Stability regions in the parameter space: D-decomposition revisited ⋆

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Abstract

The challenging problem in linear control theory is to describe the total set of parameters (controller coefficients or plant characteristics) which provide stability of a system. For the case of one complex or two real parameters and SISO system (with a characteristic polynomial depending linearly on these parameters) the problem can be solved graphically by use of so called D-decomposition. Our goal is to extend the technique and to link it with general $M-\Delta$ framework. On this way we investigate the geometry of D-decomposition for polynomials and estimate the number of root invariant regions. Several examples verify that these estimates are tight. We also extend D-decomposition for the matrix case, i.e. for MIMO systems. For instance, we partition real axis or complex plane of the parameter k into regions with invariant number of stable eigenvalues of the matrix $A+kB$. Similar technique can be applied to double-input double-output systems with two parameters.

Key words: Stability analysis; Stability domain; Linear systems; Parameter space.

1 INTRODUCTION

Consider a linear system depending on a vector parameter k with a characteristic polynomial $p(s,k)$. The boundary of a stability domain (in the space k) is given by the equation

$$p(j\omega,k)=0,\ -\infty<\omega<\infty, \tag{1}$$

that is the imaginary axis (the boundary of instability in the root plane) is mapped into the parameter space. If $k\in\mathbb{R}^2$ (or $k\in\mathbb{C}$) then we have two equations (real and imaginary part of (1)) in two variables and (in general) can define the parametric curve $k(\omega), -\infty<\omega<\infty$ defining a boundary of the stability domain. Moreover, the curve $k(\omega)$ divides the plane into root invariant regions (i.e. regions with a fixed number of stable and unstable roots of $p(s,k)$). This is the basic idea of D-decomposition approach. The idea can be traced to (Vishnegradsky, 1876) who reduced a cubic polynomial to the form $p(s,k)=s^3+k_1s^2+k_2s+1$ and treated the coefficients $k_1,\ k_2$ as parameters. Then equation (1) yields $k_1\omega^2=1,\ \omega(k_2-\omega^2)=0$. Eliminating ω we get that D-decomposition is given by the hyperbola $k_1k_2=1$. The stability domain is the set $k_1k_2>1$.

For the general case similar ideas were exploited by (Frazer & Duncan, 1929; Sokolov, 1946; Andronov & Mayer, 1946) (the last two papers deal with time-delay systems). Moreover, Nyquist plot can be considered as the realization of the same idea. But it was Yu. Neimark (Neimark, 1948; Neimark, 1949) who developed the rigorous algorithm (and coined the name "D-decomposition"). In the Western literature the technique is described first by (Mitrovic, 1959); he also proposed the mapping of contours other that imaginary axis. This line of research was significantly developed by Siljak (1964, 1966, 1969). He extended the approach for nonlinear systems and for the case of nonlinear parameter dependence. In his works D-decomposition (which he calls the parameter plane method) was broadened to become a useful tool for design purposes. Neimark's method is also investigated in (Lehnigk, 1966), where some results on multi-parameter cases can be found. More recent presentation of D-decomposition technique can be found in (Ackermann, 2002) (Section 4.5); the technique is often exploited for the design of low-order controllers; e.g. (Bhattacharyya *et al.*, 2003; Ackermann & Kaesbauer, 2003).

In this paper we extend the approach to systems presented at state space form. More specifically, given a

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class $\mathcal{K}$ of $r \times m$ matrices K, find all the matrices $K \in \mathcal{K}$ such that $A + BKC$ is stable:

$$D = \{K \in \mathcal{K} : \; A + BKC \text{ is stable}\}. \qquad (2)$$

Here A, B, C are given real matrices of dimensions $n \times n, \; n \times r, \; m \times n$ respectively; stability is understood either in continuous-time sense (all eigenvalues are in the open LHP) or discrete-time sense (all eigenvalues are in the open unit disc). Class $\mathcal{K}$ may be different; below we analyze in detail the simplest cases:

$$K = k \in \mathbb{R}^n \text{ or } K = k^T, \; k \in \mathbb{R}^n \;\; (\, m = 1 \text{ or } r = 1) \qquad (3.a)$$

$$K = kI, \quad k \in \mathbb{R} \text{ or } k \in \mathbb{C}, \quad m = r, \qquad (3.b)$$

$$K \in \mathbb{R}^{2\times 2}, \qquad (3.c)$$

where all calculations can be performed explicitly in the graphical form. Case (3.a) is equivalent to the polynomial framework, two others are essentially matrix ones. Nevertheless we present general description of D-decomposition. It is closely related to the standard $M - \Delta$ setting.

