near the limit point (x^*, z^*).

Example 2. To illustrate properties of the elaborated approach, let us consider a representative example of a conflict situation, in which analysis of the solvability condition allows to show directly all admissible manoeuvres for conflicting aircrafts. The following numerical input data are given: $x_{1\text{o}} = -20002$ m, $z_{1\text{o}} = \ 0$ m, $x_{2\text{o}} = 0$ m, $z_{2\text{o}} = 19998$ m, $V_1 = V_2 = 100$ m/sec, $\Psi_{\text{Tr}} = 1.5\pi$ rad, the aircrafts are of the same type, the maximal lateral acceleration $k = 4.87$ m/sec^2, the maximal angular velocity $\omega_{\max} = 0.0487$ rad/sec, the minimal radius of turn $R_{1\text{m}} = 2050$ m; the value of the safe distance tolerance $R_{\text{s}} = 1000$ m.

The forecast results: $T_{\text{p}} = 220.2$ sec, $R_{\text{p}} = 2.8$ m, $x_{1\text{p}} = -2$ m, $z_{1\text{p}} = 0$ m, $x_{2\text{p}} = 0$ m, $z_{2\text{p}} = 2$ m, $\varphi_1 = 3\pi/4$ rad, $\varphi_2 = \pi/4$ rad. It is seen that practically a collision occurs since the minimal distance is estimated as $R_{\text{p}} = 2.8$ m. So, $R_{\text{p}} \gg R_{\text{s}}$ and a resolving manoeuver is needed.

Let now the aircraft 1 be prescribed for manoeuver, but the aircraft 2 does not manoeuver.

To construct a resolving manoeuver, follow the algorithms described above. The auxiliary point $(x_{\text{s}}, z_{\text{s}})$ (Fig. 7) has coordinates $x_{\text{s}} = -353$ m and $z_{\text{s}} = -351$ m, the corresponding lateral deviation is $\Delta B_{\text{s}} = 353$ m.

Since in this example the aircraft 1 has reserves (8) in distance and time for avoidance manoeuver, the solvability condition (condition of existence of this manoeuver) for a resolving manoeuver to the right-towards the aircraft 2 is satisfied (Fig. 7). Here, there are trajectories of the resolving manoeuvres (solid thin curves) to the limit corner point (x_1^*, z_1^*) and the right limit point $(x^{*}{*}_1, z_1^{**})$ of the terminal set $\widetilde{M}$, its frontiers M and L_{s}, and possible positions of the line of sight for various values of the manoeuver time length. Recall that the safe distance tolerance is given $R_{\text{s}} = 1000$ m, and the minimal radius of turn for the aircraft 1 is $R_{1\text{m}} = 2050$ m.

Under such a manoeuver, the aircraft 1 resolves the conflict situation by coming into the back half-sphere of the non-manoeuvering aircraft 2. It is useful to carry out analysis of relation between the value of the given safe tolerance R_{s} and the value of the realized minimal lateral deviation ΔB^* needed for resolution of the conflict situation. In Table 2, positions of the limit point $(x_1^*(R_{\text{s}}), z_1^*(R_{\text{s}}))$ of the resolving S-wise manoeuver are shown for various values R_{s} of the safe distance tolerance. For such a manoeuver, solution of the time equality equation (30) and (32) exists up to the value $R_{\text{s}}^{**} = 4552$ m that corresponds to the maximal realizable lateral deviation $\Delta B^{**} = 2R_{1\text{m}} = 4100$ m, which is constrained by the given structure of the avoidance manoeuver.

Remark 4.6. *Elaborated type of the resolving manoeuver bringing the manoeuvering aircraft onto the peculiar terminal set* $\mathbf{M}$ *(or onto its projection* $\widetilde{M}$*, or at its limit point* (x_2^*, z_2^*)*) at the instant* T^* *seems to be new. Usually in constructing a resolving manoeuver for providing the safe distance tolerance, researchers allow variation of a number of parameters of the worst encounter: the instant, the coordinates of the aircrafts, orientation of the line of sight, and so on. As a result, it leads to increasing the number of free variables, complicate formulation of a problem of optimal control and constructing the resolving manoeuver. For example, this is why in* [10 − 12]*, to formulate the resolving problem strictly and to achieve a strict mathematical solution, the authors make assumptions on changing the structure of the aerial space in a zone of intersection of traces and fixing the schemes of motion for conflicting aircrafts. Similarly, an approach based on simple engineering reasons* [3−6] *can lead to complicated computational procedures. In all these approaches technological demands are not taken into account completely in constructing the resolving manoeuvres. In contrary, the presented approach takes directly* (Section 2) *into account the* Technological Demands [7] *in formulation of the resolution problem.*

