Weak Stationarity of Set Systems

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Outline

- Extremal Set Systems
- Extremality/Stationarity/Regularity
- Dual Criteria

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- **Extremal Set Systems**
- **Extremality/Stationarity/Regularity**
- **Dual Criteria**
- Metric Inequality
- Boundary Condition
- Regularity of Multifunctions

Extremal Set Systems

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Kruger, Mordukhovich, 1980

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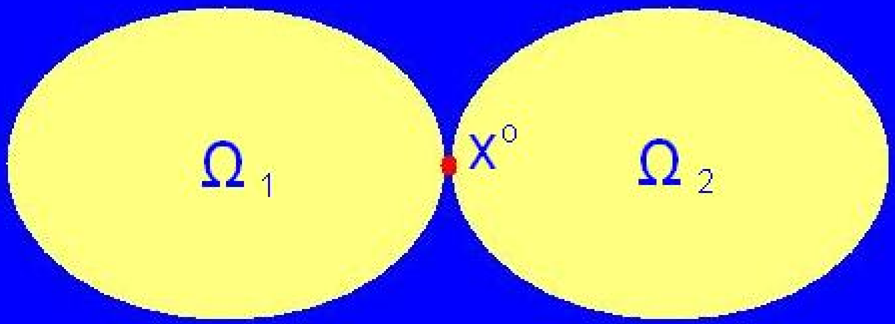

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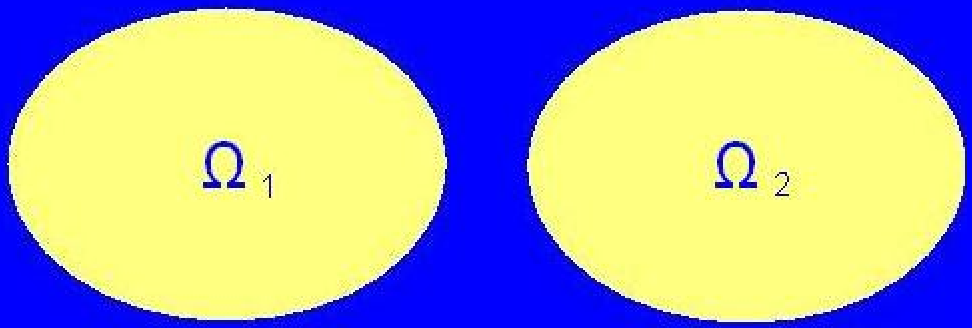

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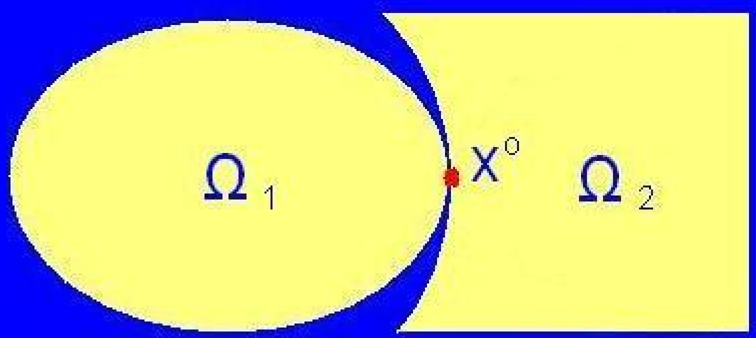

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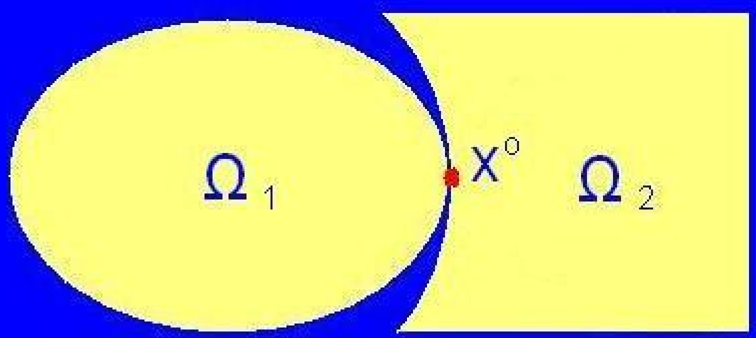