Problem (2) arises in the design or robustness studies. For instance, to find all stabilizing static output controllers for the system

$$\dot{x} = Ax + Bu, \; y = Cx \qquad (4)$$

one can construct the set D (2) with $\mathcal{K} = \mathbb{R}^{r\times m}$; here K plays the role of the feedback gain. On the other hand, if A is a nominal stable matrix and it is perturbed as $A + BKC$, where K is a constant $r \times m$ matrix, then (2) provides all admissible perturbations which preserve stability. Of course, if we know the boundary of the stability domain ∂D, then we can find the distance to it:

$$\rho = \min_{K\in\partial D} \|K\|. \qquad (5)$$

The quantity ρ^{-1} is closely related to μ (structured singular value) (Zhou *et al.*, 1996). If $\mathcal{K}$ is the set of all $\mathbb{C}^{r\times m}$ ($\mathbb{R}^{r\times m}$) matrices, then ρ is complex (real) stability radius (Hinrichsen & Pritchard, 1986; Qiu *et al.*, 1995). Of course, the knowledge of the entire set D provides much more information than the value of ρ. For instance, for design purposes a designer can solve performance or specification problems on the set of the stabilizing controllers D.

Here is a simple example demonstrating the advantages of the closed form description of the stability domain D. Consider a SISO plant with the transfer function $G(s) = \dfrac{(s-1)(s-2)}{(s+1)(s^2+s+1)}$. It was studied in (Francis, 1987) to design a controller $C(s)$ that minimizes sensitivity for low frequencies: $\min F, \; F = \max\limits_{0\le\omega\le 0.01} |S(j\omega)|\,, \; S(s) = \dfrac{1}{1 + C(s)G(s)}$. The problem was replaced with H^∞ optimization: $\min\limits_{C} J, \; J = \|WS\|_\infty, \; W(s) = \dfrac{s+1}{10s+1}$ and optimal controller of the fourth order was found; the original performance index for this controller was $F_\infty = 0.1202$.

Let us try to stabilize the plant by PI controller of the form $C(s) = k_1 + k_2/s$. The characteristic polynomial is $p(s,k) = (s-1)(s-2)(k_1 s + k_2) + s(s+1)(s^2+s+1)$ and D-decomposition of $\{k_1, k_2\}$-plane is given by the curve $k_1(\omega) = -\text{Re}G^{-1}(j\omega)$, $k_2(\omega) = \omega \text{Im}G^{-1}(j\omega)$, $\omega \neq 0$ and the straight line (corresponding to $\omega = 0$) $k_2 = 0$, as shown in Fig. 1. The numbers 0, ..., 4 indicate the number of stable roots in the corresponding region, thus 4 refers to the stability domain D (we shall use this notation elsewhere). The direct optimization of F subject to $k \in D$ has been performed in (Kiselev & Polyak, 1999). For $K^* = (-0.04747, \; 0.1328)$ one has $F = 0.0373$; it is much smaller than the result obtained by H^∞ optimization. Note that this is the first order controller versus the fourth order one for H^∞ setting. The reason is that we deal with the original performance index F and do not need the auxiliary index J. The shape of the stability domain in this example is very simple. We shall encounter much more complicated situations in the examples below.

