Ellipsoidal Estimation for Dynamic Systems with Model Uncertainty

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Problem Statement

$$\begin{aligned} x_{k+1} &= A_k x_k + w_k, \\ y_k &= C_k x_k + v_k, \end{aligned}$$

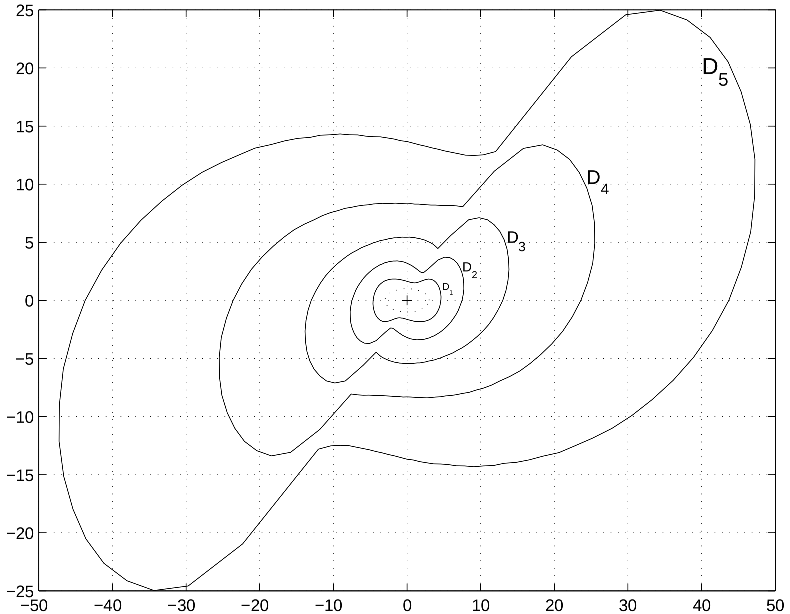
$A_k,\ C_k,\ \ w_k,\ v_k$ are uncertain (unknown-but-bounded)

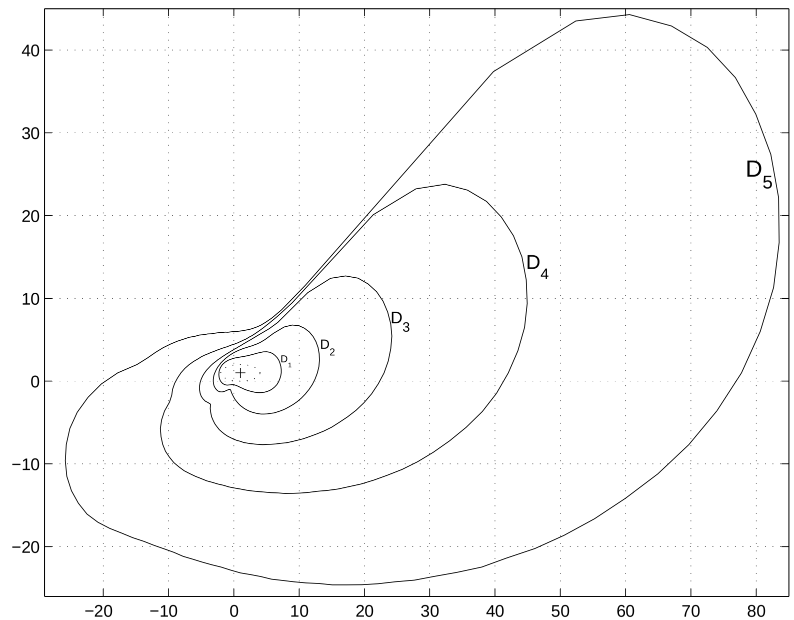
The problem is to find the guaranteed estimate for x_k under given measurements $y_1, ..., y_k$ and initial state x_0 .

