Stackelberg-Nash Equilibrium in Coflicts with A Lider: Numerical Procedure

Alex Poznyak, Samuel Moya and Martin Godoy *†

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Abstract

In this paper we suggest a new numerical procedure for the calculation of a Stackelberg-Nash equilibrium point in a static N-person noncooperative unconstrained finite game with one leader and $(N-1)$ followers. This procedure involves a random search in stochastic approach to find the leader's strategy combined with the time-step solution of a linear programming problem that finds an unique Nash-equilibrium in the game played by the followers. The optimality conditions for a startegy be an unique Nash equilibrium in followers game are derived based on the strong convexity of the $\delta-$regularized loss function. A numerical procidure combining linear programming method with stochastic gradient search technique, shows a nice workability of the proposed approach.

1 Introduction

The Nash equilibrium solution concept provides a reasonable noncooperative equilibrium solution when the roles of the players are symmetric, that is to say, when no single player dominates the decision process. However, there are another types of noncooperative decision problems wherein one of the players has an ability to enforce his strategy on the other player(s), and for such decision problem one has to introduce a hierarchical equilibrium solution concept. Such concept leads to the, so-called, *Stackelberg equilibrium* solution where the player who holds the powerful position is called the *leader* and the others players reacting (rationally) to the leader's decision are called the *followers*. This paper follows the lead of [1], [6], and [8] and deals with a hierarchical multi-participant noncooperative finite game without constraints. This game consists

*Alex Poznyak and Samuel Moya Ochoa are with CINVESTAV, IPN, Department of Automatic Control, México D.F., AP 14-740, México, apoznyak(smoya)@ctrl.cinvestav.mx

†Martin Godoy Alcantar is from Instituto Mexicano del Petróleo, Eje Central Lázaro Cárdenas 152, CP 07730, México D.F., México, mgodoy@imp.mx

of $N-$players: one leader and $(N-1)$ followers. Each player has a finite number of actions and the complete information on the corresponding losses for the followers is assumed to be available.

The Stackelberg solution in nonzero-sum static games was first introduced by H. Von Stackelberg (1934) within the context of economic competition and it was later extended to dynamic games primarily in the works of Chen and Cruz, Jr (1972) and Simaan and Cruz, Jr (1973a,b) who also introduced the concept of a feedback Stackelberg solution with the restriction that the follower's response was unique for each strategy of a leader. Its counterpart in the context of infinite games was first studied in Basar and Olsder (1980). For a discussion on a unified solution concept, which Nash, Stackelberg, minimax solution (and also cooperative solution) follow as special cases from, see Blaquière (1976).

Some news issues are presented in this paper. *First*, a random search technique of the stochastic projection gradient approach is used to find the leader's strategy that provides a Stackelberg equilibrium, i.e., that minimizes his losses when the followers are playing a Nash equilibrium strategies. *Second*, the "$\delta-$regularized joint payoff function" is introduced to characterize the unique Nash equilibrium for the followers and the problem of the determination of a **unique** Nash equilibrium point is shown to be equivalent to the solution of a geometric (polylinear) programming minimization problem (with quadratic regularizing terms) subject to geometric constraints. *Third,* a linear programming (LP) approach in the complementary variables of the followers as well as in the original players' strategy variables is suggested to transform the obtained specific polylinear programming minimization problem into a LP problem and to compute a unique Nash equilibrium point for followers.

2 Game Description

Here is considered two levels of hierarchy in decision making one leader, that always is considered as the player one, and $(N-1)$-players are followers. The followers react to the leader's announced strategy by playing according to the Nash equilibrium concept among themselves.

2.1 Stackelberg Equilibrium Point

The *loss function* $V_\delta^1(p)$ for the leader depends on the strategies of all the other players as well as on his own strategy, and is given by

$$V_\delta^1\left(p^1,...,p^N\right) := \sum_{j_1=1}^{N_1}\cdots\sum_{j_N=1}^{N_N} V^1_{j_1,\ldots,j_N}\left(p^1\right)\prod_{i=1}^{N} p^i_{j_i} + \frac{\delta}{2}\left\|p^k\right\|^2 \\ \left(k=\overline{1,N}\right),\ \delta>0 \tag{1}$$

where $V^1_{j_1,\ldots,j_N}$ denotes the loss incurred by the leader if the participant i selects the corresponding action j_i, the value $p^i_{j_i}$ $\left(j_i=\overline{1,N_i}, i=\overline{1,N}\right)$ is the preference corresponding to the probability for the i^{th} player to select the action j_i. Using a mixed strategy, the leader wishes to minimize his losses and can enforce his strategy over the rest of the players. In the next definition a precise definition of the hierarchical equilibria is given.

Definition 1 *For the $N-$person finite game with one leader and $(N-1)$ followers $(p^i, i=\overline{2,N})$ $p^{1,*} \in S^{N_1}$ is a hierarchical equilibrium strategy for the leader if*

$$\begin{array}{c} \max_{p^i \in R^F(p^{1,*}) \times S^{N_i}, i=\overline{2,N}} V^1(p^{1,*}, p^2, ..., p^N) \\ = \min_{p^1 \in S^{N_1}} \max_{p^i \in R^F(p^1) \times S^{N_i}, i=\overline{2,N}} V^1(p^1, p^2, ..., p^N) \end{array} \tag{2}$$

where $R^F(p^1)$ is the optimal response set of the followers' group and is defined for each $p^1 \in S^{N_i}$ by

$$R^F(p^1) = \left\{ \left(\bar{p}^2, ..., \bar{p}^N\right) \in \prod_{i=2}^{N} S^{N_i} : V^i(p^1, \bar{p}^2, ..., \bar{p}^N) \le V^i(p^1, \bar{p}^2, ..., p^i, ..., \bar{p}^N) \right\}$$

Any $(p^{2,*}, ..., p^{N,*}) \in R^F(p^{1,*})$ is a corresponding optimal strategy pair for the followers' group.

2.2 Followers' game description

The $(N-1)$-person matrix Nash game to be considered can be described in terms of the *individual mixed strategy vector* $p^k \in R^{N_k}$ $\left(k=\overline{2,N}\right)$. Let us denote by $p \in R^{\overline{N}}$ the *simultaneous stationary mixed strategies* of all players, where $R^{\overline{N}} := R^{N_2} \times R^{N_3} \times \cdots \times R^{N_N}$ and $\overline{N} := \sum_{k=2}^{N} N_k$. The *allowed strategies* will be limited by requirement that each vector p^k represents a stationary mixed strategy and, hence, it is a probability vector belonging to the *simplex* S^{N_k} defined by

$$S^{N_k} := \left\{ \begin{array}{c} p^k = \left(p_1^k, ..., p_{N_k}^k\right) \in R^{N_k} \mid p_{j_k}^k \ge 0, \sum_{j_k=1}^{N_k} p_{j_k}^k = 1 \\ \left(k=\overline{2,N}\right) \end{array} \right\} \tag{3}$$

So, the joint strategy vector p belongs to the set S $\left(p \in S \subset R^{\overline{N}}\right)$ where

$$S := S^{N_2} \times S^{N_3} \times \cdots \times S^{N_N} \tag{4}$$

The δ-regularized *loss function* $V^k(p)$ for the k^{th} player depends on the strategies of all the other players as well as on his own strategy, and is given by

$$\begin{array}{c} V_\delta^k\left(p^2, ..., p^N \mid p^1\right) := \sum_{j_2=1}^{N_2} \cdots \sum_{j_N=1}^{N_N} \tilde{V}_{j_2,...,j_N}^k\left(p^1\right) \prod_{i=2}^{N} p_{j_i}^i + \frac{\delta}{2}\left\|p^k\right\|^2 \\ \tilde{V}_{j_2,...,j_N}^k\left(p^1\right) := \sum_{j_1=1}^{N_1} V_{j_1,j_2,...,j_N}^k \, p_{j_1}^1, \qquad k=\overline{2,N} \end{array} \tag{5}$$

where $(\delta > 0)$ and $V^k_{j_2,...,j_N}$ denotes the loss incurred by the k^{th} player if the participant i selects the corresponding action j_i, the value $p^i_{j_i}$ $\left(j_i = \overline{1, N_i}, i = \overline{2, N}\right)$ is the preference corresponding to the probability for the i^{th} player to select the action j_i. Equation (5) show that the function $V^k(p)$ is quadratic in $R^{\overline{N}}$. Using his stationary mixed strategy, each player wishes to minimize his losses.

With this formulation, the *Nash-equilibrium point* of the $(N-1)$-person matrix game without constraints, according to [2], is given by a point $p^* \in R^{\overline{N}}$ satisfying

$$\begin{array}{c} V^k_\delta\left(p^{2,*}\left(p^1\right), ..., p^{N,*}\left(p^1\right) \mid p^1\right) = \\ \min\limits_{p^k \in S^{N_k}} \left\{V^k_\delta\left(p^{2,*}\left(p^1\right), ..., p^{k-1,*}\left(p^1\right), p^k, p^{k+1,*}\left(p^1\right), ..., p^{N,*}\left(p^1\right)\right)\right. \\ \left. \mid \left(p^{2,*}\left(p^1\right), ..., p^{k-1,*}\left(p^1\right), p^k, p^{k+1,*}\left(p^1\right), ..., p^{N,*}\left(p^1\right)\right) \in Q^k\right\} \end{array} \quad (6)$$

This equilibrium point is unique that follows from the strong convexity property of (5). The uniqueness of the Nash-equilibrium point with $\delta = 0$ can be guaranteed only for the games satisfying the so -called *strict concavity condition* [1], [3]. When $\delta > 0$ this specific condition holds automatically.

