INTERNATIONAL CONFERENCE
OPTIMIZATION AND CONTROL
**Dedicated to B.T. POLYAK's 70th anniversary**
Moscow May 19-20, 2005

# On a Variational Problem Associated with a Model of Welfare Economics

A.D. Ioffe,
Technion, Haifa,
ISRAEL

**Plan of the talk:**

1. Statement of the problem
2. A brief account of the theory
3. The model. Two main theorems of welfare economics
4. Variational formulation.
5. Preference mapping vs. utility function
6. Generalized equilibria and unjust distributions

## 1. The variational problem

$$(VP) \qquad \begin{aligned} &minimize \quad \int_T f(t, u(t)) d\mu \\ &s.t. \qquad\qquad \int_T u(t) d\mu = a \end{aligned}$$

Here: $(T, \Sigma, \mu)$ is a measure space;
$f(t, x)$ a normal integrand (extended-real-valued) on $T \times I\!R^n$.

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$$\dot{x} = A(t)x + b(t,u), \quad u \in U(t);$$

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(d) Lipschitz regularity of autonomous variational problems
(Ambrosio-Ascenzi-Buttazzo (1989)).

(Also Berliocchi-Lasri (1973), Artstein (1974), Fusco-Marcellini-Ornelas (1998), Balder (2000) and many others.)

## 2. **Theory** (Ioffe-Tikhomirov (1969))

Let $S(x)$ denote the value function of the problem:

$$S(x) = \inf\{\int_T f(t, x(t))d\mu : \quad \int_T x(t)d\mu = x\}.$$

Then

$$S^*(p) = \int_T f^*(t, p)d\mu;$$

(**A**) Regular case

**Theorem** 1. *Assume that $f(t, \cdot)$ is convex whenever $t$ is an atom. If $a \in \partial S(\bar{p})$ for some $\bar{p} \in \mathrm{int}(\mathrm{dom}\ S^*)$, then $(VP)$ has a solution, and $\bar{x}(t)$ is a solution of $(VP)$ if and only if (it is admissible in the problem and)*

$$f(t, \bar{x}(t)) + f^*(t, \bar{p}) = \bar{p} \cdot \bar{x}(t) \quad \text{a.e..}$$

In other words, $\bar{x}(t)$ maximizes $x \to f(t, x) - \bar{p} \cdot x$ for almost every $t$.

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(**B**) General case

ASSUMPTION: $T$ is a compact metric space and $\mu$ is a regular Borel measure on $T$ . (Not very restrictive in view of Rohlin's characterization of Lebesgue spaces!)

**Definition**. $t$ is a *p-critical* point ($p \in \mathrm{dom}\ S^*$) if

$$p \notin \mathrm{int}(\mathrm{dom} \int_U f^*(t, p)d\mu), \ \forall \text{ nbd } U \ni t.$$

Notation $t \in T(p)$.

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$$p \notin \text{int}\Big(\text{dom } \int_{B(t,\varepsilon)} f^*(t,p)d\mu\Big), \ \forall\, \varepsilon > 0.$$

Notation: $t \in T(p)$.

For a $t \in T(p)$ set

$$Q(t) = \lim_{\varepsilon \to 0} \Big(\text{dom } \int_{B(t,\varepsilon)} f^*(t,\cdot)d\mu\Big).$$

*The following is a relaxation of $(VP)$ in the weak* topology of the space of $\mathbb{R}^n$-valued Radon measures on $T$ whose singular parts consists of at most $n$ atoms in $T(p)$ for a certain $p \in$* dom $S^*$:

$$(GP) \quad \begin{cases} \text{minimize} & \displaystyle\int_T f(t,x(t))d\mu + \sum_{i=1}^{k} s(Q(t_i),w_i) \\ s.t. & \displaystyle\int_T (x(t)d\mu + \sum_{i=1}^{k} w_i d\varepsilon_{t_i}) = a, \end{cases}$$

(Here $s(P,\cdot)$ stands for the support function of $P$, $t_i \in T(p)$ and $\varepsilon_t$ is the unit jump at $t$.)

**Theorem** 2. *Suppose that* $\operatorname{int}(\operatorname{dom} S^*) \neq \emptyset$*. If* $a \in \partial S^*(\bar{p})$ *and* $\bar{p}$ *does not belong to the interior, then (GP) has a solution, and* $y = (x(\cdot), w_1, ..., w_k, t_1, ..., t_k)$ *is a solution of (GP) if and only if (it is admissible in (GP) and)*

$$f(t, x(t)) + f^*(t, \bar{p}) = \bar{p} \cdot x(t) \quad \text{a.e..}$$

*and*

$$\bar{p} \cdot w_i = s(Q(t_i), w_i), \ \forall \ i = 1, ...k.$$

## 3. A model of welfare economics with a measure space of agents

(Hildenbrand (1969)

GIVEN:

a **measure space of agents** $(T, \Sigma, \mu)$ which is a disjoint union of two parts, the **consumer** part $T_c$ and and the **production part** $T_p$;

an **allocation map** which is a set-valued mapping $X(t)$ from $T$ into $I\!R^n$;

a **preference map** which is a set-valued mapping $P(t, x)$ from the restriction of Graph$X$ to $T_c$ into $I\!R^n$ such that $P(t, x) \subset X(t)$ for all $x \in X(t)$ almost everywhere on $T_c$,