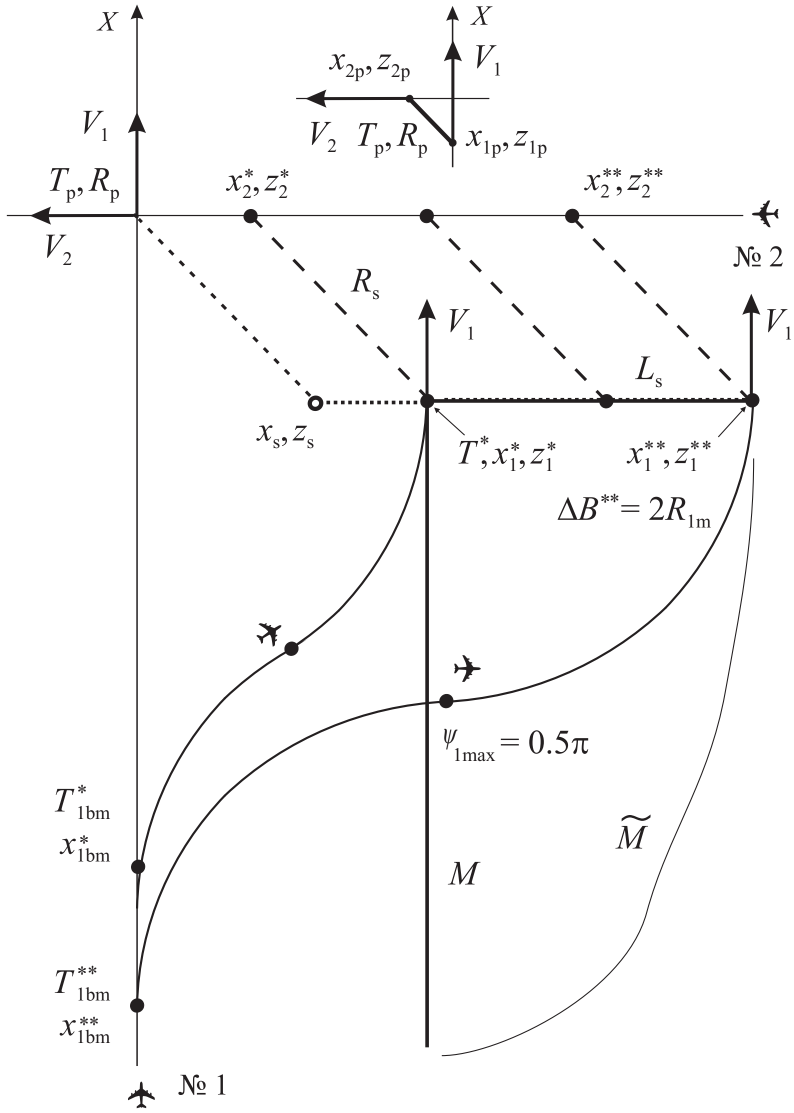

Figure 7: The resolving manoeuver of the aircraft 1 to the right–towards the aircraft 2; aircrafts are of the same type $V_1 = V_2$

As was shown in Example 2, the resolving manoeuver of the aircraft 1 to the right–towards the aircraft 2 exists. But in contrary, the equation of the time equality (30) has no solution for the manoeuver of the aircraft 1 to the left onto the direction of motion of the aircraft 2; and the solution is absent for any values of the realizing lateral deviation and any values of the safe distance tolerance. This fact has natural explanation for the aircrafts of the same type and the

Table 2. Relation between the value of the given safe tolerance R_s and the value of the realized minimal lateral devation ΔB^*

R_s, m	500	1000	1500	2000	2500	3000	3500	4000	4250	4552
$\Delta B^* = z_1^*$, m	596	1123	1612	2071	2567	2925	3320	3700	3884	4100
x_1^*, m	-353	-707	-1060	-1414	-1767	-2121	-2474	-2828	-3005	-3219