3 Joint Loss Function and Nash Equilibrium

Following the approach, presented in [1], let us introduce the joint δ-regularized loss function $\rho_\delta\left(p, q \mid p^1\right)$ defined as

$$\begin{array}{c} \rho_\delta\left(p, q \mid p^1\right) := \sum\limits_{k=2}^{N} \left[V^k_\delta\left(p^2, ..., p^{k-1}, q^k, p^{k+1}, ..., p^N \mid p^1\right) - \right. \\ \left. V^k_\delta\left(p^{2,*}\left(p^1\right), ..., p^{N,*}\left(p^1\right) \mid p^1\right)\right] \end{array} \quad (7)$$

for any $(p, q) \in (S \times S)$ where:

$$V^k_\delta\left(p^{\hat{k}}, q^k \mid p^1\right) = \sum_{j_2=1}^{N_2} \cdots \sum_{j_N=1}^{N_N} \tilde{V}^k_{j_2,...,j_N}\left(p^1\right) p^{\widehat{k}}_{j_2,..,j_{k-1},j_{k+1},..,j_N} q^k_{j_k} + \frac{\delta}{2}\left\|q^k\right\|^2 \quad (8)$$

with

$$\begin{array}{c} p^{\widehat{k}}_{j_2,...,j_{k-1},j_{k+1},...,j_N} = p^{\widehat{k}}_{j_{\widehat{k}}} := \prod\limits_{s=2, s\neq k}^{N} p^s_{j_s} \in S^{N_{\hat{k}}}; \\ N_{\widehat{k}} = \prod\limits_{s=2, s\neq k}^{N} N_s \\ \hat{p} = \left(p^{\hat{2},\intercal}, ..., p^{\hat{N},\intercal}\right)^\intercal \in R^{\hat{N}} := R^{N_{\hat{2}}} \times R^{N_{\hat{3}}} \times \cdots \times R^{N_{\hat{N}}} \end{array} \quad (9)$$

being referred to as the *extended counter coalition vector*.

Example 2 *For a three player game with two action for each one we have*

$$\begin{array}{l} p^{\widehat{2}} := \left(p^3_1 p^4_1 \quad p^3_1 p^4_2 \quad p^3_2 p^4_1 \quad p^3_2 p^4_2 \right)^\intercal \\ p^{\widehat{3}} := \left(p^2_1 p^4_1 \quad p^2_1 p^4_2 \quad p^2_2 p^4_1 \quad p^2_2 p^4_2 \right)^\intercal \\ p^{\widehat{4}} := \left(p^2_1 p^3_1 \quad p^2_1 p^3_2 \quad p^2_2 p^3_1 \quad p^2_2 p^3_2 \right)^\intercal \end{array} \quad (10)$$

Using Kakutani's fixed point theorem, the following important result, closed to ones in [1] and , can be proven .

Theorem 3 *A strategy $p^*\left(p^1\right)$ is an equilibrium point (in Nash sense (6)) in $(N-1)$-person matrix game (4, 5) of followers under a fixed strategy p^1 of the leader if and only if*

$$\min_{q\in S}\rho_\delta\left(p^*\left(p^1\right),q\mid p^1\right)=\rho_\delta\left(p^*\left(p^1\right),p^*\left(p^1\right)\mid p^1\right)=0 \qquad (11)$$

The equilibrium point $p^*\left(p^1\right)$ is unique for any $\delta>0$.

Proof. a) Necessity. Suppose that $p^*\left(p^1\right)$ is an equilibrium point for the $(N-1)$-person matrix game (4, 5) of followers under a fixed strategy p^1 of the leader. Then (6) holds, that is, for all $k=2,...,N$

$$V_\delta^k\left(p^{2,*}\left(p^1\right),...,p^{N,*}\left(p^1\right)\mid p^1\right)\le$$
$$V_\delta^k\left(p^{2,*}\left(p^1\right),...,p^{k-1,*}\left(p^1\right),p^k,p^{k+1,*}\left(p^1\right),...,p^{N,*}\left(p^1\right)\mid p^1\right)$$

In view of this inequalities and using (7) it follows

$$\rho_\delta\left(p^*\left(p^1\right),q\mid p^1\right)\ge 0$$

with strict inequality holding (in view of the strict convexity of ρ_δ in q if $\delta>0$) unless $q=p^*\left(p^1\right)$.

b) Sufficiency. Suppose that (11) holds. Then it follows

$$\rho_\delta\left(p^*\left(p^1\right),p^*\left(p^1\right)\mid p^1\right)\le\ \rho_\delta\left(p^*\left(p^1\right),q\mid p^1\right) \qquad (12)$$

Choosing then

$$q=\left(\left(p^{2,*}\left(p^1\right)\right)^\intercal,\dots,\left(q^k\right)^\intercal,\dots,\left(p^{N,*}\left(p^1\right)\right)^\intercal\right)$$

in (12) it follows:

$$\sum_{i=2}^N\left[\sum_{j_2=1}^{N_2}\cdots\sum_{j_N=1}^{N_N}\tilde V^i_{j_2,...,j_N}\ \left(p^1\right)\prod_{s=2}^N p_{j_s}^{s,*}\left(p^1\right)+\frac{\delta}{2}\left\|p^{i,*}\left(p^1\right)\right\|^2\right]\le$$
$$\sum_{\substack{i=2\\ i\ne k}}^N\left[\sum_{j_2=1}^{N_2}\cdots\sum_{j_N=1}^{N_N}\tilde V^i_{j_2,...,j_N}\ \left(p^1\right)\prod_{s=2}^N p_{j_s}^{s,*}\left(p^1\right)+\frac{\delta}{2}\left\|p^{i,*}\left(p^1\right)\right\|^2\right]+$$
$$\sum_{j_2=1}^{N_2}\cdots\sum_{j_N=1}^{N_N}\tilde V^k_{j_2,...,j_N}\ \left(p^1\right)q^k_{j_k}\prod_{\substack{s=2\\ s\ne k}}^N p_{j_s}^{s,*}\left(p^1\right)+\frac{\delta}{2}\left\|q^k\right\|^2$$

that for all q^k implies

$$\sum_{j_2=1}^{N_2}\cdots\sum_{j_N=1}^{N_N}\tilde V^k_{j_2,...,j_N}\ \left(p^1\right)\prod_{s=2}^N p_{j_s}^{s,*}\left(p^1\right)+\frac{\delta}{2}\left\|p^{k,*}\left(p^1\right)\right\|^2\le$$
$$\sum_{j_2=1}^{N_2}\cdots\sum_{j_N=1}^{N_N}\tilde V^k_{j_2,...,j_N}\ \left(p^1\right)q^k_{j_k}\prod_{\substack{s=2\\ s\ne k}}^N p_{j_s}^{s,*}\left(p^1\right)+\frac{\delta}{2}\left\|q^k\right\|^2$$

or, equivalently, by (8) for all $q^k \in S^{N_k}$

$$V_\delta^k \left(p^{\hat{k},*}\left(p^1\right), p^{k,*}\left(p^1\right) \mid p^1 \right) \leq V_\delta^k \left(p^{\hat{k},*}\left(p^1\right), q^k \mid p^1 \right) \tag{13}$$

Evidently, for any $\delta > 0$ the function $V_\delta^k \left(p^{\hat{k},*}\left(p^1\right), q^k \mid p^1 \right)$ is strictly convex and, hence, it has a unique minimum point which is exactly $q^{k,*} = p^{k,*}\left(p^1\right)$. Therefore, (13) takes place with strict inequality holding unless $q^{k,*} = p^{k,*}\left(p^1\right)$, that is,

$$V_\delta^k \left(p^{\hat{k},*}\left(p^1\right), p^{k,*}\left(p^1\right) \mid p^1 \right) = \min_{q^k \in S^{N_k}} V_\delta^k \left(p^{\hat{k},*}\left(p^1\right), q^k \mid p^1 \right)$$

which is true for all $k = \overline{2,N}$. So, (6) holds that means that $p^*\left(p^1\right)$ is the unique equilibrium point of the game. ■

4 Equilibrium Conditions and the Equivalent Geometric Programming Problem

The equation (11) and the constraints (3) on variables p can be expressed in other equivalent form using the Lagrange function approach (see [4]) with α, and μ being the Lagrange multipliers:

$$L_\delta \left(p^*\left(p^1\right), q, \alpha, \mu \mid p^1 \right) \to \min_{q \in S} \max_{\alpha \geq 0} \max_{\mu \geq 0} \tag{14}$$

where the $\delta-$*regularized Lagrange function* $L_\delta \left(p^*\left(p^1\right), q, \alpha, \mu \right)$ is given by

$$L_\delta \left(p^*\left(p^1\right), q, \alpha, \mu \mid p^1 \right) := \\ \rho_\delta \left(p^*\left(p^1\right), q \mid p^1 \right) - \sum_{k=2}^{N} \alpha^k \left(\sum_{j_k=1}^{N_k} q_{j_k}^k - 1 \right) - \sum_{k=2}^{N} \sum_{j_k=2}^{N_k} \mu_{j_k}^k q_{j_k}^k$$

where

$$\alpha := \begin{pmatrix} \alpha^2 & \cdots & \alpha^N \end{pmatrix}^\intercal \tag{15}$$

$$\mu := \begin{pmatrix} \mu_1^2 & \cdots & \mu_{N_2}^2 & \cdots & \mu_1^N & \cdots & \mu_{N_N}^N \end{pmatrix}^\intercal \tag{16}$$

and, using the notation of (9),

$$\rho_\delta \left(p^*\left(p^1\right), q \mid p^1 \right) = \sum_{k=2}^{N} \sum_{j_{\widehat{k}}=1}^{N_{\widehat{k}}} \sum_{j_k=1}^{N_k} V_{j_2,\ldots,j_N}^k \left(p^1\right) p_{j_{\widehat{k}}}^{\widehat{k},*}\left(p^1\right) q_{j_k}^k + \\ \frac{\delta}{2} \sum_{k=2}^{N} \sum_{j_k=1}^{N_k} \left(q_{j_k}^k\right)^2 - \sum_{k=2}^{N} V_\delta^k \left(p^{2,*}\left(p^1\right), \ldots, p^{N,*}\left(p^1\right) \mid p^1 \right) = \\ \sum_{k=2}^{N} \left(\left(p^{\widehat{k},*}\left(p^1\right) \right)^\intercal \overline{V}^k q^k + \frac{\delta}{2} \left\| q^k \right\|^2 \right) - \sum_{k=2}^{N} V_\delta^k \left(p^{2,*}\left(p^1\right), \ldots, p^{N,*}\left(p^1\right) \mid p^1 \right)$$

