Centro de Investigación y Estudios Avanzados del IPN

Stackleberg-Nash Equilibrium in Conflicts with a Leader: Numerical Procedure

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Introduction

Two levels of hierarchy in decision making are considered (***Stackelberg equilibrium*** concept):

• ***one leader*** who holds the powerful position (always considered as the player one);

• and ***(N-1) followers*** reacting to the leader's announced strategy by playing according to ***the Nash equilibrium concept*** among themselves.

The ***Nash equilibrium*** solution concept provides a reasonable non-cooperative equilibrium solution when the roles of the players are symmetric, that is to say, when no single player dominates the decision process.

Stackelberg Equilibrium Point

The δ-regularized loss function $V^1(p)$ for the ***leader***:

$$V_\delta^1(p^1,\dots,p^N) := \sum_{j_2=1}^{N_1} \dots \sum_{j_N=1}^{N_N} V^1_{j_2,\dots,j_N} \prod_{i=1}^{N} p^i_{j_i} + \frac{\delta}{2}\| p^1 \|^2$$

Definition. For the N-person finite game with one leader and *(N-1)* followers *(p^i, i=2,N) the strategy* $p^{1,*} \in S^{N1}$ is a ***hierarchical equilibrium strategy*** for the leader if

$$\max_{p^i \in R^F(p^{1,*}) \times S^{N_i},\, i=\overline{2,N}} V^1(p^{1,*}, p^2,\dots,p^N) =$$

$$\min_{p^1 \in S^{N_1}} \max_{p^i \in R^F(p^1) \times S^{N_i},\, i=\overline{2,N}} V^1(p^1, p^2,\dots,p^N)$$

where $R^F(p^1)$ is the Nash-response set of the followers' group defined for each $p^1 \in S^{Ni}$

If Nash-response is *unique,* then

$$V^1\left(p^{1,*}, p^2\left(p^{1,*}\right), \ldots, p^N\left(p^{1,*}\right)\right) =$$

$$\min_{p^1 \in S^{N_1}} V^1\left(p^1, p^2\left(p^1\right), \ldots, p^N\left(p^1\right)\right)$$

Followers' game description: Nash-responce

The δ-regularized ***loss function*** $V^k(p)$ for the k^{th} player:

$$V_\delta^k(p^2,\ldots,p^N \mid p^1) := \sum_{j_2=1}^{N_2} \ldots \sum_{j_N=2}^{N_N} \tilde{V}^k_{j_2,\ldots,j_N}\left(p^1\right)\prod_{i=2}^{N} p^i_{j_i} + \frac{\delta}{2}\| p^k \|^2$$

$$\tilde{V}^k_{j_2,\ldots,j_N}\left(p^1\right) := \sum_{j_1=1}^{N_1} V^k_{j_1,j_2,\ldots,j_N} p^1_{j_1} \qquad (k=\overline{1,N})$$

Nash-equilibrium point is given by a point $p^* \in R^N$ satisfying

$$V_\delta^k\left(p^{2,*}\left(p^1\right),\ldots,p^{N,*}\left(p^1\right) \mid p^1\right) =$$

$$\min_{p^k \in S^{N_k}} \{V_\delta^k\left(p^{2,*}\left(p^1\right),\ldots,p^{k-1,*}\left(p^1\right),p^k,p^{k+1,*}\left(p^1\right),\ldots,p^{N,*}\left(p^1\right)\right)$$

$$\mid \left(p^{2,*}\left(p^1\right),\ldots,p^{k-1,*}\left(p^1\right),p^k,p^{k+1,*}\left(p^1\right),\ldots,p^{N,*}\left(p^1\right)\right) \in Q^k\}$$

This equilibrium point is *unique* that follows from the strong convexity property.

Joint Loss Function and Optimally Condition

The joint δ -regularized **loss function** $\rho_\delta(p,q|p^1)$:

$$\rho_\delta(p,q|p^1) := \sum_{k=2}^{N} [V_\delta^k\left(p^2,\ldots,p^{k-1},q^k,p^{k+1},\ldots,p^N|p^1\right) -$$

$$V_\delta^k\left(p^{2,*}\left(p^1\right),\ldots,p^{N,*}\left(p^1\right)|p^1\right)]$$

Theorem (on Nash equilibrium point characterization):

$$\rho_\delta(p^*,p^*|p^1) = \min_{q\in S^N} \rho_\delta(p^*,q|p^1)$$

The optimality condition for *Leader*:

$$V_{\delta}^{k}(p^{2,*}\left(p^{1}\right),\ldots,p^{N,*}\left(p^{1}\right)\mid p^{1})+\frac{\delta}{2}\| p^{k,*}\left(p^{1}\right)\|^{2}-\alpha^{k,*}\left(p^{1}\right)=0$$

$$-\overline{V}^{k,T}p^{\hat{k},*}\left(p^{1}\right)-\delta p^{k,*}\left(p^{1}\right)+\alpha^{k,*}\left(p^{1}\right)e^{k}\leq 0$$

$$1-\left(e^{k},p^{k,*}\left(p^{1}\right)\right)=0 \quad for \;\; \forall k=\overline{2,N}$$

Theorem. The extended strategy vector $p^* \in R^N$ is ***a Nash equilibrium point***, if and only if the pair (p^*,α^*) is the solution to the following ***geometric (polylinear) programming problem***

$$F(p,\alpha) := \left(\rho_\delta(p,p \mid p^1) + \frac{\delta}{2}\sum_{k=2}^{N} \| p^k \|^2 - \sum_{k=2}^{N} \alpha^k \right) \to \min_{p \in R^{\overline{N}},\, \hat{p} \in R^{\hat{N}},\, \alpha^k \in R}$$

$$\rho_\delta(p,p \mid p^1) = \sum_{k=2}^{N} \left(p^{k,T} \overline{V}^{k,T} p^{\hat{k}} + \frac{\delta}{2} \| p^k \|^2 \right)$$

subject to

$$-\overline{V}^{k,T} p^{\hat{k}} - \delta p^k + \alpha^k e^k \le 0$$

$$1 - (e^k, p^k) = 0, \quad p^k_{j_k} \ge 0$$

$$\forall k = \overline{2,N}, \quad p^{\hat{k}} \in R^{N_{\hat{k}}}, \quad N_{\hat{k}} = \prod_{s=2, s \ne k}^{N} N_s$$