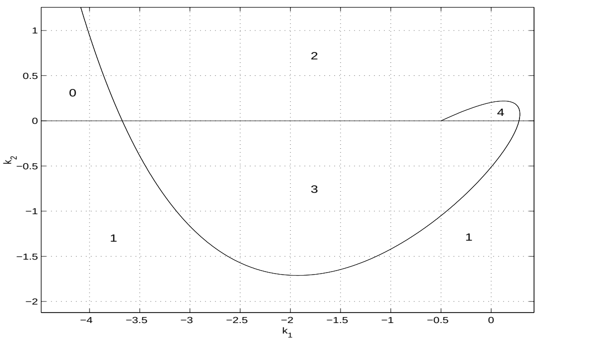

Fig. 1. Example of D-decomposition

Instead of direct optimization on stability domain D we can get more information on the roots of the characteristic polynomial inside D by similar graphical tools. For instance, we can be interested in stability degree $\sigma > 0$, this means that instead of the imaginary axis $j\omega$ we deal with the shifted axis $-\sigma + j\omega$. Then D-decomposition is generated by the mapping of this axis: $p(-\sigma + j\omega, k) = 0$. This approach has been proposed by (Neimark, 1949) and significantly developed by (Mitrovic, 1959) and Siljak (1964, 1966, 1969). For the above example the extended D-decomposition is given by the parametric curve $k_1(\omega) = \dfrac{1}{\omega} v(\omega), \; k_2(\omega) = u(\omega) + \dfrac{\sigma}{\omega} v(\omega)$, where $u(\omega) + jv(\omega) = H(-\sigma + j\omega)$, $H(s) = -sG^{-1}(s)$, and the singular line (corresponding $\omega = 0$) $k_1\sigma + k_2 = H(-\sigma)$. Such detailed decomposition is shown in Fig. 2 for several σ; the maximal available stability degree is $\sigma^* = 0.51$ (for σ^* the domain shrinks into a point). Of course we can deal with contours other than shifted imaginary axis, for instance, with a sector (as in Fig. 3)

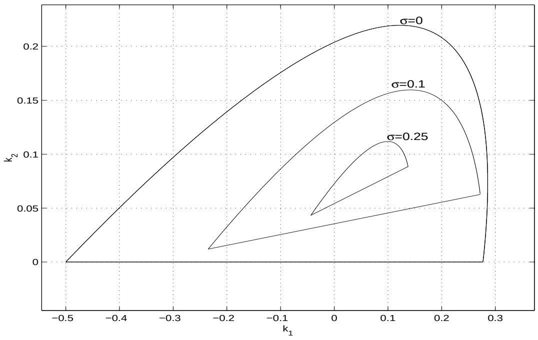

Fig. 2. Detailed D-decomposition $s = -\sigma + j\omega$

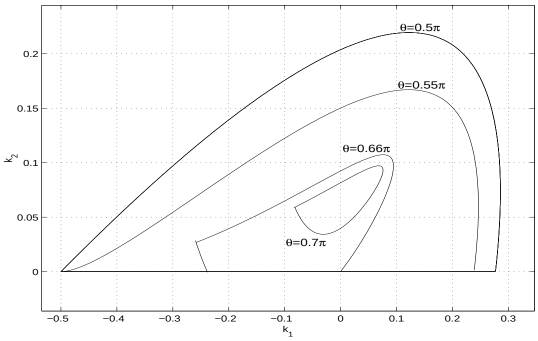

Fig. 3. Detailed D-decomposition $s = \omega e^{\pm j\theta},\ \pi/2 \le \theta < \pi$

or a hyperbolic placement of the roots. The technique of D-decomposition remains similar. This information on the root placement inside the stability domain is very helpful for the design purposes. We provide one more example with stability degree contours for discrete-time matrix case (Section 5.3 below). However, we will not develop this line of research and deal with the simplest situations of the root placement – the left half-plane (for continuous-time systems) and the unit disc (for discrete-time systems). Additional detailed decomposition can be a subject of future research. Several examples below illustrate how the technique can be extended to become a practical tool for a designer. On the other hand we restrict ourselves with a presentation of basic results of the proposed approach and do not consider practical applications.