Estimation Procedure:

1. Prediction Step (approximation of Sum),
2. Correction Step (approximation of Intersection).

Reachability Sets:





System description:

Linear discrete-time dynamic system

$$\begin{aligned} x_{k+1} &= A_k x_k + w_k, \\ y_k &= C_k x_k + v_k, \end{aligned}$$

Combined quadratic constraints

$$\frac{\|A_k - A_k^0\|^2}{\varepsilon_{A_k}^2} + \frac{\|w_k\|^2}{\delta_{w_k}^2} \leq 1,$$

$$\frac{\|C_k - C_k^0\|^2}{\varepsilon_{C_k}^2} + \frac{\|v_k\|^2}{\delta_{v_k}^2} \leq 1.$$

$$\|x\|^2 = \sum x_i^2, \quad \|A\| = \max_{\|x\| \leq 1} \|Ax\|$$

Approximation of Sum (prediction step)

$$z = (A + \Delta_A)x + w,$$

$$x \in E(c, P), \qquad \frac{\|\Delta_A\|^2}{\varepsilon^2} + \frac{\|w\|^2}{\delta^2} \leq 1. \tag{1}$$

Reachability set: $F = \{\ z : \quad x,\ (\Delta_A, w) \quad \text{from} \quad (1)\ \},$

$$F \subset E(d, Q).$$

Measures of ellipsoidal size:

$$f_1(Q) = tr\ Q^{-1}, \qquad f_2(Q) = -ln\ det\ Q.$$

Theorem 1:

Each ellipsoid in the family $E(d(\tau), Q(\tau))$ with

$$\begin{aligned} d(\tau) &= (1-\delta^2\tau)\, A\, Q_\tau^{-1} P\, c, \\ Q(\tau) &= \frac{\left\{A\, Q_\tau^{-1} A^T + \tau^{-1} I\right\}^{-1}}{1-\xi(\tau)}, \end{aligned}$$

where

$$\begin{aligned} Q_\tau &= (1-\delta^2\tau)\, P - \tau\varepsilon^2 I, \\ \xi(\tau) &= (1-\delta^2\tau)\; c^T P\, c - (1-\delta^2\tau)^2\; c^T P\, Q_\tau^{-1} P\, c, \end{aligned}$$

contains F for all τ such that $0 < \tau < \tau^* = \frac{\lambda_{\min}}{\delta^2\lambda_{\min}+\varepsilon^2}$, where $\lambda_{\min} = \min\, eig\, P$. $\square$

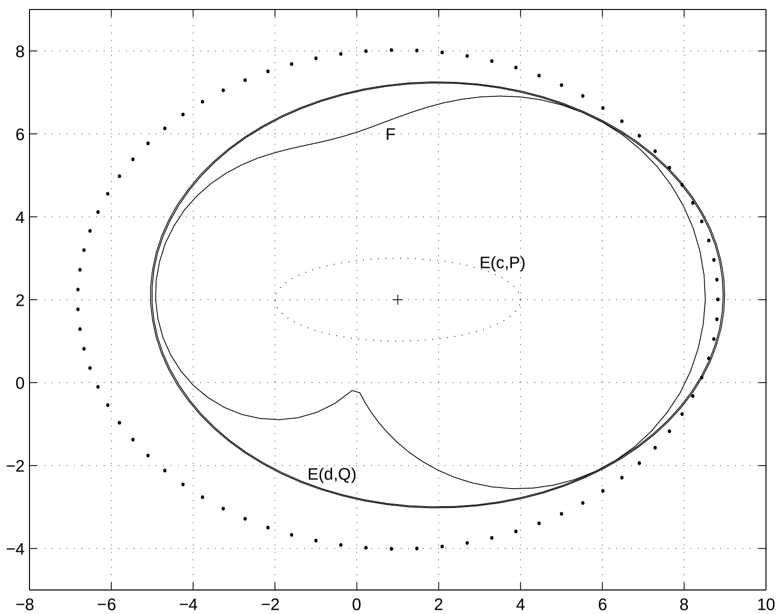
$$\tau_{min} = arg \min_{0<\tau<\tau^*} f_i(Q_\tau), \qquad i = 1, 2$$

$$z = (A + \Delta_A)x + w,$$

$$x \in E(0, P), \qquad \frac{\|\Delta_A\|^2}{\varepsilon^2} + \frac{\|w\|^2}{\delta^2} \le 1.$$