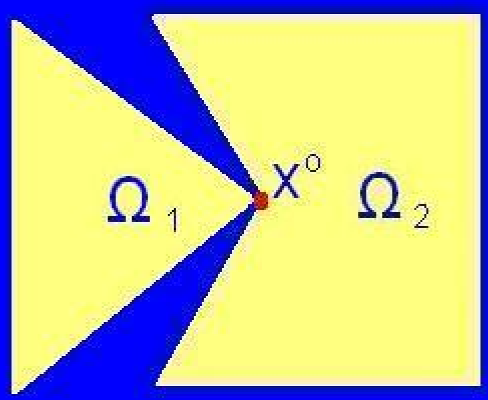

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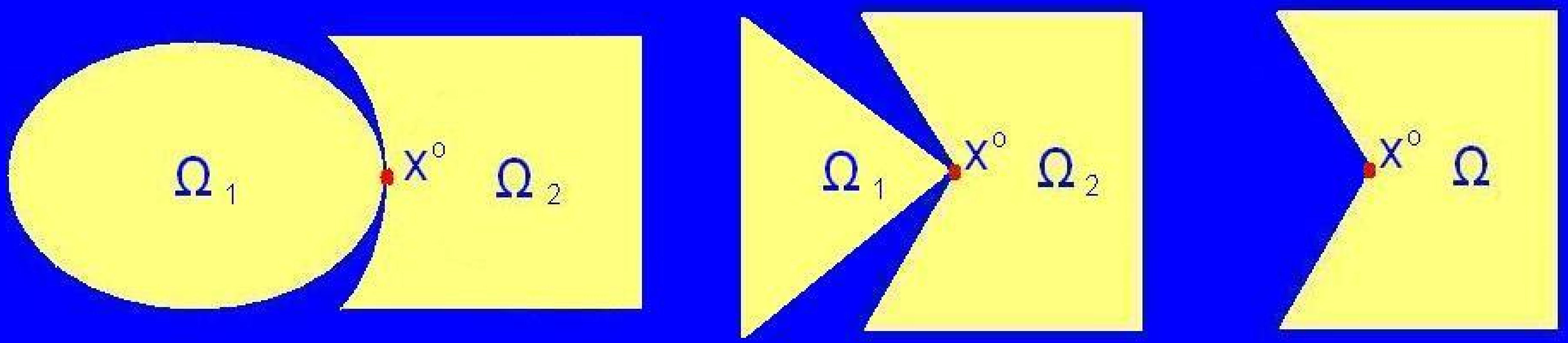

Extremality/Stationarity/Regularity

$\Omega_1, \Omega_2, \ldots, \Omega_n \subset X$, $x^\circ \in \bigcap_{i=1}^n \Omega_i$, $\omega_i \in \Omega_i$, $i = 1, 2, \ldots, n$

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$$\theta_\rho[\Omega_1, \ldots, \Omega_n](\omega_1, \ldots, \omega_n) = \sup\Big\{ r \geq 0 : \Big(\bigcap_{i=1}^n (\Omega_i - \omega_i - a_i) \Big) \bigcap B_\rho \neq \emptyset, \forall a_i \in B_r \Big\}$$

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Extremality/Stationarity/Regularity

Definition 1. $\{\Omega_1, \Omega_2, \ldots, \Omega_n\}$ is

(1) *extremal* at x° if $\theta_\rho[\Omega_1, \ldots, \Omega_n](x^\circ) = 0$ for all $\rho > 0$.

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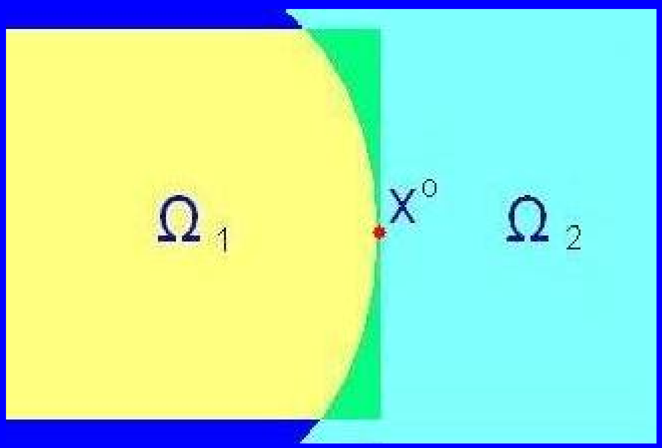
$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$$

