OUTPUT FEEDBACK STABILIZATION OF LINEAR SYSTEMS WITH PARAMETRIC NOISE

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Motivation of the Problem

Consider a control system with parameter uncertainty described by the following differential equation

$$\dot{x} = Ax(t) + Bu(t) + \sum_{i=1}^{p} \gamma_i(t)[A_i x(t) + B_i u(t)] \tag{1}$$

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and its stochastic counterpart in the form of the following Ito equation:

$$dx(t) = [(A + \alpha I)x(t) + Bu(t)]dt + \sum_{i=1}^{p} \sigma_l [A_i x(t) + B_i u(t)] dw_i(t). \tag{2}$$

Motivation of the Problem (cont.)

Bernstein (1987) used the model (2) with the right shift and multiplicative noise for investigation of robust stability of system (1) over a specified range of deterministic parameter variations. It allows to obtain both static and dynamic output feedback robust stabilizing controllers as a solution of nonstandard stochastic LQR problems.

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Let

$$y(t) = Cx(t), \quad t \geq 0, \tag{3}$$

be output vector for both the systems and uncertainties in (1) $[\gamma_1, \ldots, \gamma_p] : [0, \infty) \to \mathbb{R}^p$, are measurable and satisfying the inequalities

$$|\gamma_i(t)| \leq \delta_i, \quad i = 1, \ldots, p. \tag{4}$$

Motivation of the Problem (cont.)

Bernstein had shown that if the stochastic system (2), (3) is mean square stabilizable via static or dynamic output feedback and

$$\alpha > \frac{1}{2}\sum_{i=1}^{p}\frac{\delta_i^2}{\sigma_i^2}, \qquad (5)$$

then the uncertain deterministic system (1), (3) is robust stabilizable via the same output feedback.

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then the uncertain deterministic system (1), (3) is robust stabilizable via the same output feedback.

Unfortunately don't exists easily method of finding of stabilizing gain as a solution of mentioned nonstandard stochastic LQR problems. Moreover these problems belong to generalized "hard problems" in control theory (Polyak and Shcherbakov, 2005)

Statement of the Problem

The purpose of this paper is to describe in a parametric form the set of all mean square stabilizing linear output feedback controllers for the system (2), (3) and based on parametrization results to formulate an LMI based algorithm of synthesis of robust stabilizung controller for the system (1), (3).

Preliminaries. State space partition

The state space of the system (2) can be presented in the form of the following partition

$$\mathbb{R}^n = \mathrm{Im}(C^{\mathrm{T}}) \oplus \mathrm{Ker}(C), \tag{6}$$

where $\mathrm{Im}(C^{\mathrm{T}})$ and $\mathrm{Ker}(C)$ are orthogonal subspaces. For any $x \in \mathbb{R}^n$ we can write

$$x = x_{\mathrm{I}} + x_{\mathrm{K}},$$

where $x_{\mathrm{I}} \in \mathrm{Im}(C^{\mathrm{T}})$ and $x_{\mathrm{K}} \in \mathrm{Ker}(C)$.

State space partition (cont.)

Define the matrices

$$E_{\mathrm{I}} = C^{+}C, \quad E_{\mathrm{K}} = I - E_{\mathrm{I}}, \tag{7}$$

where C^{+} is the Moore-Penrose inverse of C. According to the partition (6) the matrices (7) are projection matrices on $\mathrm{Im}(C^{\mathrm{T}})$ and on $\mathrm{Ker}(C)$ correspondingly. We use the notation $X^{\perp}$ for full rank matrix orthogonal to X.

Stochastic stabilizability definitions

Consider a control law in the form of the static output feedback

$$u(t) = -Gy(t). \tag{8}$$

An important role in the sequel plays also the state feedback control

$$u(t) = -Kx(t). \tag{9}$$

Definition 1 The control law (8) is said to be stabilizing control if it guarantees the exponential stability in the mean square (ESMS) of the closed loop system (2), (8).

Definition 2 The control law (9) is said to be stabilizing if it guarantees ESMS of the closed loop system (2), (9).

Stabilization theorems

Theorem 1 The output feedback control (8) will be stabilizing if and only if there exist positive definite symmetric matrices H and satisfying the inequality

$$(A - BGC)^T H + H(A - BGC) + \sum_{i=1}^{p} \sigma_l^2 (A_i - B_i GC)^T H (A_i - B_i GC) < 0. \quad (10)$$

The function $V(x) = x^T Hx$ is the stochastic Lyapunov function that guarantees ESMS of the closed loop system.