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ASSUMPTIONS:

$(\mathbf{A}_1)$ there is a normal integrand $\varphi(t, x)$ on $T_c \times \mathbb{R}^n$ which assumes finite values on the graph of $X(\cdot)$ and which **represents the preference map** in the sense that for any $x \in X(t)$ the preference set, if nonempty, is defined by

$$P(t, x) = \{u \in X(t) : \ \varphi(t, u) < \varphi(t, x)\}.$$

$(\mathbf{A}_2)$ for any $(t, x) \in$ Graph$X$ either $P(t, x) = \emptyset$ or

$$\{u \in X(t) : \ \varphi(t, u) \le \varphi(t, x)\} = \overline{P(t, x)}.$$

(Here the bar denotes the closure.)

We shall denote by $M(t)$ the set of **satiation (full satisfaction) consumptions** of consumer $t$. This is the set of $x$ at which $P(t, x) = \emptyset$. For $x \in M(t)$ we set by definition

$$\overline{P(t, x)} = M(t).$$

$(\mathbf{A}_3)$ the restriction of $\mu$ to $T_c$ is non-atomic; if $t \in T_p$ is an atom, then $X(t)$ is a convex set.

**Some definitions**:

$x(t)$ a **feasible allocation** if

$$x(t) \in X(t) \quad \text{a.e.}; \qquad \int_{T_c} x(t)d\mu = a + \int_{T_p} x(t)d\mu.$$

A feasible allocation $x(t)$ is a **Pareto optimum in the model** if for any other feasible allocation $u(t)$

– either $x(t) \in \overline{P(t, u(t))}$ a.e. on $T_c$,
– or $x(t) \in P(t, u(t))$ on a set of positive measure;

(that is, if either no consumer would prefer $u(t)$ to $x(t)$ or a solid fraction of consumers would prefer $x(t)$ to $u(t)$),

or, in terms of the utility function, if
– either $\varphi(t, u(t)) \geq \varphi(t, x(t))$ a.e. on $T_c$,
– or $\varphi(t, u(t)) > \varphi(t, x(t))$ on a subset of $T_c$ of positive measure.

Given a $p \in \mathbb{R}^n$, a feasible allocation $x(t)$ is a **(pseudo-)equilibrium** with respect to $p$ if for almost every $t \in T_c$

$$p \cdot w \geq p \cdot x(t), \ \forall \ w \in P(t, x(t))$$

and for almost every $t \in T_p$

$$p \cdot w \leq p \cdot x(t), \ \forall w \in X(t).$$

**Two basic questions**:

– whether there is a Pareto optimum in the model;

– whether there is an equilibrium in the model

## Variational formulation

Set

$$f(t,x) = f_\varphi(t,x) = \begin{cases} \varphi(t,x), & \text{if } \ t \in T_c,\ x \in X(t); \\ 0, & \text{if } \ t \in T_p,\ x \in -X(t); \\ \infty, & \text{if } \ t \in T_c,\ x \notin X(t); \\ \infty, & \text{if } \ t \in T_p,\ x \notin -X(t). \end{cases}$$

and consider the problem

$$(VP_\varphi) \qquad \begin{array}{ll} \text{minimize} & \displaystyle\int_T f(t, y(t))d\mu, \\ \text{s.t.} & \displaystyle\int_T y(t)d\mu = a. \end{array}$$

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**Proposition 3**. *Assume* ($\mathbf{A}_1$) , ($\mathbf{A}_2$) . *If* $\bar{y}(t)$ *is a solution of* ($\mathbf{P}$), *then* $\bar{x}(t)$ *defined by*

$$\bar{x}(t) = \begin{cases} \bar{y}(t), & \text{if } \ t \in T_c; \\ -\bar{y}(t), & \text{if } \ t \in T_p \end{cases}$$

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Let as earlier $S_\varphi(x)$ be the value function of $(VP_\varphi)$ (with $x$ replacing $a$ in the right-hand side of the constraint).

**Proposition 4**. *If there is* $\bar{p} \in \operatorname{intdom} S_\varphi^*$ *such that* $a \in \partial S_\varphi^*(\bar{p})$, *then feasible allocation* $\bar{x}(t)$ *which is a Pareto optimum and an equilibrium with respect to* $\bar{p}$.

(The proof is immediate from Theorem 1 and Proposition 3.)

## 5. Preference map vs. utility functions

**Question**: *assuming that the preference map can be represented by a utility function, which properties of $X$ and $P$ may guarantee the existence of a $\varphi$ for which the basic conditions of the propositions hold?* These are:

(a) there is a $\bar{p}$ such that $a \in \partial S^*(\bar{p})$;

(b) this $\bar{p}$ belongs to the interior of dom $S^*$.