with the matrix $\overline{V}^k$ given by

$$\overline{V}^k = \left[\tilde{V}^k_{j_2,\ldots,j_{k-1},j_k=1,j_{k+1},\ldots,j_N}\left(p^1\right) \vdots\ \tilde{V}^k_{j_2,\ldots,j_{k-1},j_k=2,j_{k+1},\ldots,j_N}\left(p^1\right) \vdots\ \ \ldots \vdots \right.$$
$$\left. \tilde{V}^k_{j_2,\ldots,j_{k-1},j_k=N_k,j_{k+1},\ldots,j_N}\left(p^1\right) \right]$$
$$\forall j_2, \ldots, j_{k-1}, j_{k+1}, \ldots, j_N \tag{17}$$

where the index of the actions of player k grows in the columns and the combination of indexes of the counter coalition grows in the rows of $\overline{V}^k\left(p^1\right)$. Further defining

$$\overline{V} := \begin{bmatrix} \overline{V}^2\left(p^1\right) & 0 & 0 & 0 \\ 0 & \overline{V}^3\left(p^1\right) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \overline{V}^N\left(p^1\right) \end{bmatrix} \tag{18}$$

and

$$\widehat{p} := \left[\begin{array}{ccc} \left(\widehat{p^2}\right)^\intercal & \cdots & \left(\widehat{p^N}\right)^\intercal \end{array}\right]^\intercal \tag{19}$$

the min-max problem (14) can be reformulated as follows:

$$L_\delta\left(p^*\left(p^1\right), q, \alpha, \mu \mid p^1\right) = \widehat{p}^{*,\intercal}\left(p^1\right)\overline{V}\ q + \frac{\delta}{2}\left\|q\right\|^2 - \sum_{k=2}^{N}\alpha^k\left(1 - \sum_{j_k=1}^{N_k} q^k_{j_k}\right)$$
$$-\sum_{k=2}^{N}\sum_{j_k=1}^{N_k}\mu^k_{j_k} q^k_{j_k} \to \min_{q\in S}\max_{\alpha\geq 0}\max_{\mu\geq 0} \tag{20}$$

Defining

$$e^k := \Bigg(\underbrace{\begin{array}{ccc} 1 & \cdots & 1 \end{array}}_{N_k}\Bigg)^\intercal$$

and

$$\overline{e}^k := \Bigg(\underbrace{\begin{array}{ccc} 0 & \cdots & 0 \end{array}}_{\substack{1\times N_2 \\ 1}}, \cdots, \underbrace{\begin{array}{ccc} 1 & \cdots & 1 \end{array}}_{\substack{1\times N_k \\ k}},\ 0\ \cdots\ 0, \cdots, \underbrace{\begin{array}{ccc} 0 & \cdots & 0 \end{array}}_{\substack{1\times N_N \\ N}}\Bigg)^\intercal$$

the Kuhn-Tucker optimality conditions for the Lagrangian (20) can be written as

$$\nabla_q L_\delta\left(p^*\left(p^1\right), q^*\left(p^1\right), \alpha^*\left(p^1\right), \mu^*\left(p^1\right) \mid p^1\right) =$$
$$\overline{V}^\intercal\ \widehat{p}^*\left(p^1\right) + \delta q^*\left(p^1\right) - \sum_{k=2}^{N}\alpha^{k,*}\left(p^1\right)\overline{e}^k - \mu^*\left(p^1\right) = 0 \tag{21}$$

$$\nabla_{\alpha^k} L_\delta\left(p^*\left(p^1\right), q^*\left(p^1\right), \alpha^*\left(p^1\right), \mu^*\left(p^1\right) \mid p^1\right) = 1 - \left(e^k, q^{k,*}\left(p^1\right)\right) = 0$$
$$k = \overline{2, N} \tag{22}$$

$$\begin{array}{c} \mu_{j_k}^{k,*}\left(p^1\right) \cdot q_{j_k}^{k,*}\left(p^1\right) = 0 \\ k = \overline{2, N}, \qquad j_k = \overline{1, N_k} \end{array} \tag{23}$$

or, in view of (18), $\forall k = \overline{2, N} \quad j_k = \overline{1, N_k}$

$$\begin{array}{c} \overline{V}^{k,\intercal}\; p^{\widehat{k},*}\left(p^1\right) + \delta q^{k,*}\left(p^1\right) - \alpha^{k,*}\left(p^1\right) e^k = \mu^{k,*}\left(p^1\right) \geq 0 \\ 1 - \left(e^k, q^{k,*}\left(p^1\right)\right) = 0 \\ \mu_{j_k}^{k,*}\left(p^1\right) \cdot q_{j_k}^{k,*}\left(p^1\right) = 0 \end{array} \tag{24}$$

Remark 4 *Substituting $\mu^{k,*}\left(p^1\right)$ from the first equation in (24) in the third equation it can be shown that*

$$\begin{array}{c} \mu_{j_k}^{k,*}\left(p^1\right) \cdot q_{j_k}^{k,*}\left(p^1\right) = \\ \left(\overline{V}^{k,\intercal}\; p^{\widehat{k},*}\left(p^1\right) + \delta q^{k,*}\left(p^1\right) - \alpha^{k,*}\left(p^1\right) e^k\right)_{j_k} \cdot q_{j_k}^{k,*}\left(p^1\right) = 0 \end{array}$$

that implies

$$\begin{array}{c} \left(p^{\widehat{k},*}\left(p^1\right)\right)^\intercal \overline{V}^k\; q^{k,*}\left(p^1\right) + \frac{\delta}{2}\left\|q^{k,*}\left(p^1\right)\right\|^2 = \alpha^{k,*}\left(p^1\right) - \frac{\delta}{2}\left\|q^{k,*}\left(p^1\right)\right\|^2 \\ k = \overline{2, N} \qquad j_k = \overline{1, N_k} \end{array} \tag{25}$$

So, the minimum in (14) is equal to

$$\sum_{k=2}^{N}\left(\alpha^{k,*}\left(p^1\right) - \frac{\delta}{2}\left\|q^{k,*}\left(p^1\right)\right\|^2\right)$$

Finally, from (11) we have $q^{k,*}\left(p^1\right) = p^{k,*}\left(p^1\right)$ and after substituting this equality in (24) and (25) the optimality conditions can be written as:

$$\begin{array}{c} V_\delta^k\left(p^{2,*}\left(p^1\right), ..., p^{N,*}\left(p^1\right) \mid p^1\right) + \frac{\delta}{2}\left\|p^{k,*}\left(p^1\right)\right\|^2 - \alpha^{k,*}\left(p^1\right) = 0 \\ -\overline{V}^{k,\intercal}\; p^{\widehat{k},*}\left(p^1\right) - \delta p^{k,*}\left(p^1\right) + \alpha^{k,*}\left(p^1\right) e^k \leq 0 \\ 1 - \left(e^k, p^{k,*}\left(p^1\right)\right) = 0 \text{ for } \forall k = \overline{2, N} \end{array} \tag{26}$$

where $V_\delta^k\left(p^*\right)$ as in (5).

Remark 5 *When $(N-1) \geq 3$, the vector $p^{k,*}\left(p^1\right)$ can be computed from vectors $p^{\widehat{s},*}\left(p^1\right)$ with $s = \overline{2, N}$ $s \neq k$.For example, for $(N-1) = 3$ and $N_k = 2$ (as it is* *in the previous example) it follows:*

$$p^2 = \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{M^{2,\widehat{4}}} \underbrace{\begin{bmatrix} p_1^2 p_1^3 \\ p_1^2 p_2^3 \\ p_2^2 p_1^3 \\ p_2^2 p_2^3 \end{bmatrix}}_{p^{\widehat{4}}} = \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{M^{2,\widehat{3}}} \underbrace{\begin{bmatrix} p_1^2 p_1^4 \\ p_1^2 p_2^4 \\ p_2^2 p_1^4 \\ p_2^2 p_2^4 \end{bmatrix}}_{p^{\widehat{3}}}$$

$$p^3 = \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}}_{M^{3,\widehat{4}}} \underbrace{\begin{bmatrix} p_1^2 p_1^3 \\ p_1^2 p_2^3 \\ p_2^2 p_1^3 \\ p_2^2 p_2^3 \end{bmatrix}}_{p^{\widehat{4}}} = \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{M^{3,\widehat{2}}} \underbrace{\begin{bmatrix} p_1^3 p_1^4 \\ p_1^3 p_2^4 \\ p_2^3 p_1^4 \\ p_2^3 p_2^4 \end{bmatrix}}_{p^{\widehat{2}}}$$

$$p^4 = \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}}_{M^{4,\widehat{3}}} \underbrace{\begin{bmatrix} p_1^2 p_1^4 \\ p_1^2 p_2^4 \\ p_2^2 p_1^4 \\ p_2^2 p_2^4 \end{bmatrix}}_{p^{\widehat{3}}} = \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}}_{M^{4,\widehat{2}}} \underbrace{\begin{bmatrix} p_1^3 p_1^4 \\ p_1^3 p_2^4 \\ p_2^3 p_1^4 \\ p_2^3 p_2^4 \end{bmatrix}}_{p^{\widehat{2}}}$$

i.e., each vector $p^{k,*}\left(p^1\right)$ can be obtained in $(N-1)$-different forms by summing up the appropriate terms in vectors $p^{\widehat{s},*}\left(p^1\right)$. For sure, these computations must be equal when calculated from two different vectors $p^{\widehat{s},*}\left(p^1\right)$ and $p^{\widehat{r},*}\left(p^1\right)$. This property can be expressed as follows

$$\begin{gathered} p^{k,*}\left(p^1\right) = M^{k,\widehat{s}}\ p^{\widehat{s},*}\left(p^1\right) = M^{k,\widehat{r}}\ p^{\widehat{r},*}\left(p^1\right) \\ s = \overline{2,N},\ \ s \neq k, s \neq r \end{gathered} \tag{27}$$

where $M^{k,\widehat{s}}, M^{k,\widehat{r}}$ are, so-called, ***extracting matrices*** *(or product of matrices) which extract the values $p^{k,*}\left(p^1\right)$ from the complementary vectors $p^{\widehat{s},*}\left(p^1\right)$ $(s \neq k)$.*