Stabilization theorems

Theorem 2 The output feedback control (9) will be stabilizing if and only if there exist positive definite symmetric matrices H and satisfying the inequality

$$(A - BK)^T H + H(A - BK) + \sum_{i=1}^{p} \sigma_l^2 (A_i - B_i K)^T H (A_i - B_i K) < 0. \qquad (11)$$

The function $V(x) = x^T H x$ is the stochastic Lyapunov function that guarantees ESMS of the closed loop system.

Stabilization sets

$$\mathcal{L}_o = \{H = H^T > 0, \exists G \text{ such that} (A - BGC)^T H + H(A - BGC) + \sum_{l=1}^{p} \sigma_l^2 (A_l - B_l GC)^T H (A_l - B_l GC) < 0\},$$

$$\mathcal{L}_s = \{H = H^T > 0, \exists K \text{ such that} (A - BK)^T H + H(A - BK) + \sum_{l=1}^{p} \sigma_l^2 (A_l - B_l K)^T H (A_l - B_l K) < 0\}.$$

Lyapunov matrices sets

$$\mathcal{X} = \{X = X^T > 0,\ \ B^{\perp}(AX + XA^T + \sum_{l=1}^{p} \sigma_l^2 X A_l^T X^{-1} A_l X) B^{\perp T} < 0\},$$

$$\mathcal{Y} = \{Y = Y^T > 0,\ \ C^{T\perp}(A^T Y + YA + \sum_{l=1}^{p} \sigma_l^2 A_l^T Y A_l) C_i^{T\perp T} < 0\}$$

Riccati matrices sets

$$\mathcal{U}(X) = \{R > 0,\ Q > 0,\ \ A^T X + XA - (XB + \sum_{l=1}^{p} \sigma_l^2 A_l^T X B_l) R_\sigma(X)^{-1} (B^T X + \sum_{l=1}^{p} \sigma_l^2 B_l^T X A_l) + \sum_{l=1}^{p} \sigma_l^2 A_l^T X A_l + Q = 0\}$$

$$\mathcal{W}(Y) = \{V_i > 0,\ W_i > 0,\ \ AY + YA^T - Y(C^T V C - \sum_{l=1}^{p} \sigma_l^2 A_l^T Y^{-1} A_l) Y + W = 0\}$$

Main parametrization theorem

Theorem 3 Let a matrix H be given. Then the following statements are equivalent:

$$H \in \mathcal{L}_o, ; \tag{12}$$

$$H > 0,\ \mathcal{U}(H) \neq \emptyset, \text{ and } \mathcal{W}(H^{-1}) \neq \emptyset; \tag{13}$$

$$H^{-1} \in \mathcal{X} \text{ and } H \in \mathcal{Y}. \tag{14}$$

All the stabilizing static output feedback gains are given by

$$G = R_\sigma(H)^{-1}(B^T H + \sum_{l=1}^{p} \sigma_l^2 B_l^T H A_l^T) Q^{-1} C^T (C Q^{-1} C^T)^{-1} + \Theta^{\frac{1}{2}} \Lambda (C Q^{-1} C^T)^{-\frac{1}{2}}, \tag{15}$$

Main parametrization theorem (cont.)

where Λ is arbitrary matrix such that

$\| \Lambda \| < 1, \;\; H \in \mathcal{L}_o, \;\; \{R, Q\} \in \mathcal{U}(H)$ and $\Theta > 0$ is defined by

$$\Theta = R_\sigma(H)^{-1} - R_\sigma(H)^{-1}(B^T H + \sum_{l=1}^{p} \sigma_l^2 B_l^T H A_l) Q^{-1} [Q -$$

$$C^T (C Q^{-1} C^T)^{-1} C] Q^{-1} [R_\sigma(H)^{-1}(B^T H + \sum_{l=1}^{p} \sigma_l^2 B_l^T H A_l)]^T.$$

State feedback case

Theorem 4 Let a matrix H be given. Then the following statements are equivalent.