The paper is organized as follows. In Section 2 we present a general equation of D-decomposition and exhibit its links with $M - \Delta$ approach to robustness. The rank-one case (i.e. single-input or single-output systems) will be addressed in Section 3. This case has much in common with the classical D-decomposition for polynomials. The main contribution here is the study of the D-decomposition geometry. In particular, we estimate the number of all root invariant regions as well as the number of simply connected stability regions and verify that the estimates are not conservative. The related results can be found in our papers (Gryazina, 2004; Gryazina & Polyak, 2005). Of course the number of root invariant regions is mainly of theoretical interest, however the knowledge of this number can be a helpful indicator on the complexity of various situations for a designer. The situation with $K = kI$, k being a real or complex scalar, is analyzed in Section 4. Section 5 is devoted to double-input double-output systems with two parameters in a gain matrix. Final Section 6 contains conclusions.

2 EQUATION OF D-DECOMPOSITION

Let $A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times r},\ C \in \mathbb{R}^{m\times n}$ be fixed real matrices while $\mathcal{K}$ is a class of real or complex $r \times m$ matrices. The class will be specified later. The only property required at the moment is: $\mathcal{K}$ is a connected set, i.e. $K_0 \in \mathcal{K},\ K_1 \in \mathcal{K}$ imply the existence of a parametric family $K(t) \in \mathcal{K},\ 0 \le t \le 1,\ K(0) = K_0,\ K(1) = K_1$ with $K(t)$ continuously depending on t. We also assume that A has no imaginary eigenvalues in the continuous-time case and no eigenvalues on the unit circumference in the discrete-time case.

Define the transfer function

$$M(s) = C(A - sI)^{-1}B \qquad (6.a)$$

for continuous-time case and

$$M(z) = C(A - zI)^{-1}B \qquad (6.b)$$

for discrete-time case, where variables s and z are used to distinguish continuous-time and discrete-time settings.

Definition. The set $D(l) = \{K \in \mathcal{K} :\ A + BKC$ has l stable eigenvalues$\}$, $l = 0, ..., n$, is called an eigenvalue invariant domain (thus $D(n)$ is the set of stabilizing matrices D). An equation for the boundaries of $D(l),\ l = 0, ..., n$, is called D-decomposition of the parameter space. Simply connected components of $D(l)$ are eigenvalue invariant regions.

The natural question arises: what is the use of finding all eigenvalue invariant domains, while in practice we are interested in the stability domain only? The first answer to the question is very simple – we are unable to construct the stability domain separately, the only way suggested by the proposed technique is to provide the complete decomposition of the parameter space and then to pick out the stability regions (if they exist). While stability regions are found, we can continue their investigation by construction of equal stability degree or similar contours (see Fig. 2, 3 above). On the other hand there are some situations when eigenvalue invariant domains are also of interest. For instance, if one deals with Nyquist diagram under uncertainty, one should guarantee that all transfer functions under consideration have the same number of stable poles.

The following result is the base for D-decomposition.

Theorem 1

The equation

$$\det(I + M(j\omega)K) = 0, \quad -\infty < \omega < \infty \qquad (7.a)$$

or

$$\det(I + M(e^{j\omega})K) = 0, \quad 0 \le \omega < 2\pi \qquad (7.b)$$

defines D-decomposition of the class $\mathcal{K}$, i.e. if $Q \subset \mathcal{K}$ is a connected set and $\det(I + M(j\omega)K) \neq 0, -\infty < \omega < \infty$, $\forall K \in Q$ (continuous time) or $\det(I + M(e^{j\omega})K) \neq 0$, $0 < \omega < 2\pi$, $\forall K \in Q$ (discrete time), then $A + BKC$ has the same number of stable and unstable eigenvalues for all matrices K in Q.

Proof. Consider the continuous-time case first. Assume that for $K_0 \in Q$ and $K_1 \in Q$ matrices $A + BK_0C$ and $A + BK_1C$ have different number of stable eigenvalues. Due to continuous dependence of eigenvalues on matrices there exist $0 < t^* < 1$ such that $K^* = K(t^*) \in Q$ and the matrix $A + BK^*C$ has a pure imaginary eigenvalue. That is the matrix $A + BK^*C - j\omega I$ is singular for some real ω. Hence $0 = \det(A + BK^*C - j\omega I) = \det(A - j\omega I)\det(I + (A - j\omega I)^{-1}BK^*C) = \det(A - j\omega I)\det(I + C(A - j\omega I)^{-1}BK^*) = \det(A - j\omega I)\det(I + M(j\omega)K^*)$ and thus $\det(I + M(j\omega)K^*) = 0$.