$\Delta B^{**} = 2R_{1m} = 4100$ m; max: $R_s^{**} = 4552$ m

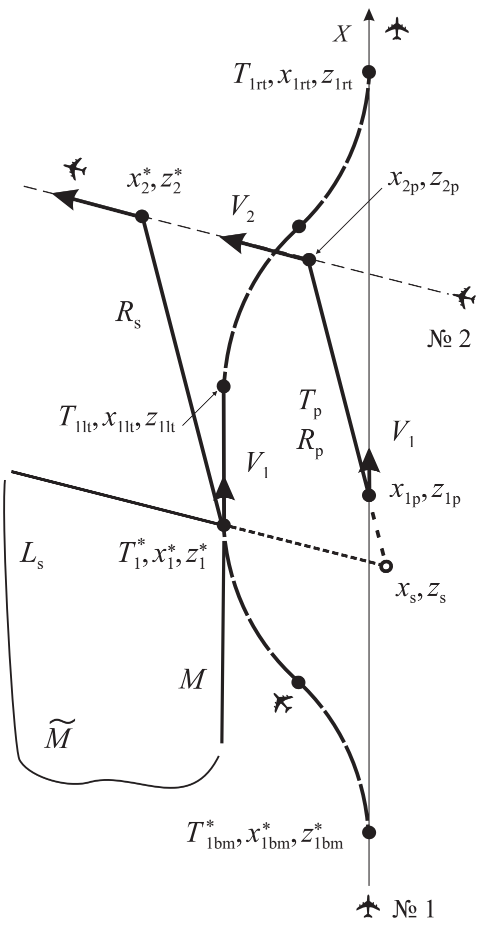

Figure 8: Resolving manoeuver of the aircraft 1 to the left

angle of intersection of their traces $\Psi_{2\mathrm{Tr}} = 1.5\pi$ rad as in the data of Example 2.

But for conflict situations similar to one shown in Fig. 7 and other input data, the resolving manoeuver can exist. Figure 8 shows one picture from simulation, where a conflict situation can be resolved under superiority of the non-manoeuvering aircraft 2 in the velocity ($V_2 > V_1$) and for large angle $\Psi_{2\mathrm{Tr}}$ of intersection of the traces.

4.4. Computation of parameters of the the line-wise segment for safe flight-by and the S-wise returning manoeuver

To provide safe flight-by, the returning manoeuvering must not cause decreasing the distance between the aircrafts smaller than the given safe distance tolerance R_s. In the considered Example 1 (Fig. 3) the aircraft 2 has superiority in the velocity value ($V_2 > V1$), and let it be more agile.

Then if the returning manoeuver begins just after the instant T^* and is performed with the maximal angular velocity $\omega_{2\max}$ at the maximal value of the control $u_2 = 1$, this would bring to unacceptable (albeit for a short time) decreasing the current distance $R(t)$ below the tolerance R_s.

If so, then strictly speaking, the fast return (Fig. 9) of the aircraft 2 onto its original trace (the thin line in dashes) must be realized on some time interval (after the instant T^*) by a complicated control proving the equality

$$u_2(t, x_1(t), z_1(t), x_2(t), z_2(t)) : R(u_2, t) = R_\mathrm{s}. \tag{34}$$

To exclude complicated computational procedures, in Technological Rules [7] some special time