(I) $H \in \mathcal{L}_s$;

(II) H is the unique positive definite solution to the generalized Riccati equation

$$A^T H + HA - (HB + \sum_{l=1}^{p} \sigma_l^2 A_l^T H B_l) R_\sigma(H)^{-1} (B^T H + \sum_{l=1}^{p} \sigma_l^2 B_l^T H A_l) + \sum_{l=1}^{p} \sigma_l^2 A_l^T H A_l + Q = 0$$

for some $Q > 0$ and $R > 0$;

(III) $H > 0$ and it satisfies

$$B^{\perp}(AH^{-1} + H^{-1}A^T + \sum_{l=1}^{p} \sigma_l^2 H^{-1} A_l^T H A_l H^{-1}) B^{\perp T} < 0,$$

if $BB^T \not> 0$.

All the stabilizing state feedback gains are given by

$$K = R_\sigma(H)^{-1}(B^T H + \sum_{l=1}^{p} \sigma_l^2 B_l^T H A_l)^T) + R_\sigma^{-\frac{1}{2}}(H) \Lambda Q^{\frac{1}{2}}, \qquad (16)$$

where the matrices $H,\ Q,\ R$ are the ones in (II) and Λ is an arbitrary matrix such that $\| \Lambda \| < 1$.

Preliminary remarks

- The set of all the Lyapunov matrices for the stabilizing output feedback $\mathcal{L}_0 = \{H : \ H^{-1} \in \mathcal{X}, \ H \in \mathcal{Y}\}$ is not convex and the problem of finding H remains open as in deterministic case. In the case of the state feedback the set of all Lyapunov matrices is convex: $\mathcal{L}_s = \{H : \ H^{-1} \in \mathcal{X}\}$ and it is equivalent to the set of all the positive definite solutions of the generalized Riccati equation.

Preliminary remarks

- The set of all the Lyapunov matrices for the stabilizing output feedback $\mathcal{L}_0 = \{H : \; H^{-1} \in \mathcal{X}, \; H \in \mathcal{Y}\}$ is not convex and the problem of finding H remains open as in deterministic case. In the case of the state feedback the set of all Lyapunov matrices is convex: $\mathcal{L}_s = \{H : \; H^{-1} \in \mathcal{X}\}$ and it is equivalent to the set of all the positive definite solutions of the generalized Riccati equation.
- The idea is helpful to find some simple additional conditions to the set of solutions of the generalized Riccati equation to be a convex part of the set of Lyapunov matrices for the stabilizing output feedback.

- The case of fixed order dynamic output feedback

$$\dot{x}_c(t) = A_c x_c(t) + B_c y_(t), \tag{17}$$

$$u(t) = C_c x_c(t) + D_c y(t), \tag{18}$$

is easily transformed to the static output feedback for the system

$$d\bar{x}(t) = [(\bar{A}\bar{x}(t) + \bar{B}\bar{u}(t)]dt + \sum_{i=1}^{p} \sigma_l[\bar{A}_i\bar{x}(t) + \bar{B}\bar{u}(t)]dw_i(t), \tag{19}$$

$$\bar{y} = \bar{C}\bar{x}, \tag{20}$$

$$\bar{u} = -\bar{G}\bar{y}, \tag{21}$$

where

$$\bar{x} = \begin{bmatrix} x \\ x_c \end{bmatrix}, \bar{A} = \begin{bmatrix} A + \alpha I & 0 \\ 0 & 0 \end{bmatrix}, \bar{B} = \begin{bmatrix} B & 0 \\ 0 & I_n \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} C & 0 \\ 0 & I_n \end{bmatrix}, \bar{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix},$$

$$\bar{B}_i = \begin{bmatrix} B_i & 0 \\ 0 & 0 \end{bmatrix}, \bar{G} = -\begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}.$$

Mean square stabilization via output feedback

Theorem 5 The system (2), (3) is mean square stabilizable via output feedback if and only if there exist matrices $M = M^T$, $R = R^T > 0$ and L such that the generalized Riccati equation:

$$A^T H + HA - H_\sigma^T R_\sigma(H)^{-1} H_\sigma + \sum_{j=1}^{p} \sigma_j^2 A_j^T H A_j + M = 0, \qquad (22)$$

where $H_\sigma = B^T H + \sum_{j=1}^{p} \sigma_j^2 B_j^T H A_j$ has positive definite solution