The proof for the discrete-time case is similar. □

Note that we do not assume $M(s)$ (or $M(z)$) to be a stable matrix as it is common in the theory of MIMO systems.

The assumption on the lack of eigenvalues of A on the boundary of the LHP or the unit disk is significant not only from formal point of view (otherwise $M(j\omega)$ or $M(e^{j\omega})$ is not defined), but to avoid degenerate situations as follows. Suppose $m = n$ and A, C have a common eigenvector h, corresponding to a zero eigenvalue: $Ah = Ch = 0, h \neq 0$. Then $A + BKC$ is singular for any K, thus equation of D-decomposition $\det(A + BKC - j\omega I) = 0$ is satisfied (with $\omega = 0$) for any K.

Equations (7.a-7.b) define D-decomposition in implicit form. Our main goal below is to point out some particular cases, where the boundaries can be constructed explicitly. This is in contrast with μ -analysis, where the problem $\min_{K \in \mathcal{K},\ \det(I+M(j\omega)K)=0} \|K\|$ is under consideration (i.e. one is seeking for the largest ball contained in D). Better approximation of D was proposed in (Barmish & Polyak, 1993); it was the ellipsoid of the largest volume, inscribed in D.

The complete description of D-decomposition is possible for exceptional cases only. They will be addressed in the following Sections.

3 SINGLE-INPUT OR SINGLE-OUTPUT SYSTEMS

Suppose we deal with a single-input system i.e. $r = 1$. Then K is a row vector: $K = [k_1, ..., k_m]$ while M is a column vector $M = [M_1, ..., M_m]^T \in \mathbb{C}^m$. For $a, b \in \mathbb{C}^m$ one has $\det(I + ab^T) = 1 + \sum_{i=1}^m a_i b_i$. Thus (7.a) is reduced to

$$1 + \sum_{i=1}^{m} k_i M_i(j\omega) = 0 \qquad (8)$$

We conclude that in this case equation (7.a) is linear in K. Similarly, for single-output systems $m = 1$, $K = [k_1, ..., k_r]^T$ is a column vector and we obtain the same equation (8).

Linearity of D-decomposition equation allows to find easily the stability radius (5) for various l^p norms of the vector k. This link between robust stability and D-decomposition has been emphasized in (Neimark, 1992). On the other hand the connection between μ-analysis and the known results on parametric robustness (Ackermann, 2002; Barmish, 1995; Bhattacharyya *et al.*, 1995; Kogan, 1995; Tsypkin & Polyak, 1991) has been pointed out by Chen *et al*, (1994). Scalar constant gain problem has been solved semi-analytically in (Ozguler & Kocan, 1994), the result was extended to compute stabilizing controllers of a given order (Saadaoui & Ozguler, 2005). We will not highlight these links, but will focus on the simplest cases, when equation (8) provides graphical tools to describe D-decomposition in the space of parameters k.

3.1 ONE REAL PARAMETER

For a single-input single-output system with a real scalar gain k (7.a) reads as

$$1 + kM(j\omega) = 0 \qquad (9)$$

with a scalar transfer function $M(s) = \frac{b(s)}{a(s)}$, where $a(s)$, $b(s)$ are polynomials of degree n. We avoid the situations when $a(s)$, $b(s)$ have a common imaginary (or zero) root. Thus (7.a) is equivalent to $-1/k = M(j\omega)$ or to the standard Nyquist diagram: the critical values of the gain k (such that correspond to a change of the stable roots number for the polynomial $p(s, k) = a(s) + kb(s)$) are defined by the intersections of the Nyquist plot $M(j\omega)$ with the real axis. Note that the alternative way to analyze the situation is root locus technique (Evans, 1954); see also `rlocus` command in MATLAB. However, D-decomposition approach allows to find critical points analytically and to estimate their number; that is not possible in the framework of root locus. On the contrary root locus provides more information on the location of the roots inside the stability region.