For any Nash equilibrium point *(p*,α*)* $\quad F\left(p^*\left(p^1\right), \alpha^*\left(p^1\right)\right) = 0$

Linear Programming Approach

Theorem. A ***necessary*** and ***sufficient*** condition that the triple *(p^*,p*, α*)* be a solution of a geometric programming problem is that the tuple *(p^*,p*, α*,t*)* is a solution to the following *linear programming problem*

$$\overline{F}(t,\alpha)=\sum_{k=2}^{N} t^k-\sum_{k=2}^{N}\alpha^k \to \min_{\hat{p}\in R^{\widehat{N}},\, p\in R^{\overline{N}},\, \alpha\in R,\, t^k\in R}$$

subject to

$$-\overline{V}^{k,T} p^{\hat{k}}-\delta p^k+\alpha^k e^k \le 0 \qquad k,r,s=\overline{1,N},\ s\neq k,\ s\neq r$$

$$\overline{V}^{k,T} p^{\hat{k}}+\delta p^k-t^k e^k \le 0 \qquad p^k\in S^{N_k},\ p^{\hat{k}}\in S^{N_{\hat{k}}}$$

$$M^{k,\hat{s}} p^{\hat{s}}=M^{k,\hat{r}} p^{\hat{r}} \qquad N_{\hat{k}}=\prod_{s=2,s\neq k}^{N} N_s,\ \widehat{N}=\prod_{i=2}^{N} N_{\hat{i}}$$

$$p^k=M^{k,\hat{s}} p^{\hat{s}}$$

The following property holds: $F\left(t^*\left(p^1\right),\alpha^*\left(p^1\right)\right)=0$

Main Problem: *how realize the optimization procedure for Leader*?

Leader's loss function:

$$\widetilde{V}^1\left(p^1\right) := V^1\left(p^1, p^{2,*}\left(p^1\right), \ldots, p^{N,*}\left(p^1\right)\right) \to \min_{p^1 \in S^{N_1}}$$

The main problem consists in the absence of the analytical expressions:

$$\nabla_{p^1} V^1 \to ?$$

$$\frac{\partial p^{i*}\left(p^1\right)}{\partial p^1} \to ?$$

Estimation of gradient without differentiation

1) $$\xi_n : \widetilde{V}^1\left(p^1_{n-1} + \alpha_n \xi_n\right) = \widetilde{V}^1\left(p^1_{n-1}\right) + \alpha_n \left(\nabla \widetilde{V}^1\left(p^1_{n-1}\right), \xi_n\right) + o\left(\alpha_n\right)$$

$$\xi_n = \left(\xi^1_n, \ldots, \xi^{N_1}_n\right)^T; \quad \xi^i_n \Rightarrow Unif\,[-1,1]; \alpha_n \downarrow 0$$

2) $$\overline{\xi_n \widetilde{V}^1\left(p^1_{n-1} + \alpha_n \xi_n\right)} =$$

$$\widetilde{V}^1\left(p^1_{n-1}\right)\underbrace{\overline{\xi_n}}_{=0} + \alpha_n \underbrace{\overline{\xi_n \xi^T_n}}_{=\frac{1}{3}} \nabla \widetilde{V}^1\left(p^1_{n-1}\right) + o\left(\alpha_n\right)$$

Indeed,

$$\overline{\xi_n \xi_n^T} = \| \overline{\xi_n^i \xi_n^j} \| \quad ; \quad \overline{\xi_n^i \xi_n^j} = \begin{cases} \overline{\xi_n^i}\,\overline{\xi_n^j} = 0, & i \neq j \\ \overline{\left(\xi_n^i\right)^2} = \frac{1}{3}, & i = j \end{cases}$$

$$\overline{\left(\xi_n^i\right)^2} = \int_{-1}^{1} x^2 \cdot \underbrace{p_{unif}(x)}_{\frac{1}{2}} dx = \frac{1}{2}\int_{-1}^{1} x^2 dx = \frac{1}{2} \cdot \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{6}\left[1-(-1)\right] = \frac{1}{3}$$

3) $$\overline{\xi_n \widetilde{V}^1\left(p_{n-1}^1 + \alpha_n \xi_n\right)} = \frac{1}{3}\alpha_n \nabla \widetilde{V}^1\left(p_{n-1}^1\right) + o(\alpha_n)$$

4) $$\nabla \widetilde{V}^1\left(p_{n-1}^1\right) = 3\alpha_n^{-1} \overline{\xi_n \widetilde{V}^1\left(p_{n-1}^1 + \alpha_n \xi_n\right)} - \underbrace{\frac{o(\alpha_n)}{\alpha_n}}_{o(1)\downarrow 0}$$

Projection Gradient Procedure
(*Stochastic Approximation Technique*)

$$p_n^1 = \prod\nolimits_{S^{N_1}} \left\{ p_{n-1}^1 - \frac{\gamma_n}{\alpha_n} \xi_n \widetilde{V}^1 \left(p_{n-1}^1 + \alpha_n \xi_n \right) \right\}$$

$$\sum_n \gamma_n = \infty, \qquad \sum_n \left(\frac{\gamma_n}{\alpha_n} \right)^2 = \infty$$

$$0 < \alpha_n \downarrow 0, \quad 0 < \gamma_n \downarrow 0$$

The recursion is realized as

1. $\gamma_n = \gamma_0$ and $\alpha_n = \alpha_0$ untill $n \leq n_0$

2. $\gamma_n = \gamma_0/(n-n_0)$ and $\alpha_n = \alpha_0/(n-n_0)^{1/4}$ from $n > n_0$

(*n* is the recursions' number).