Extremality/Stationarity/Regularity



$$\theta[\Omega_1, \Omega_2](x^\circ) = 0$$

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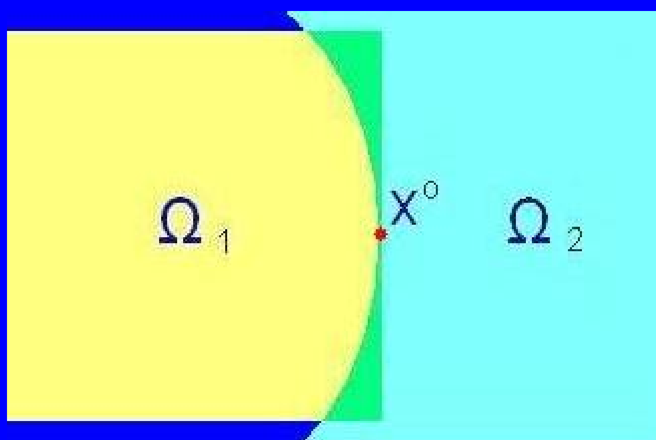

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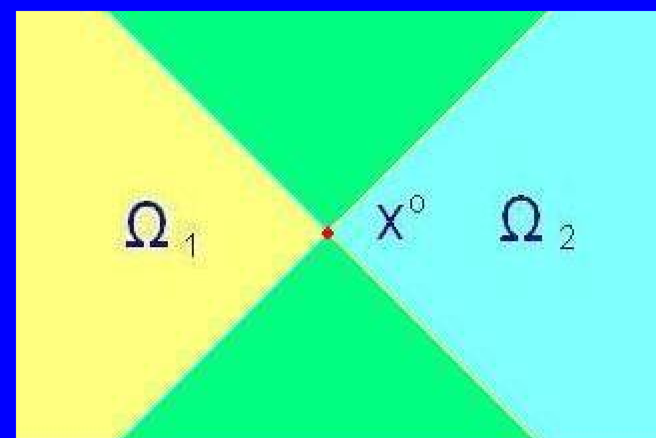

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Extremality/Stationarity/Regularity

Proposition 1. *Let* $\Omega_1, \Omega_2, \ldots, \Omega_n$ *be* **convex**. *Then* *(1)* $\Leftrightarrow$ *(2)* $\Leftrightarrow$ *(3)* $\Leftrightarrow$ *(4)*.

Extremality/Stationarity/Regularity

Proposition 1. *Let* $\Omega_1, \Omega_2, \ldots, \Omega_n$ *be* **convex**. *Then* (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

If $\operatorname{int}\Omega_i \neq \emptyset$, $i = 1, 2, \ldots, n-1$, *then these conditions are equivalent to*

$$\bigcap_{i=1}^{n-1} \operatorname{int}\Omega_i \bigcap \Omega_n = \emptyset.$$

Dual Criteria

Fréchet normal cone to Ω at $x^\circ \in \Omega$

$$N(x^\circ|\Omega) = \left\{ x^* \in X^* : \ \limsup_{x \xrightarrow{\Omega} x^\circ} \frac{\langle x^*, x - x^\circ \rangle}{\|x - x^\circ\|} \le 0 \right\}$$

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Ω_1, Ω_2, . . . , Ω_n are closed, $x^\circ \in \bigcap_{i=1}^n \Omega_i$, $(0/0)_\infty = \infty$

$$\eta[\Omega_1, \ldots, \Omega_n](x^\circ) = \lim_{\delta \to +0} \inf \left\{ \left(\left\| \sum_{i=1}^n x_i^* \right\| / \sum_{i=1}^n \|x_i^*\| \right)_\infty : \right.$$

$$\left. x_i^* \in N(x_i|\Omega_i),\ x_i \in \Omega_i \cap B_\delta(x^\circ),\ i = 1, \ldots, n \right\}$$

Dual Criteria

Theorem 1. $\hat{\theta}[\Omega_1, \ldots, \Omega_n](x^\circ) \leq \eta[\Omega_1, \ldots, \Omega_n](x^\circ)$.