Theorem 2

The real axis can be divided into $m \leq n + 2$ root invariant intervals $(-\infty, k_1), (k_1, k_2), ..., (k_m, \infty)$ with $-\infty < k_1 < k_2 < ... < k_m < k_{m+1} < \infty$ such that for $k_i < k < k_{i+1}$ the polynomial $p(s, k)$ has the invariant number ν_i of stable roots. Moreover, the number of stability intervals (i.e. intervals (k_i, k_{i+1}) with $\nu_i = n$) is no more than $\lfloor \frac{n}{2} \rfloor + 1$ ($\lfloor \alpha \rfloor$ is the biggest integer smaller or equal α).

The proof of this and consequent theorems is given in the Appendix.

The algorithm for finding the critical values k_i and the numbers ν_i is as follows.

Algorithm 1.

a. Find (e.g. via Routh table) the number of stable roots for $b(s)$, this is ν_0.
b. Solve the polynomial equation

$$\text{Im } M(j\omega) = 0; \quad (10)$$

we can restrict ourselves with $0 \le \omega \le \infty$ because $\text{Im } M(j\omega) = \text{Im } M(-j\omega)$. Let $0 = \omega_0 < \omega_1 < \omega_2 < ... < \omega_m$ be the roots of (10). Calculate $u_i = \text{Re}(M(j\omega_i)),\ i = 0, ..., m,\ u_{m+1} = b_n/a_n$.
c. Order quantities $-\frac{1}{u_i},\ i = 0, ..., m+1$ and denote them $k_1 < ... < k_{m+2}$ such that $k_s = -\frac{1}{u_{i_s}}$.
d. Numbers ν_s are calculated successfully: $\nu_{s+1} = \nu_s + 2\text{sign}\varphi(\omega_{i_s}),\ \varphi(\omega_{i_s}) = \frac{d}{d\omega}\text{Im } M(j\omega_{i_s})$, if $\varphi(\omega_{i_s}) \neq 0$. Otherwise ω_{i_s} is the multiple root for (10). If the multiplicity of ω_{i_s} is odd then $\nu_{s+1} = \nu_s$; if the multiplicity of ω_{i_s} is even then φ is the lowest nonvanishing derivative.

The examples below verify that the estimates of the number of root invariant intervals and stability intervals provided by Theorem 2 are not conservative. But we start with the example, where D-decomposition is lacking — for any k the polynomial $p(s,k)$ has the same number of stable and unstable roots.

Example 1

Let for $n = 4m$, $p(s,k) = s^n + ks + 1$. Then $p(j\omega,k) = \omega^n + kj\omega + 1$, and $\text{Re}\, p(j\omega,k) \neq 0$ for all k, ω. Thus there are no critical values of ω, and the entire real axis is the single root invariant region for the polynomial $p(s,k)$ (indeed it has $2m$ stable and $2m$ unstable roots for any k). Minor variation of the example, $p(s,k) = k(s^n+1)+s$ provides real axis with the exception of the origin as the root invariant region: for any $k \neq 0, p(s,k)$ has $2m$ stable and $2m$ unstable roots.

Note that this example (as well as many consequent ones) is very simple and a bit artificial. Our aim is not to consider practical problems but to exhibit the simplest or the most sophisticated diagrams of D-decomposition.

Example 2

This is the modification of a $2D$ example from (Nikolayev, 2002). The polynomial $p(z,k) = z^n + kz^{n-1} + \alpha z^{n-2} + \beta$ with $1 < \alpha < 1 + \frac{2}{(n-2)^2}$, $\beta = 1 - \alpha - \frac{1}{n^2}$ has $\lfloor \frac{n}{2} \rfloor$ stability intervals in k.