Theorem 6 *The extended strategy vector $p^*\left(p^1\right) \in R^{\overline{N}}$ is a Nash equilibrium point, i.e., (26) are satisfied, if and only if the pair $\left(p^*\left(p^1\right), \alpha^*\left(p^1\right)\right)$ is the solution to the following geometric programming problem*

$$\begin{gathered} F(p,\alpha) := \left(\rho_\delta\left(p,p \mid p^1\right) + \frac{\delta}{2}\sum_{k=2}^{N}\left\|p^k\right\|^2 - \sum_{k=2}^{N}\alpha^k\right) \to \min_{p\in R^{\overline{N}}, \hat{p}\in R^{\hat{N}}, \alpha^k\in R} \\ \rho_\delta\left(p,p \mid p^1\right) = \sum_{k=2}^{N}\left(p^{k,\intercal}\overline{V}^{k,\intercal}\ p^{\widehat{k}} + \frac{\delta}{2}\left\|p^k\right\|^2\right) \end{gathered} \tag{28}$$

subject to

$$\left.\begin{gathered} -\overline{V}^{k,\intercal}\ p^{\widehat{k}} - \delta p^k + \alpha^k e^k \leq 0 \\ 1 - \left(e^k, p^k\right) = 0,\ \ p^k_{j_k} \geq 0 \\ k = \overline{2,N},\ \ p^{\widehat{k}} \in R^{N_{\widehat{k}}}, N_{\widehat{k}} = \prod_{s=2, s\neq k}^{N} N_s \end{gathered}\right\} \tag{29}$$

For any Nash equilibrium point $\left(p^{*}\left(p^{1}\right), \alpha^{*}\left(p^{1}\right)\right)$ *the following property holds:*

$$F\left(p^{*}\left(p^{1}\right), \alpha^{*}\left(p^{1}\right)\right)=0 \tag{30}$$

Remark 7 *This result for non-regularized case (when* $\delta=0$*) is proven in [7] and called "Generalized Mangasarian-Stone Theorem" (MGMST). So, this theorem is a regularized version of MGMST.*

Proof. a) Necessity. First, note that two first constraints in (29) are the same as the last two in (26). So, from (24) it follows

$$\begin{gathered}\rho\left(p^{*}\left(p^{1}\right), q^{*}\left(p^{1}\right) \mid p^{1}\right)+\delta\left\|q^{k,*}\left(p^{1}\right)\right\|^{2}-\sum_{k=2}^{N} \alpha^{k,*}\left(p^{1}\right)= \\ \sum_{k=2}^{N}\left(q^{k,*}\left(p^{1}\right)\right)^{\intercal}\left(\overline{V}^{k,\intercal}\ p^{\widehat{k},*}\left(p^{1}\right)+\delta q^{k,*}\left(p^{1}\right)-\alpha^{k,*}\left(p^{1}\right) e^{k}\right) \geq 0\end{gathered} \tag{31}$$

and, hence,

$$\min_{q, \alpha} \sum_{k=2}^{N} q^{k,\intercal}\left(\overline{V}^{k,\intercal} p^{\widehat{k},*}\left(p^{1}\right)+\delta q^{k}-\alpha^{k} e^{k}\right) \geq 0 \tag{32}$$

Now, let $q^{*}\left(p^{1}\right)$ be a Nash equilibrium point. Then $\left(q^{*}\left(p^{1}\right), \alpha^{*}\left(p^{1}\right), \mu^{*}\left(p^{1}\right)\right)$ satisfies (26) and the first equation in (26) implies

$$\sum_{k=2}^{N}\left(q^{k,*}\left(p^{1}\right)\right)^{\intercal}\left(\overline{V}^{k,\intercal} p^{\widehat{k},*}\left(p^{1}\right)+\delta q^{k,*}\left(p^{1}\right)-\alpha^{k,*}\left(p^{1}\right) e^{k}\right)=0 \tag{33}$$

and

$$\begin{gathered}\sum_{k=2}^{N}\left(q^{k,*}\left(p^{1}\right)\right)^{\intercal}\left(\overline{V}^{k,\intercal} p^{\widehat{k},*}\left(p^{1}\right)+\delta q^{k,*}\left(p^{1}\right)-\alpha^{k,*}\left(p^{1}\right) e^{k}\right)= \\ \min_{q \in R^{\overline{N}}, \alpha^{k} \in R} \sum_{k=2}^{N} q^{k,\intercal}\left(\overline{V}^{k,\intercal} p^{\widehat{k},*}\left(p^{1}\right)+\delta q^{k}-\alpha^{k} e^{k}\right)\end{gathered} \tag{34}$$

that corresponds (29).

b) Sufficiency. Let $\left(p^{*}\left(p^{1}\right), \alpha^{*}\left(p^{1}\right)\right)$ be a solution of (28) and (29), that is, (34) is satisfied. From (32) it follows

$$\sum_{k=2}^{N}\left(q^{k,*}\left(p^{1}\right)\right)^{\intercal}\left(\overline{V}^{k,\intercal} p^{\widehat{k},*}\left(p^{1}\right)+\delta q^{k,*}\left(p^{1}\right)-\alpha^{k,*}\left(p^{1}\right) e^{k}\right) \geq 0 \tag{35}$$

and, taking into account that

$$\overline{V}^{k,\intercal}\ p^{\widehat{k},*}\left(p^{1}\right)+\delta q^{k,*}\left(p^{1}\right)-\alpha^{k,*}\left(p^{1}\right) e^{k}=\mu^{k,*}\left(p^{1}\right) \tag{36}$$

$$\left(\mu^{k,*}\left(p^{1}\right),\ p^{k,*}\left(p^{1}\right)\right)=0 \tag{37}$$

$$q^{k,*}\left(p^{1}\right)=p^{k,*}\left(p^{1}\right) \tag{38}$$

we derive that

$$\sum_{k=2}^{N}\left(q^{k,*}\left(p^1\right)\right)^{\intercal}\left(\overline{V}^{k,\intercal}p^{\widehat{k},*}\left(p^1\right)+\delta q^{k,*}\left(p^1\right)-\alpha^{k,*}\left(p^1\right)e^k\right)=$$
$$\sum_{k=2}^{N}\left(\left(p^{k,*}\left(p^1\right)\right)^{\intercal}\overline{V}^{k,\intercal}\;p^{\widehat{k},*}\left(p^1\right)+\delta\left\|p^{k,*}\left(p^1\right)\right\|^2-\alpha^{k,*}\left(p^1\right)\right)=0$$

or, in view of (35),

$$\left(p^{k,*}\left(p^1\right)\right)^{\intercal}\overline{V}^{k,\intercal}\;p^{\widehat{k},*}\left(p^1\right)+\delta\left\|p^{k,*}\left(p^1\right)\right\|^2-\alpha^{k,*}\left(p^1\right)=0 \quad k=\overline{2,N} \tag{39}$$

that corresponds the conditions in the first equation in (26) and, hence, it is an equilibrium point. ■

5 Computational Procedure for Unconstrained Finite Static Games: A Linear Programming Approach

Since the variables $p\in R^{\overline{N}}$ and $\widehat{p}\in R^{\widehat{N}}$ are functionally connected (see, for example, (10)), the optimization problem (28)-(29), given in the variables p, can be rewritten as another optimization problem, but with respect to the complementary variables $p^{\widehat{k}}\left(k=\overline{2,N}\right)$ as well as the original variables p^k. It is done in the next theorem which also demonstrates that the original quadratic (geometric) programming problem is equivalent to a linear programming problem (LPP).

Theorem 8 (LPP-representation) *A necessary and sufficient condition that the triple $\left(\widehat{p}^*\left(p^1\right),p^*\left(p^1\right),\alpha^*\left(p^1\right)\right)$ be a solution of (28) subject to (29) is that the tuple $\left(\widehat{p}^*\left(p^1\right),p^*\left(p^1\right),\alpha^*\left(p^1\right),t^*\left(p^1\right)\right)$ is a solution of the following linear programming problem*

$$\overline{F}(t,\alpha)=\sum_{k=2}^{N}t^k-\sum_{k=2}^{N}\alpha^k\to\min_{\widehat{p}\in R^{\widehat{N}},p\in R^{\overline{N}},\alpha^k\in R,t^k\in R} \tag{40}$$

subject to

$$-\overline{V}^{k,\intercal}\;p^{\widehat{k}}-\delta p^k+\alpha^k e^k\leq 0 \tag{41}$$

$$\overline{V}^{k,\intercal}p^{\widehat{k}}+\delta p^k-t^k e^k\leq 0 \tag{42}$$

$$\begin{gathered} M^{k,\widehat{s}}\;p^{\widehat{s}}=M^{k,\widehat{r}}\;p^{\widehat{r}} \\ p^k=M^{k,\widehat{s}}\;p^{\widehat{s}} \\ k,r,s=\overline{1,N},\;\; s\neq k, s\neq r \\ p^k\in S^{N_k},\;\; p^{\widehat{k}}\in S^{N_{\widehat{k}}} \\ N_{\widehat{k}}=\prod_{s=2,s\neq k}^{N}N_s,\quad \widehat{N}=\prod_{i=2}^{N}N_{\widehat{i}} \end{gathered} \tag{43}$$

The following property holds:

$$\overline{F}\left(t^{*}\left(p^{1}\right), \alpha^{*}\left(p^{1}\right)\right)=0 \tag{44}$$

Proof. a) Necessity. Consider a vector $p^{*}\left(p^{1}\right) \in Arg\min_{p \in S} F(p, \alpha)$ in (28) - (29), then by (39) it follows

$$\sum_{k=2}^{N}\left[\left(p^{k,*}\left(p^{1}\right)\right)^{\intercal} \overline{V}^{k,\intercal} \; p^{\widehat{k},*}\left(p^{1}\right)+\delta\left\|p^{k,*}\left(p^{1}\right)\right\|^{2}\right]=\sum_{k=2}^{N} \alpha^{k,*}\left(p^{1}\right)$$