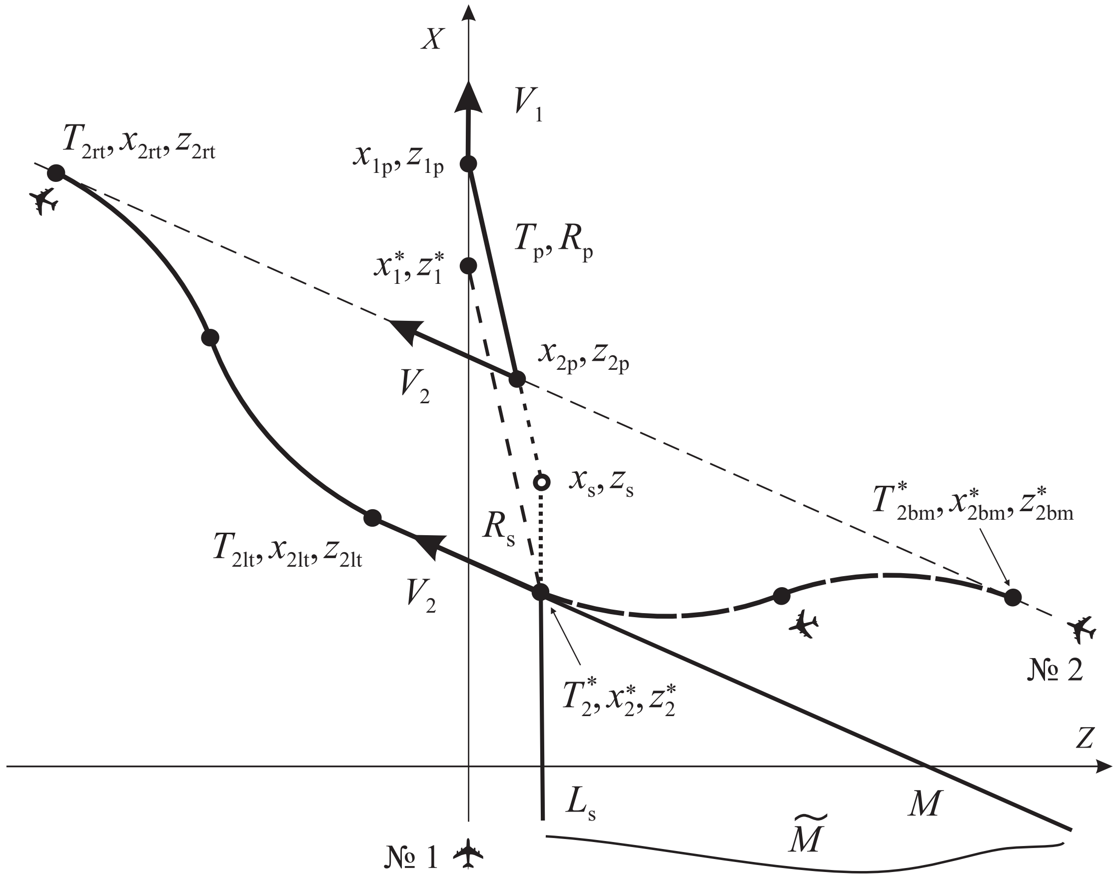

Figure 9: The segment for safe flight-by and the returning manoeuver

interval of the length $\tau_{\rm lt}$ is introduced for *obligatory keeping* the course (of the manoeuvering aircraft) unchanged after the worst encounter.

In practice [3–6], as a consequence of high velocities of contemporary aircrafts, necessary interval for the safe flight-by is prescribed on the value $\tau_{\rm lt} \sim 15 - 20$ seconds. Such a length does not essentially increase the resultant time length of the whole resolving manoeuver.

Under the given value $\tau_{\rm lt}$ of the flight-by segment, the following computations of the instant $T_{\rm 2lt}$ and coordinates $x_{\rm 2lt}, z_{\rm 2lt}$ of the point of its termination are performed:

$$T_{\rm 2lt} = T^* + \tau_{\rm lt}, \quad x_{\rm 2lt} = x_2^* + V_2\tau_{\rm lt}{\rm Cos}\Psi_{\rm Tr}, \quad z_{\rm 2lt} = z_2^* + V_2\tau_{\rm lt}{\rm Sin}\Psi_{\rm Tr}. \tag{35}$$

For the input data of Example 1, parameters of the segment of the safe flight-by are: the time length $\tau_{\rm lt} = 10$ sec, $T_{\rm 2lt}^* = 214.0$ sec, $x_{\rm 1lt}^* = 570.1$ m, $z_{\rm 1lt}^* = 0$ m, $x_{\rm 2lt}^* = -2936.0$ m, $z_{\rm 2lt}^* = 128.8$ m, the distance between the aircrafts at the end of this segment $R_{1-2}^*(T_{\rm 2lt}^*) = 3508.4$ m. Recall that at the instant of the worst encounter the provided distance was $R_{\rm min} = R_{\rm s} = 3000$ m. The segment provides the increasing of the distance (since here the derivative in time of the distance is strictly positive), and both on the segment and at its end the safe distance tolerance is not violated.

The further returning manoeuver is carried out with the minimal time length equal to the length of the previous avoidance manoeuver τ^*.

The instant $T_{\rm 2rt}$ and coordinates $(x_{\rm 2rt}, z_{\rm 2rt})$ of the return point of the manoeuvering aircraft 2 on its original trace are computed

$$\begin{gathered} T_{\rm 2rt} = T_{\rm 2lt} + \tau^*, \quad \widetilde{x} = x_{\rm 2lt} + \Delta B^*/|{\rm Sin}\Psi_{\rm Tr}|, \\ x_{\rm 2rt} = \widetilde{x} - (x_{\rm 2bm} - x_2^*), \quad z_{\rm 2rt} = z_{\rm 2lt} - (z_{\rm 2bm} - z_2^*). \end{gathered} \tag{36}$$

For the input data of Example 1, parameters of the returning manoeuver are: the time length $\tau_{\rm 2rt}^* = \tau^* = 63.2$ sec, $T_{\rm 2rt}^* = 277.2$ sec, $x_{\rm 2rt}^* = 0$ m, $z_{\rm 2rt}^* = -8850.8$ m. The resultant time length of the whole manoeuver resolving this conflict situation (i.e., the resultant time of the aircraft 2 being out of its trace) is 136.4 sec and is minimal by construction.