Example 1:

$$c = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \; P = \begin{pmatrix} 1/9 & 0 \\ 0 & 1 \end{pmatrix}, \; A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \varepsilon = \delta = 0.5.$$



Approximation of Intersection (correction step)

$$y = (C + \Delta_C)x + v,$$

$$x \in E(c, P), \qquad \frac{\|\Delta_C\|^2}{\varepsilon^2} + \frac{\|v\|^2}{\delta^2} \le 1. \tag{2}$$

$$y - Cx = \Delta_C x + v,$$
$$\|y - Cx\|^2 \le \varepsilon^2 \|x\|^2 + \delta^2.$$

$$(x - d)^T M (x - d) \le 1,$$

where

$$M = \frac{R}{y^T C R^{-1} C^T y - y^T y + \delta^2}, \quad d = R^{-1} C^T y, \quad R = C^T C - \varepsilon^2 I.$$

Theorem 2:

If $x \in E(c, P),\ P > 0$ and $y = (C + \Delta_C)x + w$

where Δ_C, w satisfy (2),

then $x \in E(g(\tau), Q(\tau))$ with

$$\left.\begin{aligned} Q(\tau) &= (1-\nu)^{-1} Q_\tau, \\ Q_\tau &= (1-\tau)P + \tau M, \\ g(\tau) &= Q_\tau^{-1}[(1-\tau)Pc + \tau M d], \\ \nu &= (1-\tau)\, c^T P c + \tau d^T M d - g(\tau)^T Q_\tau g(\tau), \end{aligned}\right]$$

$$\forall \tau : \quad 0 \le \tau < \tau^* = \min\left\{1, \frac{1}{1-\lambda_{min}}\right\},$$

where $\lambda_{min} = \min eig(M, P)$. □

$$\tau_{min} = arg \min_{0 \le \tau < \tau^*} f_i(Q_\tau), \qquad i = 1, 2$$

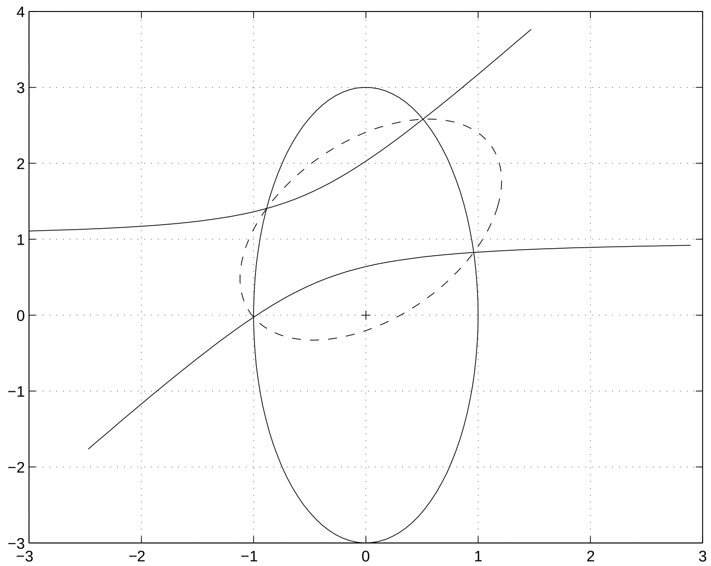
$$y = (C + \Delta_C)x + v,$$

$$x \in E(c, P), \qquad \frac{\|\Delta_C\|^2}{\varepsilon^2} + \frac{\|v\|^2}{\delta^2} \le 1.$$

Example 2:

$y = 1,\ C = (1,\ 2)\ ,\quad \varepsilon = 1.5$ and $\delta = 0.5;$

$$E(c, P):\quad c = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ P = \begin{pmatrix} 1 & 0 \\ 0 & 1/9 \end{pmatrix}.$$



Some Extensions

1. All the results holds true if the spectral matrix norm is replaced with the Frobenius norm in the model uncertainty constraints
2. Separate uncertainties

Conclusions

1. one-step optimal estimates for systems with no measurements
2. the correction step based on similar principles is suboptimal

Paper:

Polyak, Nazin, Durieu, & Walter (2004).
Ellipsoidal parameter or state estimation under model uncertainty.
Automatica, **40**(7)