On the other hand, function $\rho_{\delta}\left(p, p \mid p^{1}\right)+\frac{\delta}{2} \sum_{k=2}^{N}\left\|p^{k}\right\|^{2}$ in (28) can be estimated from above by new slack variables t^{k} as

$$\rho_{\delta}\left(p, p \mid p^{1}\right)+\frac{\delta}{2} \sum_{k=2}^{N}\left\|p^{k}\right\|^{2}=\sum_{k=2}^{N} p^{k,\intercal}\left(\overline{V}^{k,\intercal} \; p^{\widehat{k}}+\delta p^{k}\right) \leq \sum_{k=2}^{N} t^{k}$$

if t^{k} satisfies the inequality

$$\overline{V}^{k,\intercal} \; p^{\widehat{k}}+\delta p^{k} \leq t^{k} e^{k}$$

or, in equivalent form,

$$\overline{V}^{k,\intercal} \; p^{\widehat{k}}+\delta p^{k}-t^{k} e^{k} \leq 0 \tag{45}$$

that coincides with (42). So, we have

$$\min_{p \in S}\left[\rho_{\delta}\left(p, p \mid p^{1}\right)+\frac{\delta}{2} \sum_{k=2}^{N}\left\|p^{k}\right\|^{2}\right] \leq \min_{t^{k} \in R} \sum_{k=2}^{N} t^{k}$$

subject to (45). Then, multiplying (45) by $p^{k,*}$ in the optimal point $p^{k}=p^{k,*}$ and summing the results by k we obtain

$$\begin{gathered}\sum_{k=2}^{N}\left(p^{k,*}\left(p^{1}\right)\right)^{\intercal}\left(\overline{V}^{k,\intercal} \; p^{\widehat{k},*}\left(p^{1}\right)+\delta p^{k,*}\left(p^{1}\right)-t^{k,*}\left(p^{1}\right) e^{k}\right)= \\ \sum_{k=2}^{N}\left[\left(p^{k,*}\left(p^{1}\right)\right)^{\intercal} \overline{V}^{k,\intercal} \; p^{\widehat{k},*}\left(p^{1}\right)+\delta\left\|p^{k,*}\left(p^{1}\right)\right\|^{2}\right]-\sum_{k=2}^{N} t^{k,*}\left(p^{1}\right) \leq 0\end{gathered}$$

and

$$\begin{gathered}\overline{F}\left(p^{*}\left(p^{1}\right), \alpha^{*}\left(p^{1}\right)\right)=\sum_{k=2}^{N} t^{k,*}\left(p^{1}\right)-\sum_{k=2}^{N} \alpha^{k,*}\left(p^{1}\right) \geq \\ \sum_{k=2}^{N}\left[\left(p^{k,*}\left(p^{1}\right)\right)^{\intercal} \overline{V}^{k,\intercal} \; p^{\widehat{k},*}\left(p^{1}\right)+\delta\left\|p^{k,*}\left(p^{1}\right)\right\|^{2}\right]-\sum_{k=2}^{N} \alpha^{k,*}\left(p^{1}\right)=0\end{gathered}$$

So, the minimum in (40) is equal to zero and reached when (45) is met with equality. Evidently, the minimum in (28) is also reached with the same arguments.

b) Sufficiency. Now consider the tuple $\left(\widehat{p}^{*}\left(p^{1}\right), p^{*}\left(p^{1}\right), \alpha^{*}\left(p^{1}\right), t^{*}\left(p^{1}\right)\right)$ which is a solution of (40). Multiplying (41) and (42) by $p^{k,*}$ and summing by k, it follows that

$$\sum_{k=2}^{N}\left(p^{k,*}\left(p^{1}\right)\right)^{\intercal}\left(-\overline{V}^{k,\intercal}\ p^{\widehat{k}}-\delta p^{k}+\alpha^{k} e^{k}\right)=$$
$$-\sum_{k=2}^{N}\left[\left(p^{k,*}\left(p^{1}\right)\right)^{\intercal} \overline{V}^{k,\intercal}\ p^{\widehat{k},*}\left(p^{1}\right)+\delta\left\|p^{k,*}\left(p^{1}\right)\right\|^{2}\right]+\sum_{k=2}^{N} \alpha^{k,*}\left(p^{1}\right) \leq 0$$

and, analogously,

$$\sum_{k=2}^{N}\left(p^{k,*}\left(p^{1}\right)\right)^{\intercal}\left(\overline{V}^{k,\intercal}\ p^{\widehat{k},*}\left(p^{1}\right)+\delta p^{k,*}\left(p^{1}\right)-t^{k,*}\left(p^{1}\right) e^{k}\right)=$$
$$\sum_{k=2}^{N}\left[\left(p^{k,*}\left(p^{1}\right)\right)^{\intercal} \overline{V}^{k,\intercal}\ p^{\widehat{k},*}\left(p^{1}\right)+\delta\left\|p^{k,*}\left(p^{1}\right)\right\|^{2}\right]-\sum_{k=2}^{N} t^{k,*}\left(p^{1}\right) \leq 0$$

These two inequalities lead to the following inequality:

$$\sum_{k=2}^{N} t^{k,*}\left(p^{1}\right) \leq \sum_{k=2}^{N}\left[p^{k,*}\left(p^{1}\right) \overline{V}^{k,\intercal}\ p^{\widehat{k},*}\left(p^{1}\right)+\delta\left\|p^{k,*}\left(p^{1}\right)\right\|^{2}\right] \leq \sum_{k=2}^{N} \alpha^{k,*}\left(p^{1}\right)$$

The minimum in (40) is equal to zero and reached when

$$\sum_{k=2}^{N} p^{k,*}\left(p^{1}\right) \overline{V}^{k,\intercal}\ p^{\widehat{k},*}\left(p^{1}\right)+\delta\left\|p^{k,*}\left(p^{1}\right)\right\|^{2}=\sum_{k=2}^{N} \alpha^{k,*}\left(p^{1}\right)$$

that corresponds to the minimum of $F\left(p^{*}\left(p^{1}\right), \alpha^{*}\left(p^{1}\right)\right)$ in (28). ■

6 Stochastic Projection Gradient Technique

To obtain the optimal strategy for the leader one has to solve the following optimization problem:

$$\tilde{V}^{1}(p^{1}) := V_{\delta}^{1}\left(p^{1}, p^{2,*}(p^{1}) \ldots, p^{N,*}(p^{1})\right) \rightarrow \min_{p^{1} \in S^{N_{1}}} \tag{46}$$

Considering $\tilde{V}^{1}(p^{1})$ as a convex function (the numerical calculations confirms this supposition), for this optimization problem resolution the following iterative procedure, corresponding to the projection gradient method (see [4]), may be applied:

$$p_{n}^{1}=\Pi_{S^{N_{1}}}\left\{p_{n-1}^{1}-\gamma_{n} \nabla \tilde{V}^{1}\left(p_{n-1}^{1}\right)\right\}$$

where $\Pi_{S^{N_{1}}}\{\cdot\}$ is the projector to the simplex $S^{N_{1}}$. The realization of this procedure is related to the possibility of the calculation of the components of the matrices $\dfrac{\partial}{\partial p^{1}} p^{k,*}(p^{1}),\ k=2, \ldots, N$. Since the vector functions $p^{k,*}(p^{1})$

result from the corresponding linear programming problem (40), their exact analytical expressions are not available in principle. To overcome this problem the following stochastic gradient technique may be applied.

First, notice that for any point $p^1 \in S^{N_1}$, in view of the differentiability of $p^{k,*}(p^1)$ for any $\delta > 0$, the following Taylor expansion holds for small enough scalar $\alpha > 0$:

$$\tilde{V}^1(p^1 + \alpha\xi_n) = \tilde{V}(p^1) + \alpha\left(\nabla\tilde{V}^1(p^1), \xi_n\right) + o(\alpha)\, o(\xi_n)$$

where ξ_n is a vector from R^{N_1} with random components $\xi_{i,n}$ uniformly distributed within the closed interval $[-1, 1]$. Notice that

$$E\{\xi_n\} = 0, E\{\xi_n \xi_n^\intercal\} = \frac{1}{3} I \in R^{N_1 \times N_1}$$

Here $E\{\cdot\}$ is the operator of mathematical expectation on ξ_n under a fixed value of p^1. Premultiplying the last identity by ξ_n and applying the operator of mathematical expectation, one has

$$E\left\{\xi_n \tilde{V}^1(p^1 + \alpha\xi_n)\right\} = \tilde{V}(p^1) E\{\xi_n\} + \alpha E\{\xi_n \xi_n^\intercal\} \nabla\tilde{V}^1(p^1) + \\ o(\alpha)\, E\{\xi_n o(\xi_n)\} = \frac{\alpha}{3} \nabla\tilde{V}^1(p^1) + o(\alpha)$$

that implies

$$\nabla\tilde{V}^1(p^1) = \frac{3}{\alpha} E\left\{\xi_n \tilde{V}^1(p^1 + \alpha\xi_n)\right\} + \frac{o(\alpha)}{\alpha}$$

This means that $\frac{3}{\alpha}\xi_n \tilde{V}^1(p^1 + \alpha\xi_n)$ can be considered as a realization of the stochastic gradient of the function $\tilde{V}(p^1)$ in the point p^1. Using the stochastic projection gradient technique [4] the following procedure may be suggested for the resolution of the problem (46):

$$\begin{gathered} p_n^1 = \Pi_S\left\{p_{n-1}^1 - \gamma_n \alpha_n^{-1} \xi_n V^1\left(p_{n-1}^1 + \alpha_n \xi_n\right)\right\} \\ \xi_n \in U[-1, 1],\ 0 < \alpha_n \downarrow 0,\ 0 < \gamma_n \downarrow 0 \\ \sum_n \gamma_n \alpha_n = \infty,\ \sum_n \left(\frac{\gamma_n}{\alpha_n}\right)^2 < \infty \end{gathered} \tag{47}$$

Remark 9 *For the numerical simulation realizations may be fulfilled using the following sequences satisfying (47):*

$$\gamma_n = \begin{cases} \gamma_0 & for \quad n \le n_0 \\ \dfrac{\gamma_0}{(n - n_0)^{3/4}} & for \quad n > n_0 \end{cases},$$
$$\alpha_n = \begin{cases} \alpha_0 & for \quad n \le n_0 \\ \dfrac{\alpha_0}{(n - n_0)^{1/4-\varepsilon}} & for \quad n > n_0, \varepsilon > 0 \end{cases}$$