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If X is Asplund and $\hat{\theta}[\Omega_1,\ldots,\Omega_n](x^\circ) < 1$ then

$$\eta[\Omega_1,\ldots,\Omega_n](x^\circ) \leq \frac{\hat{\theta}[\Omega_1,\ldots,\Omega_n](x^\circ)}{1-\hat{\theta}[\Omega_1,\ldots,\Omega_n](x^\circ)}.$$

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Corollary 1.1. $\eta[\Omega_1,\ldots,\Omega_n](x^\circ) = 0 \Rightarrow$
$\{\Omega_1,\Omega_2,\ldots,\Omega_n\}$ is weakly stationary at x°.

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$\{\Omega_1, \Omega_2, \ldots, \Omega_n\}$ is weakly stationary at x°.
If X is Asplund then the Extended extremal principle *holds:*

- $\{\Omega_1, \Omega_2, \ldots, \Omega_n\}$ *is weakly stationary at x°* $\Leftrightarrow$ $\eta[\Omega_1, \Omega_2, \ldots, \Omega_n](x^\circ) = 0$.

Dual Criteria

Extremal principle (Kruger, Mordukhovich, 1980; Mordukhovich, Shao, 1996).

$\{\Omega_1, \Omega_2, \ldots, \Omega_n\}$ is locally extremal at x°
$\Rightarrow \eta[\Omega_1, \ldots, \Omega_n](x^\circ) = 0.$

Dual Criteria

Extremal principle (Kruger, Mordukhovich, 1980; Mordukhovich, Shao, 1996).

$\{\Omega_1, \Omega_2, \ldots, \Omega_n\}$ is locally extremal at x°
$\Rightarrow \eta[\Omega_1, \ldots, \Omega_n](x^\circ) = 0$.

Theorem 2. *The following assertions are equivalent:*

- *X is an Asplund space.*
- *The Extremal principle is valid in X.*
- *The Extended extremal principle is valid in X.*

Metric Inequality

Ioffe, 1989; Bauschke, Borwein, 1993; Ngai, Théra, 2001.

$(0/0)_\circ = 0$

$$\vartheta[\Omega_1,\ldots,\Omega_n](x^\circ) = \limsup_{x\to x^\circ} \left(\frac{d(x, \bigcap_{i=1}^n \Omega_i)}{\max_{1\le i\le n} d(x,\Omega_i)} \right)_\circ$$

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$$\hat{\vartheta}[\Omega_1,\ldots,\Omega_n](x^\circ) = \limsup_{\substack{x\to x^\circ\\ x_i\to 0}}\left(\frac{d(x,\bigcap_{i=1}^n (\Omega_i - x_i))}{\max_{1\le i\le n} d(x+x_i,\Omega_i)}\right)_\circ$$

Metric Inequality

Definition 2. $\{\Omega_1, \Omega_2, \ldots, \Omega_n\}$ satisfies at x°

- the *metric inequality* if $\vartheta[\Omega_1, \ldots, \Omega_n](x^\circ) < \infty$.
- the *strong metric inequality* if $\hat{\vartheta}[\Omega_1, \ldots, \Omega_n](x^\circ) < \infty$.

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Theorem 3. $\hat{\vartheta}[\Omega_1, \ldots, \Omega_n](x^\circ) = 1/\hat{\theta}[\Omega_1, \ldots, \Omega_n](x^\circ)$.

Metric Inequality

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- the *strong metric inequality* if $\hat{\vartheta}[\Omega_1, \ldots, \Omega_n](x^\circ) < \infty$.

Theorem 3. $\hat{\vartheta}[\Omega_1, \ldots, \Omega_n](x^\circ) = 1/\hat{\theta}[\Omega_1, \ldots, \Omega_n](x^\circ)$.

Corollary 3.1. $\{\Omega_1, \Omega_2, \ldots, \Omega_n\}$ *is regular at* x° *if and only if it satisfies the strong metric inequality at* x°.