Numerical Examples

1) Three-persons finite game. In this game player 1 is the ***leader***, players 2 and 3 are the followers playing the "*Nash game*" between them. The payoffs for each player 1,2 and 3 are as follows:

$$V^1(:,:,1) = \begin{bmatrix} 3 & 0 \\ 2 & 4 \end{bmatrix}; V^1(:,:,2) = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$$

$$V^2(:,:,1) = \begin{bmatrix} 5 & 0 \\ 4 & 3 \end{bmatrix}; V^2(:,:,2) = \begin{bmatrix} 6 & 1 \\ 0 & 1 \end{bmatrix}$$

$$V^3(:,:,1) = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}; V^3(:,:,2) = \begin{bmatrix} 2 & 2 \\ 8 & 2 \end{bmatrix}$$

Let it be

$$p^1 - fixed : p^1 = \left(p_1^1; \left(1 - p_1^1\right)\right)$$

So, the costs for the player I, player II and the player III are

$$V^1_{j_2,j_3}\left(p^1\right)=\begin{bmatrix}3 & 0\\ 2 & 4\end{bmatrix}p^1_1+\begin{bmatrix}2 & 4\\ 0 & 1\end{bmatrix}\left(1-p^1_1\right)$$

$$V^2_{j_2,j_3}\left(p^1\right)=\begin{bmatrix}5 & 0\\ 6 & 1\end{bmatrix}p^1_1+\begin{bmatrix}4 & 3\\ 0 & 1\end{bmatrix}\left(1-p^1_1\right)$$

$$V^3_{j_2,j_3}\left(p^1\right)=\begin{bmatrix}2 & 4\\ 2 & 2\end{bmatrix}p^1_1+\begin{bmatrix}3 & 1\\ 8 & 2\end{bmatrix}\left(1-p^1_1\right)$$

Nash equilibrium for the followers results from LPP formulated for a fixed strategy selected by the leader:

$$\overline{F}(t,\alpha)=t^2+t^3-\alpha^2-\alpha^3\rightarrow\min_{\widehat{p}\in R^{\widehat{N}},\,p\in R^{\overline{N}},\,\alpha\in R,\,t^k\in R}$$

subject to

$$-\begin{bmatrix} p_1^1+4 & -3p_1^1+3 \\ -6p_1^1+8 & 1 \end{bmatrix}^T \begin{bmatrix} p_1^2 \\ p_2^2 \end{bmatrix} - \delta \begin{bmatrix} p_1^3 \\ p_2^3 \end{bmatrix} + \alpha^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \le \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-\begin{bmatrix} -p_1^1+3 & 3p_1^1+1 \\ 6p_1^1 & 2 \end{bmatrix}^T \begin{bmatrix} p_1^3 \\ p_2^3 \end{bmatrix} - \delta \begin{bmatrix} p_1^2 \\ p_2^2 \end{bmatrix} + \alpha^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \le \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} p_1^2 \\ p_2^2 \end{bmatrix} - 1 = 0$$

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} p_1^3 \\ p_2^3 \end{bmatrix} - 1 = 0$$

$$\begin{bmatrix} p_1^1+4 & -3p_1^1+3 \\ -6p_1^1+8 & 1 \end{bmatrix}^T \begin{bmatrix} p_1^2 \\ p_2^2 \end{bmatrix} + \delta \begin{bmatrix} p_1^3 \\ p_2^3 \end{bmatrix} - t^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \le \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -p_1^1+3 & 3p_1^1+1 \\ 6p_1^1 & 2 \end{bmatrix}^T \begin{bmatrix} p_1^3 \\ p_2^3 \end{bmatrix} + \delta \begin{bmatrix} p_1^2 \\ p_2^2 \end{bmatrix} - t^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \le \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$p_1^2, p_2^2, p_1^3, p_2^3 \ge 0$$

The cost function for the leader:

$$V^1\left(p^1, p^2, p^3\right) = \sum_{j_2=1}^{2}\sum_{j_3=1}^{2}\left[V^1_{j_2,j_3}\left(p^1\right)\right]p^{2,*}_{j_2}\left(p^1\right)p^{3,*}_{j_3}\left(p^1\right) + \frac{\delta}{2}\| p^1_{j_1} \|^2$$

$$= \left[p^{2,*}_1\left(p^1\right) \quad p^{2,*}_2\left(p^1\right)\right]\begin{bmatrix} p^1_1 + 2 & -4p^1_1 + 4 \\ 2p^1_1 & 3p^1_1 + 1 \end{bmatrix}\begin{bmatrix} p^3_1 \\ p^3_2 \end{bmatrix} + \frac{\delta}{2}\| p^1 \|^2$$

It has the unique minimum value

$$V^1\left(p^1, p^{2,*}\left(p^1\right), p^{3,*}\left(p^1\right)\right) = 1.7646$$

$$p^1_1 = 0.6063$$

$$p^1_2 = 0.3937$$

The Nash equilibrium for the followers:

$$V^2\left(p^2, p^3 \mid p^1\right) = 1.1535$$

$$p^2_1 = 0$$

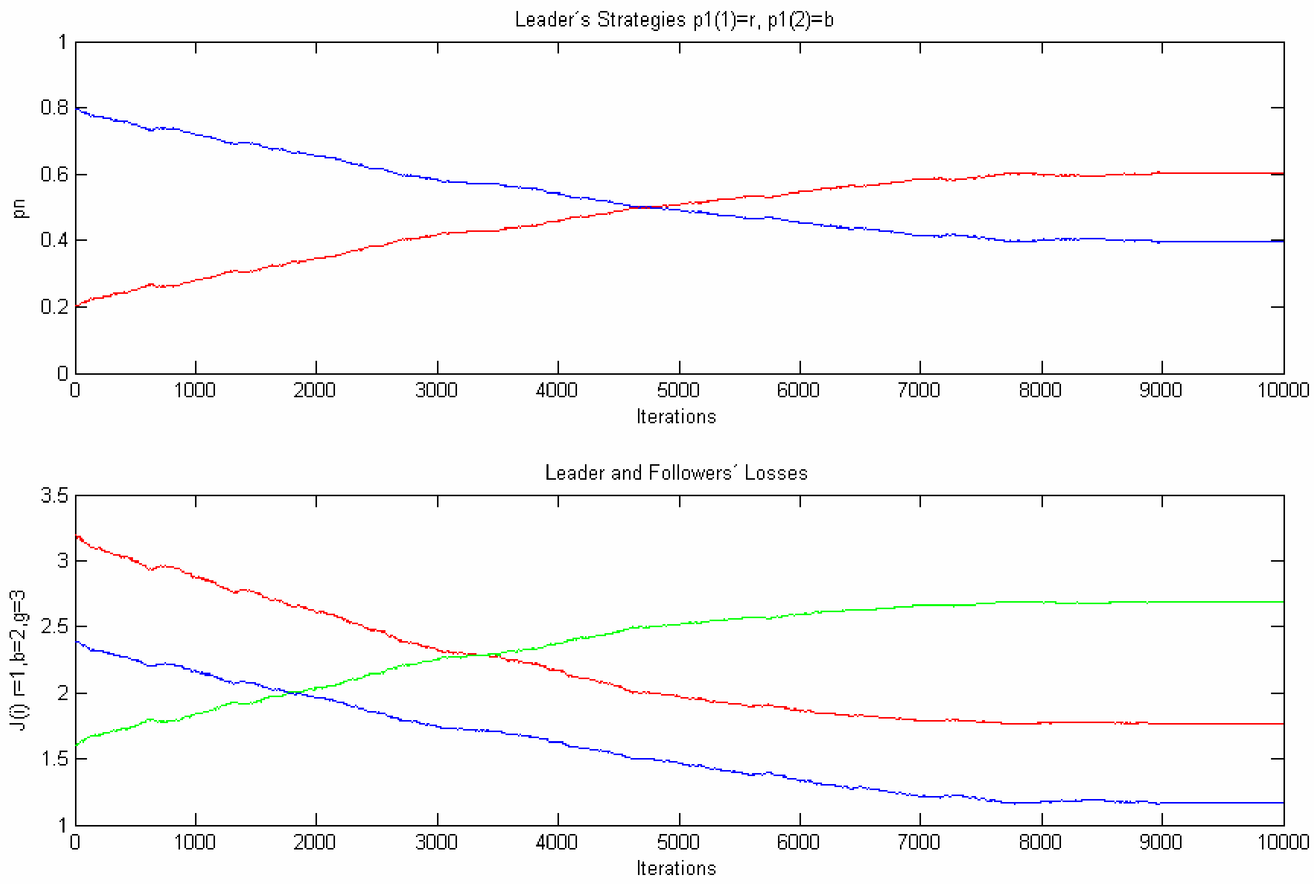
$$p^2_2 = 1$$

$$V^3\left(p^2, p^3 \mid p^1\right) = 2.6940$$

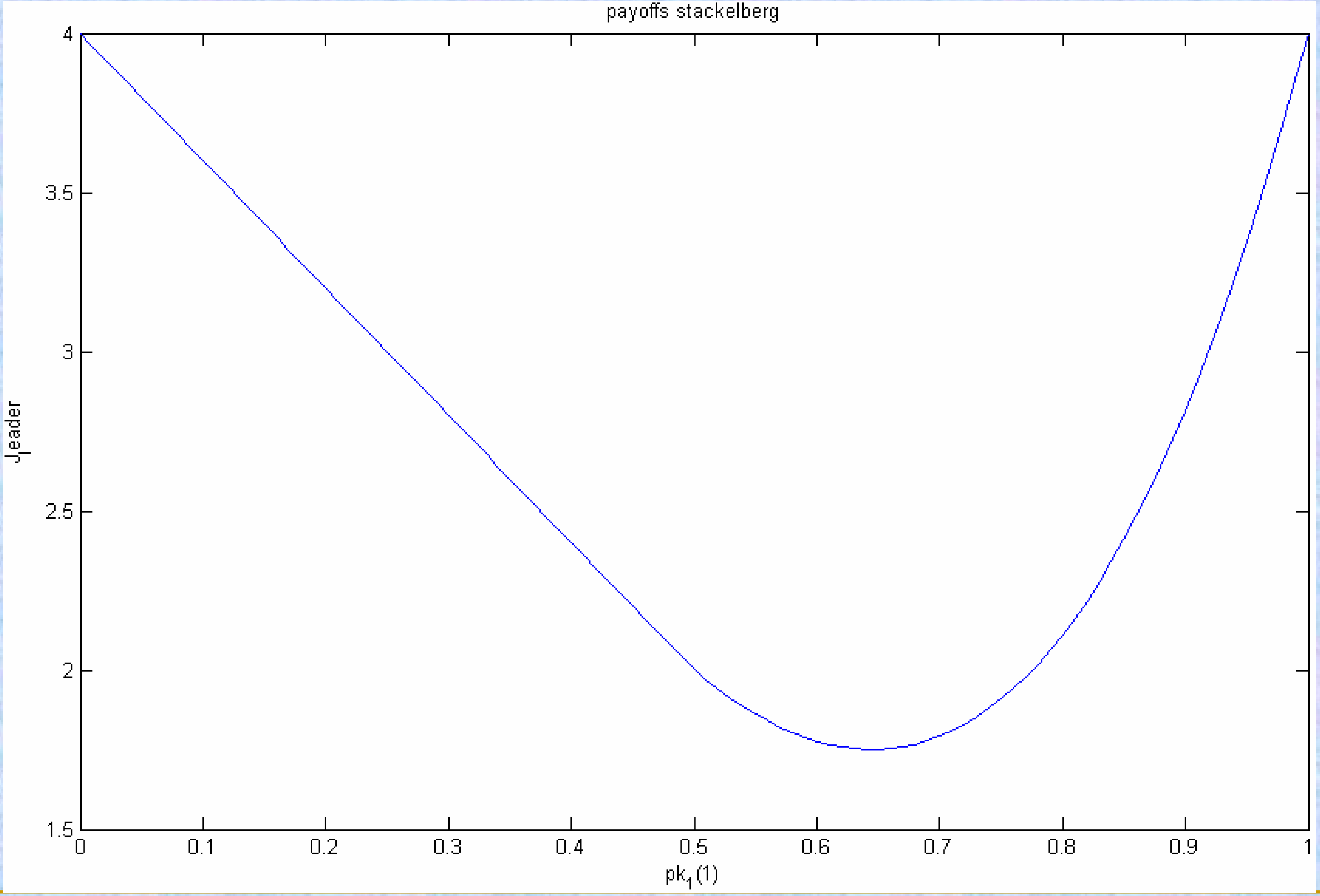
$$p^3_1 = 0.8475$$

$$p^3_2 = 0.1525$$

The evolution of the variables through the iterative process is given in the following figure:



The loss function for the leader:



2) Three-person finite game. This example is enounced in DNGT* (example 3.16). The payoffs for each player (1,2 and 3) respectively, are

$$V^1(:,:,1) = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}; V^1(:,:,2) = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

$$V^2(:,:,1) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}; V^2(:,:,2) = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$V^3(:,:,1) = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}; V^3(:,:,2) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

DNGT* T. Basar , G. J.Olsder , "Dynamic Noncooperative Game Theory", second edition, SIAM, Philadelphia.

The optimal Stackelberg payoff:

$$V^1\left(p^1, p^{2,*}\left(p^1\right), p^{3,*}\left(p^1\right)\right) = 0.1667$$

$$p_1^1 = 0.3334$$

$$p_2^1 = 0.6666$$

The Nash equilibrium for the followers:

$$V^2\left(p^2, p^3 \mid p^1\right) = 0.2525$$

$$p_1^2 = 1$$

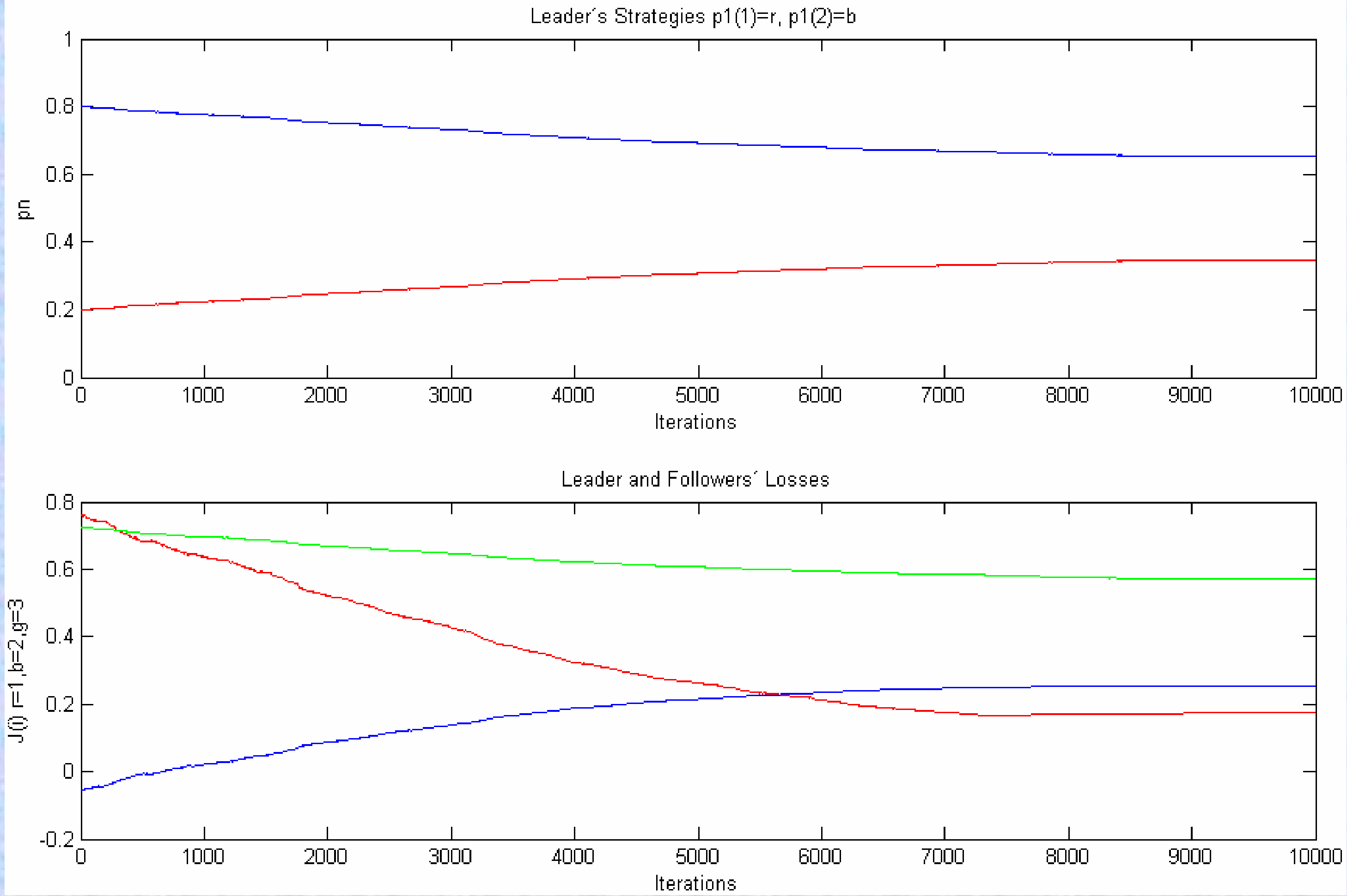
$$p_2^2 = 0$$

$$V^3\left(p^2, p^3 \mid p^1\right) = 0.5776$$

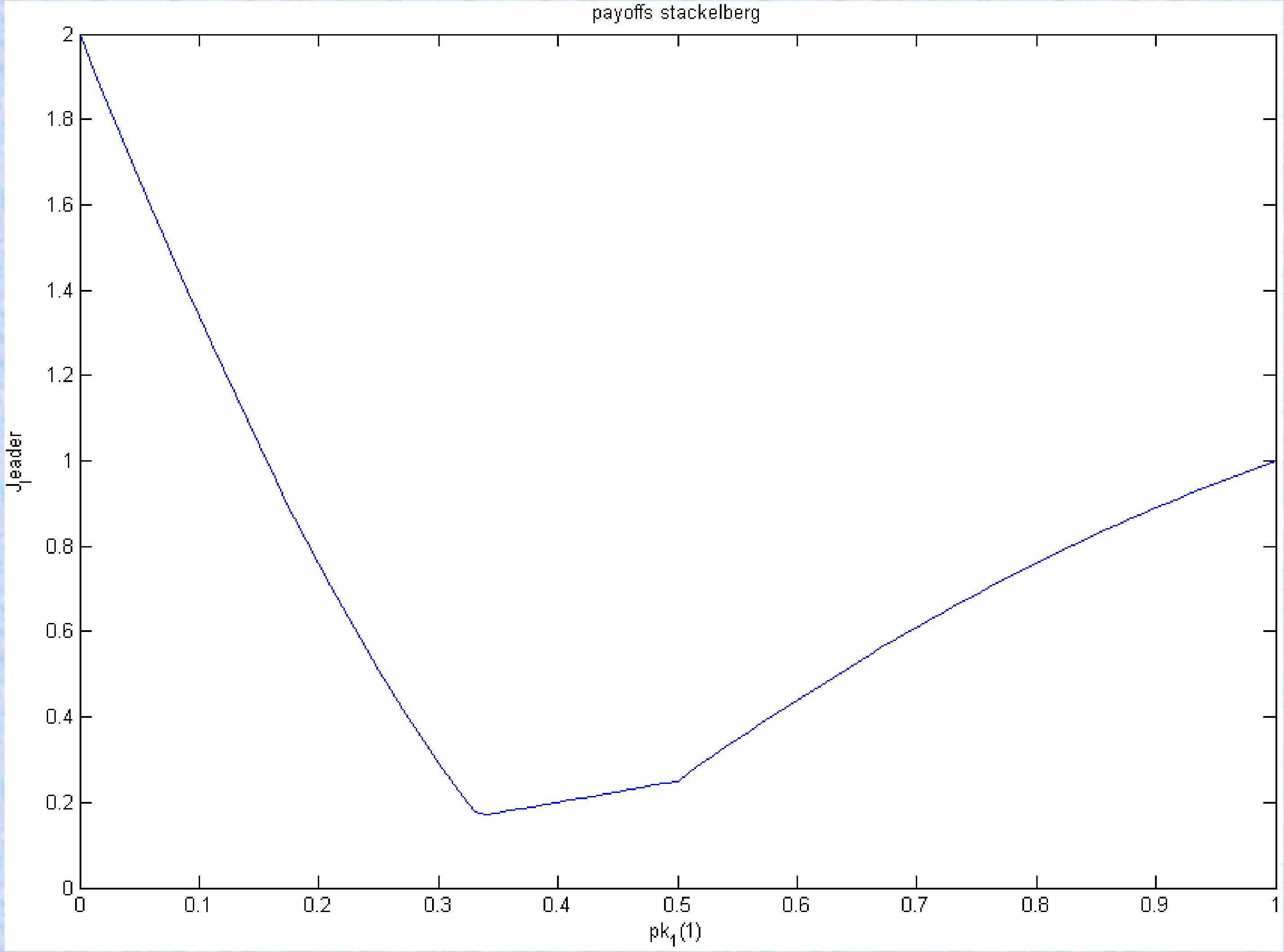
$$p_1^3 = 0.7423$$

$$p_2^3 = 0.2577$$

The evolution of the variables:



The loss function for the leader



3) Four-person finite game. The payoffs for each player (1, 2 ,3 and 4 respectively) are

$$V^1(:,:,1,1)=\begin{bmatrix}0 & 1\\ 2 & 4\end{bmatrix}; V^1(:,:,1,2)=\begin{bmatrix}1 & 6\\ 8 & 2\end{bmatrix}$$

$$V^1(:,:,2,1)=\begin{bmatrix}7 & 6\\ 2 & 8\end{bmatrix}; V^1(:,:,2,2)=\begin{bmatrix}6 & 5\\ 6 & 1\end{bmatrix}$$

$$V^2(:,:,1,1)=\begin{bmatrix}3 & 8\\ 1 & 1\end{bmatrix}; V^2(:,:,1,2)=\begin{bmatrix}2 & 9\\ 3 & 8\end{bmatrix}$$

$$V^2(:,:,2,1)=\begin{bmatrix}6 & 3\\ 1 & 9\end{bmatrix}; V^2(:,:,2,2)=\begin{bmatrix}0 & 3\\ 4 & 6\end{bmatrix}$$