4.5. The choice of an aircraft for manoeuvering

As it was mentioned earlier, four variants of uncooperative (only one aircraft is prescribed for manoeuvering) resolving manoeuvres in the horizontal plane are possible in principle:
– the aircraft 1, to the left of its original trace with counterclock-wise changing its course;
– the aircraft 1, to the right of its original trace with clock-wise changing its course;
– the aircraft 2, to the left of its original trace with counterclock-wise changing its course;
– the aircraft 2, to the right of its original trace with clock-wise changing its course.

By relations (16)–(22) a conflict situation is detected if the minimal distance in forecast of the worst encounter of these aircrafts is smaller than the given safe distance tolerance, and the type of the conflict situation is defined (similarly to (23)).

By relations similar to (27)–(30),(32) on the basis of the described iteration procedure, the condition of existence of a resolving manoeuver (the condition of solvability of a conflict situation) is analyzed for each mentioned variant of possible manoeuver.

Note that this analysis is constructive since the optimal avoidance manoeuver is found directly (if exists) by the procedure itself. For each variant with satisfied solvability condition all parameters of the resolving manoeuver are calculated by relations similar to (33),(35),(36):
– the minimal needed value of the lateral deviation ΔB^*;
– the time length τ^* of the avoidance and returning manoeuver;
– the sign $Sign(u^*)$ of the extremal control u^* on the first fragment of avoidance;
– the sign $-Sign(u^*)$ of the extremal control on the second fragment of avoidance;
– the sign $-Sign(u^*)$ of the extremal control on the first fragment of returning;
– the sign $Sign(u^*)$ of the extremal control on the second fragment of returning;
– the instant $T^*_{\rm bm}$ of the beginning of avoidance;
– the instant T^* of termination of avoidance;
– the instant $T^*_{\rm bm} + 0.5\tau^*$ of switching the control of avoidance;
– the instant $T^*_{\rm lt}$ of termination of the segment for the safe flight-by and the beginning of the returning manoeuver;
– the instant $T^*_{\rm lt} + 0.5\tau^*$ of switching the control of returning;
– the instant $T^*_{\rm rt}$ of termination of the aircraft return onto its original trace.

The calculated data on the realizable variants of the resolving manoeuvres are delivered to the air traffic manager. On the basis of these data such an aircraft and such a resolving manoeuver is chosen that, for example, has the minimal resultant time length $2\tau^* + \tau_{\rm lt}$, or provides the minimal value ΔB^* of the minimal needed lateral deviation of the manoeuvering aircraft from its original trace, or one that excludes appearing the secondary (stipulated) conflict situations with other aircrafts during the possible resolving manoeuver. In practice, often the aircraft, which forecast position happens to be in the rear half-sphere of another aircraft, is prescribed for manoeuvering.

On the last reason, in considered Example 1 the aircraft 2 is preferable for manoeuvering in the counterclock-wise direction, but in peculiar Example 2, the clock-wise resolving manoeuver of the aircraft 1 towards the aircraft 2 and the similar counterclock-wise resolving manoeuver of the aircraft 2 towards the aircraft 1 only exist.

5 SUMMARY AND CONCLUSIONS

Underline constructive character and universality of the approach considered. Analysis of existence conditions (solvability conditions), constructing the resolving manoeuver and computation of its parameters can be implemented for any parameters of conflicting aircrafts on velocity, constraints on the lateral acceleration and the maximal angular velocity of the course changing, arbitrary initial positions and angles of intersection of the traces.

Elaborated approach successfully works in situations of evident collision (as in [9, Fig. 1, Fig. 2]), in degenerated or uncertain results of the worst encounter forecast.

Since it has been succeeded to include the main demands of the Technological Rules for air traffic management [7] directly in formulation of the resolution problem, the algorithms obtained are representative and practically meaning.

Application of the standard structures to constructing the resolving manoeuvres allows to elaborate rather simple algorithms and the software that can work in the real time mode.

In cases, when system (1) can be applied to description of aircraft motion in the vertical plane, the elaborated approach and corresponding algorithms successfully work in problems of resolving conflict situation on the vertical plane [2].

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