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Let $$X_c = \int_{T_c} X(t) d\mu \qquad \text{and} \qquad X_p = \int_{T_p} X(t) d\mu$$

(the sets of possible **total** consumptions and supplies)

A sufficient condition for

(a) *there is a* $\bar{p}$ *such that* $\bar{a} \in \partial S^*(\bar{p})$

to hold:

**Proposition 5**. *Set*

$$W = X_c - X_p.$$

*Then*

*(a) If* $a \notin \mathrm{ri}W$*, then there is a price vector* $p \neq 0$ *such that every feasible allocation is a pseudo-equilibrium with respect to* $p$*;*

*(b) If* $a \in \mathrm{ri}W$*, then for any utility function associated with* $P(t,x)$ *there is a price vector* $\bar{p}$ *such that* $a \in \partial S^*_\varphi(\bar{p})$ *.*

A sufficient conditions for

(b) *the* $\bar{p}$ *belongs to the interior of* dom $S^*_\varphi$

to hold:

**Proposition 6**. *Assume that*

*(i)* $X(t)$ *is integrably bounded on* $T_p$ *(*that is the set $X_p$ of possible total supplies is bounded*)*

*(ii)* $M(t) \neq \emptyset$ *for almost all* $t \in T_c$ *and* $\delta_M(t,0) = \sup\{\|u\| : \ u \in M(t)\}$ *is* $\mu$*-summable on* $T_c$ *(*that is for almost every consumer there is a "full satisfaction" consumption and the set of total full satisfaction consumptions is bounded*)*

*(iii) for all* $x \in X(t)$*, the excess* $r(t,x) < \infty$ *of* $P(t,x)$ *from* $M(t)$ *is finite a.e.on* $T_c$ *(*that is almost every consumer is "reasonable" in defining preferences*)*

*Then there is a normal integrand* $\varphi$ *representing* $P(t,x)$ *and such that* dom $S^*_\varphi = I\!R^n$.

**Proposition 7**. *Suppose again that* $X(t)$ *is integrably bounded on* $T_p$*. If* $X_c$ *does not contain lines (*that is no two coalitions of consumers have conflicting preferences*), then there is a normal integrand* $\varphi(t,x)$ *representing* $P(t,x)$ *and such that*

$$\text{int}(\text{dom } S^*_\varphi) \neq \emptyset.$$

## 6. Generalized equilibria and "unjust distributions"

Here we assume that $T_c$ and $T_p$ are disjoint metric compact sets.

$y = (x(\cdot), w_1, ..., w_k, t_1, ...t_k), \quad (k \leq n)$ is a **generalized allocation** if

(i) $x(\cdot)$ is an allocation, that is

$$x(t) \in X(t) \quad \text{a.e.}; \quad \int_{T_c} x(t)d\mu = a + \int_{T_p} x(t)d\mu;$$

(ii) $t_i \in T_c$ and $s(Q(t_i), w_i) < \infty$, where

$$Q(t) = \lim_{\varepsilon \to 0} \Big(\text{dom} \int_{B(t,\varepsilon)} f^*(t, \cdot)d\mu\Big).$$

A generalized allocation is a **generalized (pseudo-)equilibrium** with respect to $p$ if

(i) $x(\cdot)$ is a pseudo-equilibrium, that is for almost every $t \in T_c$

$$p \cdot w \geq p \cdot x(t), \ \forall \ w \in P(t, x(t))$$

and for almost every $t \in T_p$

$$p \cdot w \leq p \cdot x(t), \ \forall w \in X(t).$$

(ii) $$p \cdot w_i \geq p \cdot w, \quad \forall \ w \in Q(t_i), \ i = 1, .., k).$$

*(That is: a generalized equilibrium consumption is an "equilibrium distribution" in which arbitrarily small coalitions (around $t_i$) grasp a sizable portion of consumer goods!)*

**Question**: *Do there exist models with "inherent injustice" in which for a certain* $a$ *only generalized equilibria appear for any utility function associated with the preference map?*

**Example**: a model of pure consumption, with $T_c = T = [0, 1/2]$,

$$X(t) = I\!R^2_+ = \{x \in I\!R^2 : \ x_i \geq 0\}, \quad \forall\, t.$$

and $P(t,x)$ defined by the utility function

$$\varphi(x) = \max\{-2(x_1 - tx_2), -2(x_2 - tx_1), -x_1 + tx_2, -x_2 + tx_1\}.$$

Here

$$P(t,x) = (z(t,x) + K(t)) \bigcap R^2_+; \quad \text{where}$$

$$K(t) = \{x \in R^2_+ : \ x_1 \geq tx_2;\ x_2 \geq tx_1\}$$

and

$$z(t,x) = -\frac{1}{1-t} \begin{cases} (\varphi(x), \varphi(x)), & \text{if } \ \varphi(x) \leq 0 \\ \frac{1}{2}(\varphi(x), \varphi(x)), & \text{if } \ \varphi(x) \geq 0 \end{cases}$$

In this model "ordinary equilibria" exist only if both components of the initial endowment vector $a$ are equal

**Open problem**: find general conditions on $X(t)$ and $P(t, x)$ which do not exclude unboudedness of preference sets and guarantee that for any initial endowment vector $a$ there is an "ordinary" equilibrium.