$$V^3(:,:,1,1)=\begin{bmatrix}2 & 6\\ 9 & 2\end{bmatrix}; V^3(:,:,1,2)=\begin{bmatrix}6 & 9\\ 4 & 4\end{bmatrix}$$

$$V^3(:,:,2,1)=\begin{bmatrix}3 & 7\\ 8 & 1\end{bmatrix}; V^3(:,:,2,2)=\begin{bmatrix}1 & 10\\ 10 & 0\end{bmatrix}$$

$$V^4(:,:,1,1)=\begin{bmatrix}3 & 6\\ 5 & 4\end{bmatrix}; V^4(:,:,1,2)=\begin{bmatrix}2 & 9\\ 4 & 2\end{bmatrix}$$

$$V^4(:,:,2,1)=\begin{bmatrix}8 & 0\\ 7 & 10\end{bmatrix}; V^4(:,:,2,2)=\begin{bmatrix}9 & 6\\ 1 & 5\end{bmatrix}$$

The cost functions for the leader:

$$V^1\left(p^1,p^2,p^3,p^4\right)=\sum_{j_2=1}^{2}\sum_{j_3=1}^{2}\sum_{j_4=1}^{2}\left[V^1_{j_2,j_3,j_4}\left(p^1\right)\right]p^2_{j_2}p^3_{j_3}p^4_{j_4}+\frac{\delta}{2}\|p^1_{j_1}\|^2$$

$$=\left[p^2_1 \quad p^2_2\right]\begin{bmatrix}7p^1_1 & 5p^1_1+1 & p^1_1+1 & -p^1_1+6\\ 2 & -2p^1_1+8 & 4p^1_1+4 & -p^1_1+2\end{bmatrix}\begin{bmatrix}p^{\hat{2}}_1\\ p^{\hat{2}}_2\\ p^{\hat{2}}_3\\ p^{\hat{2}}_4\end{bmatrix}+\frac{\delta}{2}\|p^1\|^2$$

The optimal Stackelberg payoff is

$$V^1\left(p^1,p^{2,*}\left(p^1\right),p^{3,*}\left(p^1\right),p^{4,*}\left(p^1\right)\right)=3.9323$$