7 Numerical Example

Example 10 (Three-person finite game) *In this game player 1 is the leader, players 2 and 3 are the followers the play a Nash game between them. The costs for the players 1,2 and 3 are respectively*

$$V^1(:,:,1) = \begin{bmatrix} 3 & 0 \\ 2 & 4 \end{bmatrix}; V^1(:,:,2) = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$$

$$V^2(:,:,1) = \begin{bmatrix} 5 & 0 \\ 4 & 3 \end{bmatrix}; V^2(:,:,2) = \begin{bmatrix} 6 & 1 \\ 0 & 1 \end{bmatrix}$$

$$V^3(:,:,1) = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}; V^3(:,:,2) = \begin{bmatrix} 2 & 2 \\ 8 & 2 \end{bmatrix}$$

Let

$$\underline{p^1 - fixed :} \; p^1 = (p^1_1; (1 - p^1_1))$$

So the costs matrix for the player I, player II and the player III are

$$V^1_{j_2,j_3}(p^1) = \begin{bmatrix} 3 & 0 \\ 2 & 4 \end{bmatrix} p^1_1 + \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} (1 - p^1_1)$$

$$V^2_{j_2,j_3}(p^1) = \begin{bmatrix} 5 & 0 \\ 6 & 1 \end{bmatrix} p^1_1 + \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} (1 - p^1_1)$$

$$V^3_{j_3,j_2}(p^1) = \begin{bmatrix} 2 & 4 \\ 2 & 2 \end{bmatrix} p^1_1 + \begin{bmatrix} 3 & 1 \\ 8 & 2 \end{bmatrix} (1 - p^1_1)$$

Player II minimizes his payoff over the rows of V^2 and the player III minimizes his incomes over the columns of V^3. Note that in this case $p^{\widehat{2}} = p^3$ and $p^{\widehat{3}} = p^2$. The previous theorem leads to the following linear programming formulation for each strategy selected by the leader p^1

$$F(t, \alpha) = t^2 + t^3 - \alpha^2 - \alpha^3 \quad \to \min_{\widehat{p} \in R^{\overline{N}},\; \alpha^k \in R,\; t^k \in R}$$

subject to

$$-\begin{bmatrix} p^1_1 + 4 & -3p^1_1 + 3 \\ 6p^1_1 & 1 \end{bmatrix}^T \begin{bmatrix} p^2_1 \\ p^2_2 \end{bmatrix} - \delta \begin{bmatrix} p^3_1 \\ p^3_2 \end{bmatrix} + \alpha^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-\begin{bmatrix} -p^1_1 + 3 & 3p^1_1 + 1 \\ -6p^1_1 + 8 & 2 \end{bmatrix}^T \begin{bmatrix} p^3_1 \\ p^3_2 \end{bmatrix} - \delta \begin{bmatrix} p^2_1 \\ p^2_2 \end{bmatrix} + \alpha^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} p^2_1 \\ p^2_2 \end{bmatrix} - 1 = 0$$

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} p^3_1 \\ p^3_2 \end{bmatrix} - 1 = 0$$

$$\begin{bmatrix} p_1^1+4 & -3p_1^1+3 \\ 6p_1^1 & 1 \end{bmatrix}^T \begin{bmatrix} p_1^2 \\ p_2^2 \end{bmatrix} + \delta \begin{bmatrix} p_1^3 \\ p_2^3 \end{bmatrix} - t^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -p_1^1+3 & 3p_1^1+1 \\ -6p_1^1+8 & 2 \end{bmatrix}^T \begin{bmatrix} p_1^3 \\ p_2^3 \end{bmatrix} + \delta \begin{bmatrix} p_1^2 \\ p_2^2 \end{bmatrix} - t^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$p_1^2, p_2^2, p_1^3, p_2^3 \geq 0$$

Let the cost function for the leader:

$$\begin{aligned} V^1(p^1,p^2,p^3) &= \sum_{j_2=1}^{2}\sum_{j_3=1}^{2} \left[V_{j_2,j_3}^1\left(p^1\right)\right] p_{j_2}^2 p_{j3}^3 + \frac{\delta}{2}\left\|p^k\right\|^2 \\ &= \begin{bmatrix} p_1^2 & p_2^2 \end{bmatrix} \begin{bmatrix} p_1^1+2 & -4p_1^1+4 \\ 2p_1^1 & 3p_1^1+1 \end{bmatrix} \begin{bmatrix} p_1^3 \\ p_2^3 \end{bmatrix} + \frac{\delta}{2}\left\|p^1\right\|^2 \end{aligned}$$

which has as solution applying the optimal Stackelberg costs used a technique of random search in stochastic approach (47)

$$V^1(p^1,p^2,p^3) = 1.7646$$
$$p_1^1 = 0.6063,\ p_2^1 = 0.3937$$

For the followers, the Nash equilibrium is given by

$$V^2(p^1,p^2,p^3) = 1.1535$$
$$p_1^2 = 0,\ p_2^2 = 1$$
$$V^3(p^1,p^2,p^3) = 2.6940$$
$$p_1^3 = 0.8475,\ p_2^3 = 0.1525$$

The evolution of the variables through the iterative process with

$$\alpha_0 = 0.1,\ \gamma_0 = 0.0001,\ n_0 = 9000$$

is given in the following graph (Fig. 1). After 10000 iterations, the technique of random search in stochastic approach provides the solution of the problem. The following graph gives the cost function for the leader (Fig. 2).

Example 11 (Four-person finite game) *In this game player 1 is the leader, players 2, 3 and 4 are the followers the play a Nash game between them. The payoffs for each player 1, 2, 3 and 4 are respectively*

$$V^1(:,:,1,1) = \begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix}; V^1(:,:,1,2) = \begin{bmatrix} 1 & 6 \\ 8 & 2 \end{bmatrix}$$
$$V^1(:,:,2,1) = \begin{bmatrix} 7 & 6 \\ 2 & 8 \end{bmatrix}; V^1(:,:,2,2) = \begin{bmatrix} 6 & 5 \\ 6 & 1 \end{bmatrix}$$

$$V^2(:,:,1,1) = \begin{bmatrix} 3 & 8 \\ 1 & 1 \end{bmatrix}; V^2(:,:,1,2) = \begin{bmatrix} 2 & 9 \\ 3 & 8 \end{bmatrix}$$
$$V^2(:,:,2,1) = \begin{bmatrix} 6 & 3 \\ 1 & 9 \end{bmatrix}; V^2(:,:,2,2) = \begin{bmatrix} 0 & 3 \\ 4 & 6 \end{bmatrix}$$

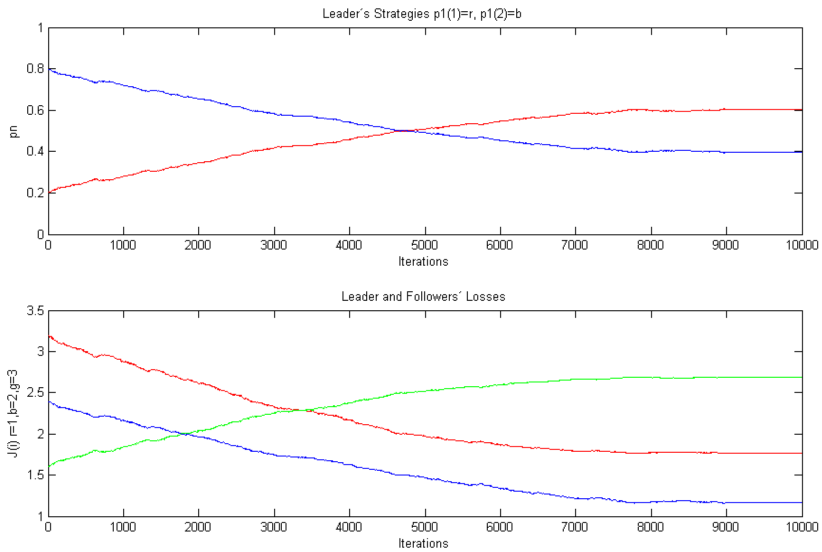

Figure 1: Optimal Stackelberg costs

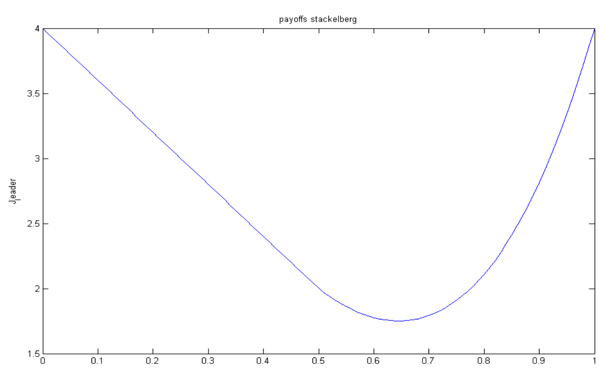

Figure 2: Cost function for the leader

$$V^3(:,:,1,1) = \begin{bmatrix} 2 & 6 \\ 9 & 2 \end{bmatrix}; V^3(:,:,1,2) = \begin{bmatrix} 6 & 9 \\ 4 & 4 \end{bmatrix}$$
$$V^3(:,:,1,1) = \begin{bmatrix} 3 & 7 \\ 8 & 1 \end{bmatrix}; V^3(:,:,1,2) = \begin{bmatrix} 1 & 10 \\ 10 & 0 \end{bmatrix}$$

$$V^4(:,:,1,1) = \begin{bmatrix} 3 & 6 \\ 5 & 4 \end{bmatrix}; V^4(:,:,1,2) = \begin{bmatrix} 2 & 9 \\ 4 & 2 \end{bmatrix}$$
$$V^4(:,:,1,1) = \begin{bmatrix} 8 & 0 \\ 7 & 10 \end{bmatrix}; V^4(:,:,1,2) = \begin{bmatrix} 9 & 6 \\ 1 & 5 \end{bmatrix}$$

In the matrices above the player 2 selects his payoffs over the rows and the player 3 selects over the corresponding columns. The player 4 have to chose over his/her actions in the rows or in the columns. Note that in this case

$$p^{\widehat{2}} = p^{\widehat{2}}\left(p^3, p^4\right), p^{\widehat{3}} = p^{\widehat{3}}\left(p^2, p^4\right), p^{\widehat{4}} = p^{\widehat{4}}\left(p^2, p^3\right)$$