Indeed, D-decomposition is given by $k = -e^{j\omega} - \alpha e^{-j\omega} - \beta e^{-(n-1)j\omega} = \psi(\omega)$. The equation $\text{Im } \psi(\omega) = 0$ reads $(\alpha - 1)\sin\omega + \beta\sin(n-1)\omega = 0$; it has n solutions on $[0,\pi]$ because $|\beta| > |\alpha - 1|$. The values $0 = \omega_1 < \omega_2 < ... < \omega_n \le \pi$ increase monotonically and critical values $k_i = \text{Re } \psi(\omega_i) = (\alpha+1)\cos\omega + \beta\cos(n-1)\omega$ also increase monotonically. The derivatives of $\text{Im } \psi(\omega)$ change sign at the points ω_i. For large k the polynomial $p(z,k)$ has $n-1$ stable roots (close to zero) and one unstable root ($z \approx -k$). Thus $\nu_0 = n - 1$ and there are $\lfloor \frac{n}{2} \rfloor$ stability intervals for k varying from $-\infty$ to $+\infty$. The case of $n = 8,\ \alpha = 1.01,\ \beta = -0.026$ is depicted in Fig. 4.

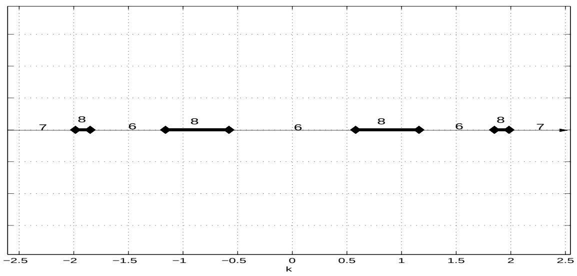

Fig. 4. Root invariant intervals in Example 2

Example 3

The idea of this example belongs to (Neimark, 1949). Consider $p(s,k) = Q(s)(1 + ks^2) + \alpha$, where Q is a Hurwitz polynomial. Then $k = \dfrac{\alpha + Q(j\omega)}{\omega^2} = \psi(\omega)$ and it can be proved that when α is small enough, the equation $\text{Im } \psi(\omega) = 0$ has $n/2$ solutions with alternating stability/instability intervals. However, these intervals tend to zero very fast when $|Q(j\omega)|$ increases. To avoid this difficulty we construct the polynomial $Q(s)$ as follows: $Q(j\omega) = U(t) + j\omega V(t),\ t = \omega^2$. Take $U(t) = T_m(1-t),\ V(t) = T_{m-1}(1-t)$, where $T_m(t)$ is Chebyshev polynomial $T_m(t) = \cos(m \arccos t)$. Then $U(t),\ V(t)$ have real alternating roots only and due to Hermite-Biehler criterion the polynomial $Q(s)$ is stable. But $|U(t)| \le 1,\ |V(t)| \le 1$ for $0 \le t \le 2$ as follows from the estimates of Chebyshev polynomials. Hence $|Q(j\omega)| \le (1 + 2 \cdot 1)^{1/2} \le \sqrt{3}$ for $0 \le \omega \le \sqrt{2}$ and $Q(j\omega)$ intersects all coordinate axes n times for this ω interval. D-decomposition for the parameter k and $n = 16, \alpha = 0.01$ is depicted in Fig. 5; there are $\lfloor \frac{n}{4} \rfloor$ stability intervals, where $n = 2m$ is the degree of $Q(s)$.

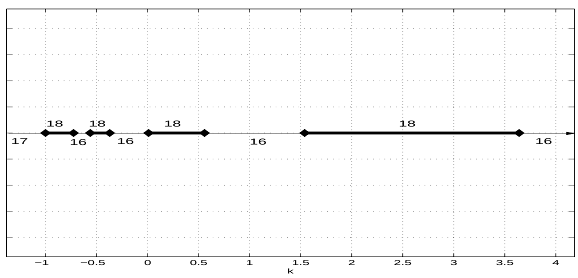

Fig. 5. Root invariant intervals in Example 3

3.2 ONE COMPLEX PARAMETER

We are in the same setting as above ($r = m = 1$) but now $k \in \mathbb{C}$. Equation (9) gives the formula

$$k(\omega) = -M(j\omega)^{-1} = -\frac{a(j\omega)}{b(j\omega)}, \quad (11)$$

where $b(j\omega)$ has no roots with zero real part. The same result follows from the direct analysis of the characteristic polynomial

$$p(s,k) = a(s) + kb(s). \quad (12)$$

It has imaginary roots $j\omega$ for k defined by (11). Curve (11) for $-\infty < \omega < \infty$ decomposes a complex plane into root invariant regions. Their number is estimated below.