Boundary Condition

Borwein, Jofré, 1998

Ω_1, Ω_2, . . . , Ω_n are closed, $x_i^\circ \in \Omega_i$, $i = 1, 2, \ldots, n$

$$\sum_{i=1}^{n} \Omega_i = \left\{ \omega = \sum_{i=1}^{n} \omega_i : \ \omega_i \in \Omega_i, \ i = 1, 2, \ldots, n \right\}$$

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$$\sum_{i=1}^{n} \Omega_i = \left\{ \omega = \sum_{i=1}^{n} \omega_i : \ \omega_i \in \Omega_i, \ i = 1, 2, \ldots, n \right\}$$

$$\zeta_\rho[\Omega_1, \ldots, \Omega_n](x_1^\circ, \ldots, x_n^\circ) =$$

$$\sup \left\{ r \geq 0 : \ B_r(\sum_{i=1}^{n} x_i^\circ) \subset \sum_{i=1}^{n} (\Omega_i \cap B_\rho(x_i^\circ)) \right\}$$

Boundary Condition

Definition 3. $\{\Omega_1, \Omega_2, \ldots, \Omega_n\}$ satisfies at $x_1^\circ, x_2^\circ, \ldots, x_n^\circ$

- the *boundary condition* if
  $\zeta_\rho[\Omega_1, \ldots, \Omega_n](x_1^\circ, \ldots, x_n^\circ) = 0$ for all $\rho > 0$.
- the *local boundary condition* if
  $\zeta_\rho[\Omega_1, \ldots, \Omega_n](x_1^\circ, \ldots, x_n^\circ) = 0$ for some $\rho > 0$.

Boundary Condition

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- the *local boundary condition* if
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Proposition 2. *Define* $\tilde{x}^\circ = (x_1^\circ, x_2^\circ, \ldots, x_n^\circ) \in X^n$, $\tilde{\Omega}_1 = \Omega_1 \times \ldots \times \Omega_n$, $\tilde{\Omega}_2 = \{(x_1, \ldots, x_n) \in X^n : \sum_{i=1}^n x_i = \sum_{i=1}^n x_i^\circ\}$. $\{\Omega_1, \ldots, \Omega_n\}$ *satisfies the (local) boundary condition at* $x_1^\circ, \ldots, x_n^\circ$ *if and only if* $\{\tilde{\Omega}_1, \tilde{\Omega}_2\}$ *is (locally) extremal at* $\tilde{x}^\circ$.

Regularity of Multifunctions

$F: \ X \rightrightarrows Y$, $\operatorname{gph} F = \{(x,y) \in X \times Y : \ y \in F(x)\}$,
$(x^\circ, y^\circ) \in \operatorname{gph} F$

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$$\hat{\theta}[F](x^\circ, y^\circ) = \liminf_{(x,y) \overset{\operatorname{gph} F}{\to} (x^\circ, y^\circ)} \theta[F](x, y)$$

Regularity of Multifunctions

Definition 4. $F: X \rightrightarrows Y$ is

(1) *extremal* at (x°, y°) if $\theta_\rho[F](x^\circ, y^\circ) = 0$ for all $\rho > 0$.

(2) *locally extremal* at (x°, y°) if $\theta_\rho[F](x^\circ, y^\circ) = 0$ for some $\rho > 0$.

(3) *stationary* at (x°, y°) if $\theta[F](x^\circ, y^\circ) = 0$.

(4) *weakly stationary* at (x°, y°) if $\hat{\theta}[F](x^\circ, y^\circ) = 0$.

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(5) *regular* at (x°, y°) if $\hat{\theta}[F](x^\circ, y^\circ) > 0$.

(5) $\Leftrightarrow$ F is metrically regular at (x°, y°)

Regularity of Multifunctions

Theorem 4. *Define* $\Omega_1 = \text{gph}\,(F)$, $\Omega_2 = X \times \{y^\circ\}$. *The following assertions hold:*

- F *is extremal at* $(x^\circ, y^\circ) \Leftrightarrow \{\Omega_1, \Omega_2\}$ *is extremal at* (x°, y°).
- F *is locally extremal at* $(x^\circ, y^\circ) \Leftrightarrow \{\Omega_1, \Omega_2\}$ *is locally extremal at* (x°, y°).
- F *is stationary at* $(x^\circ, y^\circ) \Leftrightarrow \{\Omega_1, \Omega_2\}$ *is stationary at* (x°, y°).
- F *is weakly stationary at* $(x^\circ, y^\circ) \Leftrightarrow \{\Omega_1, \Omega_2\}$ *is weakly stationary at* (x°, y°).
- F *is regular at* $(x^\circ, y^\circ) \Leftrightarrow \{\Omega_1, \Omega_2\}$ *is regular at* (x°, y°).

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