The corresponding LPP formulation is as follows

$$F(p, \alpha) = t^2 + t^3 + +t^4 - \alpha^2 - \alpha^3 - \alpha^4 \quad \rightarrow \min_{\widehat{p} \in R^{\overline{N}}, p \in R^{\overline{N}}, \alpha^k \in R, t^k \in R}$$

subject to

$$V^1_{j_2,j_3}(p^1) = \begin{bmatrix} 7 & 6 & 6 & 5 \\ 2 & 6 & 8 & 1 \end{bmatrix} p^1_1 + \begin{bmatrix} 0 & 1 & 1 & 6 \\ 2 & 8 & 4 & 2 \end{bmatrix} (1 - p^1_1)$$

$$V^2_{j_2,j_3}(p^1) = \begin{bmatrix} 6 & 0 & 3 & 3 \\ 1 & 4 & 9 & 6 \end{bmatrix} p^1_1 + \begin{bmatrix} 3 & 2 & 8 & 9 \\ 1 & 3 & 1 & 8 \end{bmatrix} (1 - p^1_1)$$

$$V^3_{j_2,j_3}(p^1) = \begin{bmatrix} 3 & 1 & 8 & 10 \\ 7 & 10 & 1 & 0 \end{bmatrix} p^1_1 + \begin{bmatrix} 2 & 6 & 9 & 4 \\ 6 & 9 & 2 & 4 \end{bmatrix} (1 - p^1_1)$$

$$V^4_{j_3,j_2}(p^1) = \begin{bmatrix} 8 & 0 & 7 & 10 \\ 9 & 6 & 1 & 5 \end{bmatrix} p^1_1 + \begin{bmatrix} 3 & 6 & 5 & 4 \\ 2 & 9 & 4 & 2 \end{bmatrix} (1 - p^1_1)$$

$$\begin{bmatrix} 3p^1_1 + 3 & -2p^1_1 + 2 & -5p^1_1 + 8 & -6p^1_1 + 9 \\ 1 & p^1_1 + 3 & 8p^1_1 + 1 & -2p^1_1 + 8 \end{bmatrix} \begin{bmatrix} p^{\hat{2}}_1 \\ p^{\hat{2}}_2 \\ p^{\hat{2}}_3 \\ p^{\hat{2}}_4 \end{bmatrix} + \delta \begin{bmatrix} p^2_1 \\ p^2_2 \end{bmatrix}$$
$$-\alpha^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} p^1_1 + 2 & -5p^1_1 + 6 & -p^1_1 + 9 & 6p^1_1 + 4 \\ p^1_1 + 6 & p^1_1 + 9 & -p^1_1 + 2 & -4p^1_1 + 4 \end{bmatrix} \begin{bmatrix} p^{\hat{3}}_1 \\ p^{\hat{3}}_2 \\ p^{\hat{3}}_3 \\ p^{\hat{3}}_4 \end{bmatrix} + \delta \begin{bmatrix} p^3_1 \\ p^3_2 \end{bmatrix}$$
$$-\alpha^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5p_1^1+3 & -6p_1^1+6 & 2p_1^1+5 & 6p_1^1+4 \\ 7p_1^1+2 & -3p_1^1+9 & -3p_1^1+4 & 3p_1^1+2 \end{bmatrix} \begin{bmatrix} p_1^{\hat{4}} \\ p_2^{\hat{4}} \\ p_3^{\hat{4}} \\ p_4^{\hat{4}} \end{bmatrix} + \delta \begin{bmatrix} p_1^4 \\ p_2^4 \end{bmatrix}$$

$$-\alpha^4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-\begin{bmatrix} 3p_1^1+3 & -2p_1^1+2 & -5p_1^1+8 & -6p_1^1+9 \\ 1 & p_1^1+3 & 8p_1^1+1 & -2p_1^1+8 \end{bmatrix} \begin{bmatrix} p_1^{\hat{2}} \\ p_2^{\hat{2}} \\ p_3^{\hat{2}} \\ p_4^{\hat{2}} \end{bmatrix} - \delta \begin{bmatrix} p_1^2 \\ p_2^2 \end{bmatrix}$$

$$+t^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-\begin{bmatrix} p_1^1+2 & -5p_1^1+6 & -p_1^1+9 & 6p_1^1+4 \\ p_1^1+6 & p_1^1+9 & -p_1^1+2 & -4p_1^1+4 \end{bmatrix} \begin{bmatrix} p_1^{\hat{3}} \\ p_2^{\hat{3}} \\ p_3^{\hat{3}} \\ p_4^{\hat{3}} \end{bmatrix} - \delta \begin{bmatrix} p_1^3 \\ p_2^3 \end{bmatrix}$$

$$+t^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-\begin{bmatrix} 5p_1^1+3 & -6p_1^1+6 & 2p_1^1+5 & 6p_1^1+4 \\ 7p_1^1+2 & -3p_1^1+9 & -3p_1^1+4 & 3p_1^1+2 \end{bmatrix} \begin{bmatrix} p_1^{\hat{4}} \\ p_2^{\hat{4}} \\ p_3^{\hat{4}} \\ p_4^{\hat{4}} \end{bmatrix} - \delta \begin{bmatrix} p_1^4 \\ p_2^4 \end{bmatrix}$$

$$+t^4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$p_1^{\hat{2}} + p_2^{\hat{2}} + p_3^{\hat{2}} + p_4^{\hat{2}} = 1$$
$$p_1^{\hat{3}} + p_2^{\hat{3}} + p_3^{\hat{3}} + p_4^{\hat{3}} = 1$$
$$p_1^{\hat{4}} + p_2^{\hat{4}} + p_3^{\hat{4}} + p_4^{\hat{4}} = 1$$

The constraints (43) in this example take the form

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} p_1^{\hat{3}} = p_1^2 p_1^4 \\ p_2^{\hat{3}} = p_1^2 p_2^4 \\ p_3^{\hat{3}} = p_2^2 p_1^4 \\ p_4^{\hat{3}} = p_2^2 p_2^4 \end{bmatrix} - \begin{bmatrix} p_1^2 \\ p_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} p_1^{\hat{4}} = p_1^2 p_1^3 \\ p_2^{\hat{4}} = p_1^2 p_2^3 \\ p_3^{\hat{4}} = p_2^2 p_1^3 \\ p_4^{\hat{4}} = p_2^2 p_2^3 \end{bmatrix} - \begin{bmatrix} p_1^2 \\ p_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \widehat{p_1^2} = p_1^3 p_1^4 \\ \widehat{p_2^2} = p_1^3 p_2^4 \\ \widehat{p_3^2} = p_2^3 p_1^4 \\ \widehat{p_4^2} = p_2^3 p_2^4 \end{bmatrix} - \begin{bmatrix} p_1^3 \\ p_2^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \widehat{p_1^4} = p_1^2 p_1^3 \\ \widehat{p_2^4} = p_1^2 p_2^3 \\ \widehat{p_3^4} = p_2^2 p_1^3 \\ \widehat{p_4^4} = p_2^2 p_2^3 \end{bmatrix} - \begin{bmatrix} p_1^3 \\ p_2^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \widehat{p_1^2} = p_1^3 p_1^4 \\ \widehat{p_2^2} = p_1^3 p_2^4 \\ \widehat{p_3^2} = p_2^3 p_1^4 \\ \widehat{p_4^2} = p_2^3 p_2^4 \end{bmatrix} - \begin{bmatrix} p_1^4 \\ p_2^4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \widehat{p_1^3} = p_1^2 p_1^4 \\ \widehat{p_2^3} = p_1^2 p_2^4 \\ \widehat{p_3^3} = p_2^2 p_1^4 \\ \widehat{p_4^3} = p_2^2 p_2^4 \end{bmatrix} - \begin{bmatrix} p_1^4 \\ p_2^4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\widehat{p_1^2}, \widehat{p_2^2}, \widehat{p_3^2}, \widehat{p_4^2}, \widehat{p_1^3}, \widehat{p_2^3}, \widehat{p_3^3}, \widehat{p_4^3}, \widehat{p_1^4}, \widehat{p_2^4}, \widehat{p_3^4}, \widehat{p_4^4} \geq 0$$

Let the cost functions for the leader:

$$V^1(p^1, p^2, p^3, p^4) = \sum_{j_2=1}^{2} \sum_{j_3=1}^{2} \sum_{j_4=1}^{2} \left[V^1_{j_2, j_3, j_4}\left(p^1\right)\right] p^2_{j_2} p^3_{j3} p^4_{j4} + \frac{\delta}{2} \left\| p^1_{j_1} \right\|^2 =$$

$$\begin{bmatrix} p_1^2 & p_2^2 \end{bmatrix} \begin{bmatrix} 7p_1^1 & 5p_1^1 + 1 & 5p_1^1 + 1 & -p_1^1 + 6 \\ 2 & -2p_1^1 + 8 & 4p_1^1 + 4 & -p_1^1 + 2 \end{bmatrix} \begin{bmatrix} \hat{p_1^2} \\ \widehat{p_2^2} \\ \widehat{p_3^2} \\ \widehat{p_4^2} \end{bmatrix} + \frac{\delta}{2} \left\| p^1 \right\|^2$$

which has solution applying the optimal Stackelberg payoff used a technique of random search in stochastic approach (47)

$$V^1(p^1, p^2, p^3, p^4) = 3.9323$$
$$p_1^1 = 0.6173,\ p_2^1 = 0.3827$$

For the followers, the Nash equilibrium is given by

$$V^2(p^1, p^2, p^3, p^4) = 4.5878$$
$$p_1^2 = 0.4965,\ p_2^2 = 0.5035$$
$$V^3(p^1, p^2, p^3, p^4) = 5.2383$$
$$p_1^3 = 0.4318,\ p_2^3 = 0.5682$$
$$V^4(p^1, p^2, p^3, p^4) = 4.9059$$
$$p_1^4 = 0.3758,\ p_2^4 = 0.6242$$

The evolution of the variables through the iterative process with

$$\alpha_0 = 0.1, \ \gamma_0 = 0.0001, \ n_0 = 14000$$

is given in the following graph (Fig. 3).After 15000 iterations, the technique of