$H = H^T$, satisfying the inequality

$$
\begin{aligned}
&(A - BK)^T H + H(A - BK) + \\
&\sum_{j=1}^{p} \sigma_j^2 (A_j - B_j K)^T H (A_j - B_j K) - \\
&S^T R_\sigma(H)^{-1} (B^T H + \sum_{j=1}^{p} \sigma_j^2 B_j^T H (A_j - B_j K)) - \\
&(HB + \sum_{j=1}^{p} \sigma_j^2 (A_j - B_j K)^T H B_j) R_\sigma(H)^{-1} S + \\
&S^T R_\sigma(H)^{-1} \sum_{j=1}^{p} \sigma_j^2 B_j^T H B_j R_\sigma(H)^{-1} S < 0,
\end{aligned} \tag{23}
$$

where

$$S = LE_I - (B^T H + \sum_{j=1}^{p} \sigma_j^2 B_j^T H A_j) E_K, \tag{24}$$

$$K = R_\sigma(H)^{-1}(B^T H + \sum_{j=1}^{p} \sigma_j^2 B_j^T H A_j) \tag{25}$$

The corresponding robust output feedback control has the form of (8), where the gain matrix is defined by the formula

$$G = R_\sigma(H)^{-1}(B^T H + \sum_{j=1}^{p} \sigma_j^2 B_j^T H A_j + L) C^+. \tag{26}$$

LMI based algorithm

1. Assign noise intensities σ_i based on given bounds of parameters uncertainties δ_i $(i = 1, \dots, p)$.

2. Find α from the inequality (5) and replace the matrix A to right shifted $A_\alpha = A + \alpha I$.

3. Find matrix X as a solution of LMIs

$$X > 0,$$

$$B^{\perp}(A_\alpha X + X A_\alpha^T + \sum_{l=1}^{p} \sigma_l^2 X A_l^T X^{-1} A_l X) B^{\perp T} < 0$$

and calculate $P = X^{-1}$

4. Find matrices Q and R, satisfying the relations

$$R_\sigma(P) > 0, \;\; Q > 0,$$

$$A_\alpha^T P + P A_\alpha - (PB + \sum_{l=1}^{p} \sigma_l^2 A_l^T P B_l) R_\sigma(P)^{-1} (B^T H + \sum_{l=1}^{p} \sigma_l^2 B_l^T P A_l) + \sum_{l=1}^{p} \sigma_l^2 A_l^T P A_l + Q = 0.$$

5. Put $H = P, \;\; \varepsilon = \varepsilon_0$.

6. Find matrix K by the formula

$$K = R_\sigma(H)^{-1} (B^T H + \sum_{j=1}^{p} \sigma_j^2 B_j^T H A_j).$$

7. If the inequality

$$(A_\alpha - BK)^T H + H(A_\alpha - BK) +$$
$$\sum_{j=1}^{p} \sigma_j^2 (A_j - B_j K)^T H (A_j - B_j K) -$$
$$S^T R_\sigma(H)^{-1} (B^T H + \sum_{j=1}^{p} \sigma_j^2 B_j^T H (A_j - B_j K)) -$$
$$(HB + \sum_{j=1}^{p} \sigma_j^2 (A_j - B_j K)^T H B_j) R_\sigma(H)^{-1} S +$$
$$S^T R_\sigma(H)^{-1} \sum_{j=1}^{p} \sigma_j^2 B_j^T H B_j R_\sigma(H)^{-1} S < 0,$$

is feasible, then calculate the gain matrix G stop;

else find matrix H as a solution of the LMI optimization problem

$$\mathrm{trace} H \to \max;$$

$$H > 0;$$

$$R_\sigma(H)^{-1}(B^T H + \sum_{l=1}^{p} \sigma_l^2 B_l^T H A_l) =$$

$$R_\sigma(P)^{-1}(B^T P + \sum_{l=1}^{p} \sigma_l^2 B_l^T P A_l) + R_\sigma(P)^{-\frac{1}{2}} \Lambda Q^{\frac{1}{2}},$$

$$\begin{bmatrix} \varepsilon I & \Lambda \\ \Lambda^T & I \end{bmatrix} > 0.$$

8. Put $\varepsilon = \varepsilon + \Delta\varepsilon$

9. If $\varepsilon > 1$, then stop, else go to step 6.

Concluding remarks

- The problems of parametrization and synthesis of mean square stabilizing output feedback controllers for a class of linear systems with multiplicative noise are studied in this paper.
- A parametrization of a class of fixed-order linear output feedback controllers that stabilize in the mean square a given system is presented.
- Necessary and sufficient conditions for output feedback controller to be mean square stabilizing one are obtained.
- Based on these results an LMI-based computational algorithm for robust control synthesis is presented.

Thank you very much for your attention!