$$p^1_1=0.6173$$

$$p^1_2=0.3827$$

The Nash equilibrium for the frollowes is given by

$$V^2\left(p^2,p^3,p^4\mid p^1\right)=4.5878$$

$$p^2_1=0.4965$$

$$p^2_2=0.5035$$

$$V^3\left(p^1,p^2,p^3,p^4\mid p^1\right)=5.2383$$

$$p^3_1=0.4318$$

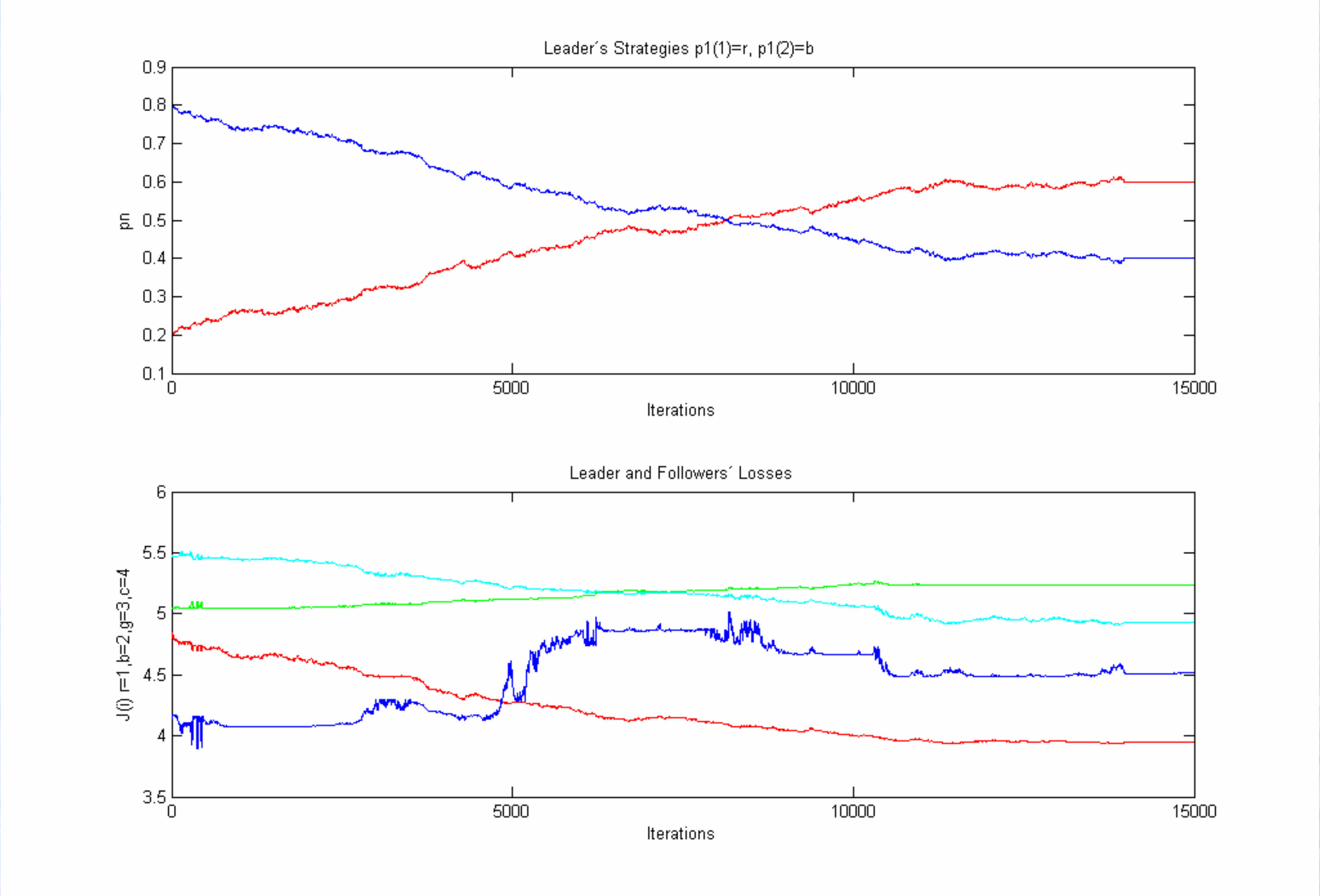
$$p^3_2=0.5682$$

$$V^4\left(p^2,p^3,p^4\mid p^1\right)=4.9059$$

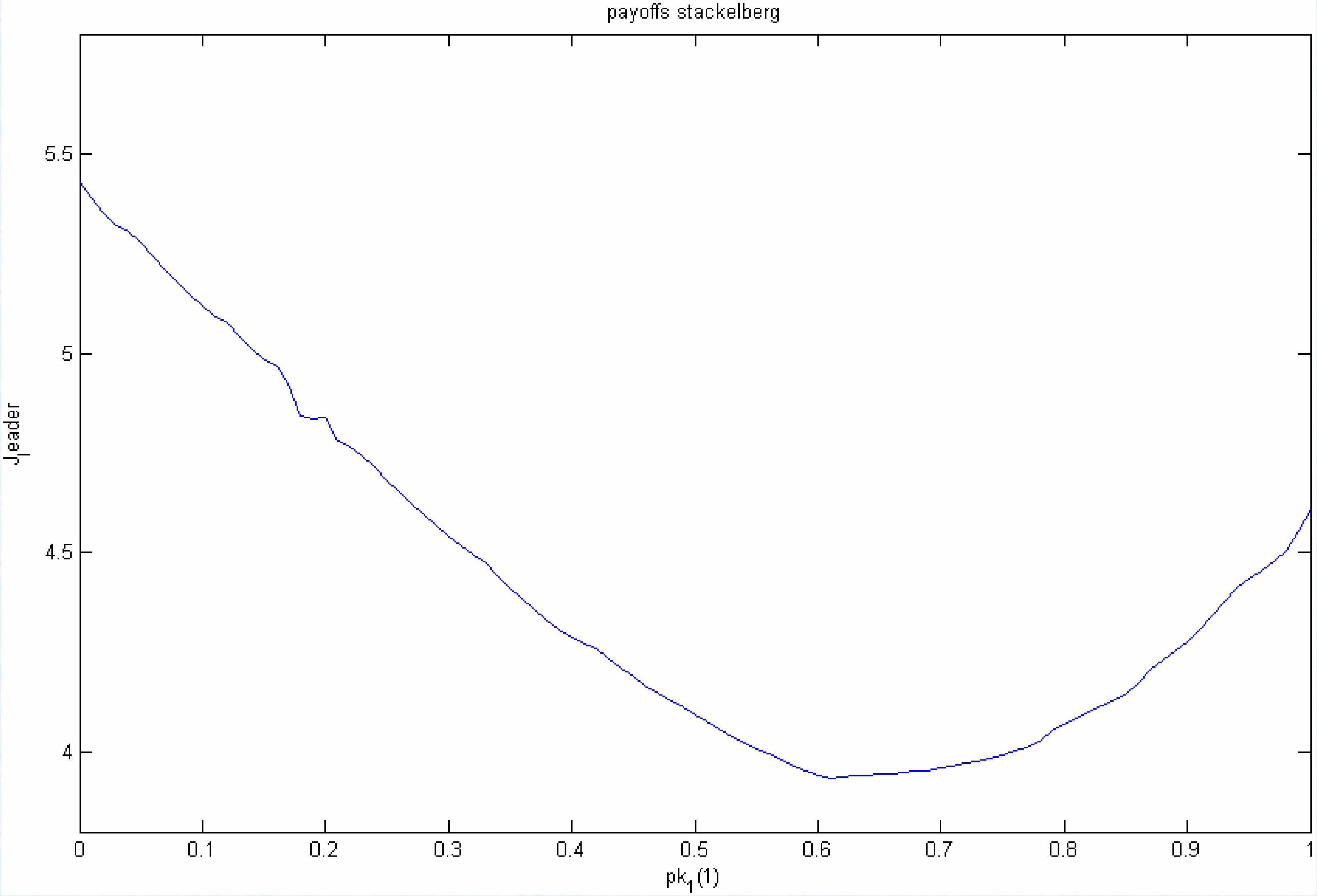
$$p^4_1=0.3758$$

$$p^4_2=0.6242$$

The evolution of the variables:



The loss function for the leader:



Conclusion

- The ***conflict situation*** among multi-particpants with a leader, where he /she has a preference to be the first in the turn of an action selection, is tackled.
- When the strategy of a leader (player 1) is selected, ***the rest of participants are playing a "standard" non-cooperative finite game*** (may be, with constraints) trying to find a Nash equilibrium.
- To guarantee the uniqueness of this equilibrium the, so-called, - ***regularized individual pay-off function*** is introduced.
- Then the ***generalized version of the Mangasarian-Stone*** theorem is applied permitting to reformulate this non-cooperative game as a poly-linear programming problem.

- The ***first main result*** of this work consists of the theorem which shows that the last nonlinear programming problem may be represented as a ***linear programming problem*** (LPP) formulated in term of ***counter-coalition strategies***.

- Finally, when the Nash-equilibrium strategies (as a functions of the strategy selected by a leader) are found, the leader optimizes his own pay-off like-random search optimization problem in ***its non-gradient form***.

- Numerical examples (compared with some results published another authors) show ***the workability*** of the suggested approach.

Thanks for attention
and
Best Regards!