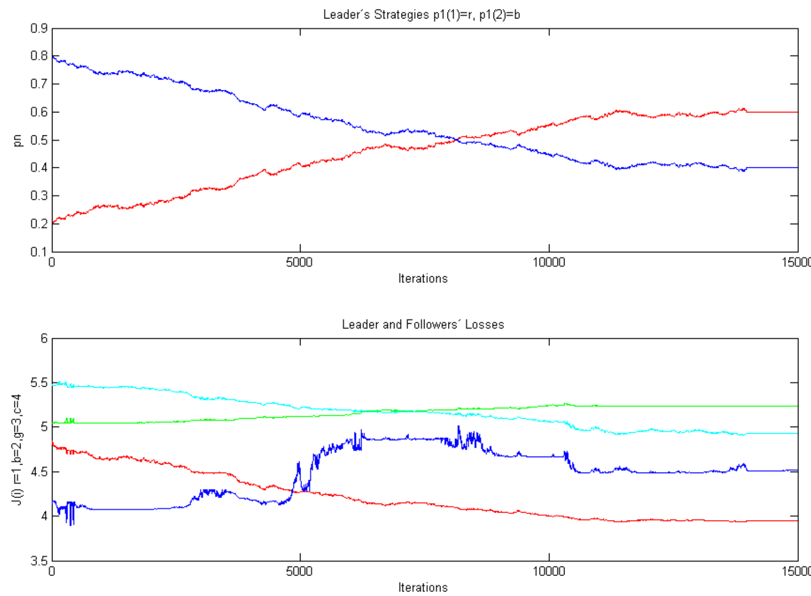

Figure 3: Optimal Stackelberg costs

random search in stochastic approach provides the solution of the problem. The following graph gives the cost function for the leader (Fig. 4).

Example 12 (Example 3.16 in [8]) *In this game player 1 is the leader, players 2 and 3 are the followers the play a Nash game between them. The costs for the players 1,2 and 3 are respectively*

$$V^1(:,:,1) = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}; V^1(:,:,2) = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

$$V^2(:,:,1) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}; V^2(:,:,2) = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$V^3(:,:,1) = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}; V^3(:,:,2) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

Let

$$\underline{p^1 - fixed:} \ p^1 = (p^1_1; (1 - p^1_1))$$

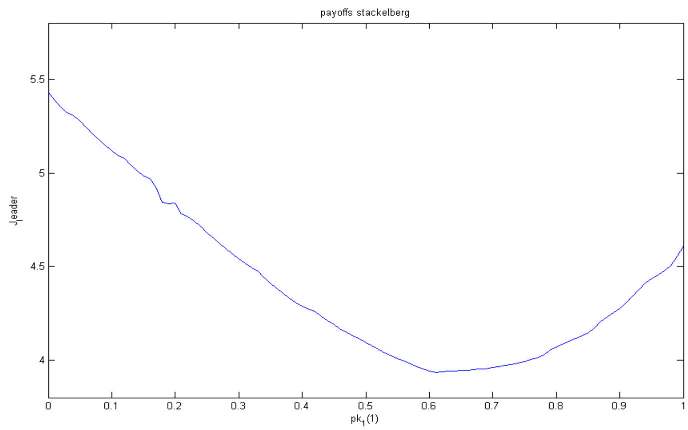

Figure 4: Cost function for the leader

So the costs matrix for the player I, player II and the player III are

$$V^1_{j_2,j_3}(p^1) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} p^1_1 + \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} (1 - p^1_1)$$

$$V^2_{j_2,j_3}(p^1) = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} p^1_1 + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} (1 - p^1_1)$$

$$V^3_{j_3,j_2}(p^1) = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} p^1_1 + \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} (1 - p^1_1)$$

Player II minimizes his payoff over the rows of V^2 and the player III minimizes his incomes over the columns of V^3. Note that in this case $p^{\widehat{2}} = p^3$ and $p^{\widehat{3}} = p^2$. The previous theorem leads to the following linear programming formulation for each strategy selected by the leader p^1

$$F(t,\alpha) = t^2 + t^3 - \alpha^2 - \alpha^3 \quad \to \min_{\widehat{p} \in R^{\overline{N}},\ \alpha^k \in R,\ t^k \in R}$$

subject to

$$-\begin{bmatrix} 0 & 2p^1_1 - 1 \\ -2p^1_1 + 1 & p^1_1 \end{bmatrix}^T \begin{bmatrix} p^2_1 \\ p^2_2 \end{bmatrix} - \delta \begin{bmatrix} p^3_1 \\ p^3_2 \end{bmatrix} + \alpha^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \le \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-\begin{bmatrix} -2p^1_1 + 2 & -2p^1_1 + 1 \\ p^1_1 & -p^1_1 + 2 \end{bmatrix}^T \begin{bmatrix} p^3_1 \\ p^3_2 \end{bmatrix} - \delta \begin{bmatrix} p^2_1 \\ p^2_2 \end{bmatrix} + \alpha^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \le \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} p^2_1 \\ p^2_2 \end{bmatrix} - 1 = 0$$

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} p^3_1 \\ p^3_2 \end{bmatrix} - 1 = 0$$

$$\begin{bmatrix} 0 & 2p_1^1-1 \\ -2p_1^1+1 & p_1^1 \end{bmatrix}^T \begin{bmatrix} p_1^2 \\ p_2^2 \end{bmatrix} + \delta \begin{bmatrix} p_1^3 \\ p_2^3 \end{bmatrix} - t^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2p_1^1+2 & -2p_1^1+1 \\ p_1^1 & -p_1^1+2 \end{bmatrix}^T \begin{bmatrix} p_1^3 \\ p_2^3 \end{bmatrix} + \delta \begin{bmatrix} p_1^2 \\ p_2^2 \end{bmatrix} - t^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$p_1^2, p_2^2, p_1^3, p_2^3 \geq 0$$

Let the cost function for the leader:

$$\begin{aligned} V^1(p^1, p^2, p^3) &= \sum_{j_2=1}^{2} \sum_{j_3=1}^{2} \left[V^1_{j_2,j_3}\left(p^1\right) \right] p^2_{j_2} p^3_{j3} + \frac{\delta}{2} \left\| p^k \right\|^2 \\ &= \begin{bmatrix} p_1^2 & p_2^2 \end{bmatrix} \begin{bmatrix} p_1^1 & 2 \\ 2p_1^1-1 & 0 \end{bmatrix} \begin{bmatrix} p_1^3 \\ p_2^3 \end{bmatrix} + \frac{\delta}{2} \left\| p^1 \right\|^2 \end{aligned}$$

which has as solution applying the optimal Stackelberg costs used a technique of random search in stochastic approach (47)

$$V^1(p^1, p^2, p^3) = 0.1667$$
$$p_1^1 = 0.3334, \; p_2^1 = 0.6666$$

For the followers, the Nash equilibrium is given by

$$V^2(p^1, p^2, p^3) = 0.2525$$
$$p_1^2 = 1, \; p_2^2 = 0$$
$$V^3(p^1, p^2, p^3) = 0.5776$$
$$p_1^3 = 0.7423, \; p_2^3 = 0.2577$$

The evolution of the variables through the iterative process with

$$\alpha_0 = 0.1, \; \gamma_0 = 0.00003, \; n_0 = 9000$$

is given in the following graph (Fig. 5).After 10000 iterations, the technique of random search in stochastic approach provides the solution of the problem. The following graph gives the cost function for the leader (Fig. 6).

8 Conclusion

This paper suggests a new numerical procedure designed for the calculation of a Stackelberg-Nash equilibrium point in a multi participant noncooperative unconstrained finite game with one leader and $(N-1)$ followers. The suggested procedure involves the use of a stochastic gradient (random search) algorithm to find the leader's strategy combined with the time-step solution of a linear programming problem formulated in terms of the extended (complementary) counter-coalition vector as well as in terms of the original player's variables. The optimality conditions for a strategy to be a Nash equilibrium are derived from the $\delta-$regularized joint pay-off function guaranteeing the existence of the unique Nash equilibrium point. The numerical examples, dealing with three and four participants games, show a good workability of the proposed approach.

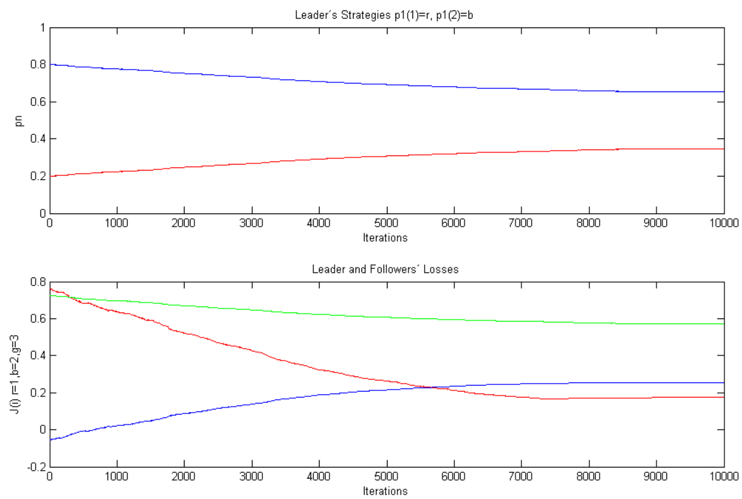

Figure 5: Optimal Stackelberg costs.

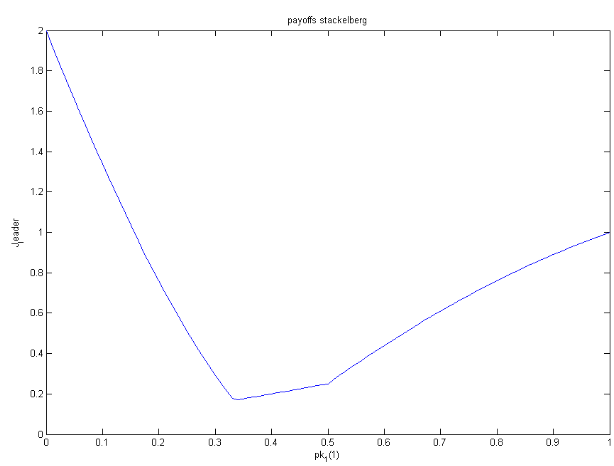

Figure 6: Cost function for the leader

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