Theorem 3

The number N of root invariant regions for the polynomial (12) on the complex plane k is: $N \leq (n-1)^2 + 2$.

This result is valid for both continuous-time and discrete-time polynomials. The proof of Theorem 3 exploits some tools of the algebraic geometry (e.g. Bezout theorem on the number of real roots for two polynomials in two variables and Euler formula).

Example 4

The polynomial $p(z,k) = z^n + kz^{n-1} + \alpha$, where $k \in \mathbb{C}$, has $(n-1)^2 + 1$ root invariant regions for $\alpha > 1$ and two root invariant regions for $\alpha < 1/(n-1)$. The D-decomposition is given by the parametric curve $k(\omega) = -e^{j\omega} - \alpha e^{-j\omega(n-1)}$, $0 \leq \omega < 2\pi$, which describes a hypotrohoid. This curve is generated by a moving point on the complex plane. This motion is a superposition of two rotations. First rotation has radius 1 and frequency 2π while the second one has radius α and frequency $(n-1)2\pi$. For $\alpha > 1$ this curve consists of n arcs, each arc intersects any other twice. Thus the calculation of the number of intersections gives $N = n^2 - 2n + 2$; this is just one region less the upper bound given by Theorem 3. For $n = 6,\ \alpha = 1.5$ the picture is shown in Fig. 6. It is interesting to note that there are no stability regions in this case. For $n = 6,\ \alpha = 0.15$ the picture is shown in Fig. 7.

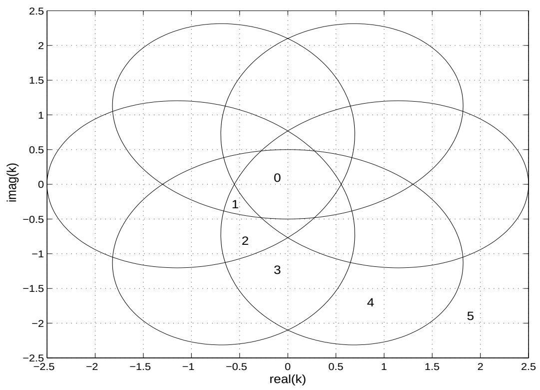

Fig. 6. Max number of root invariant regions in Example 4

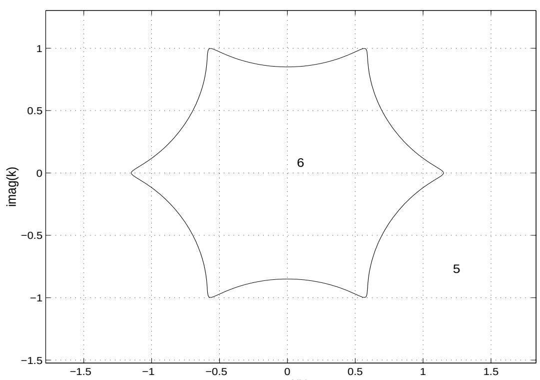

Fig. 7. Min number of root invariant regions in Example 4

Example 5

The example with large number of regions in D-decomposition and nonempty stability domain is constructed as follows. Let $p(s,k) = Q(s) + k$, where $Q(s)$ is the stable polynomial from Example 3, i.e. $Q(j\omega) = U(t) + j\omega V(t)$, $t = \omega^2$, $U(t) = T_n(1-t)$, $V(t) = T_{n-1}(1-t)$, and T_k is Chebyshev polynomial $T_k = \cos(k \arccos t)$. Then D-decomposition is generated by the curve $k(\omega) = -Q(j\omega)$ coinciding with Mikhailov plot of $-Q(s)$ (see Fig. 8 for $n = 10$). The number N of D-decomposition regions is 32 for deg $Q(s) = n = 10$. The region containing the origin is the stability domain (because p(s,0)=Q(s) is a stable polynomial). It is worth mentioning that this domain is convex due to the result of (Hamann & Barmish, 1993). Note that the minimal number of root invariant regions is one, as the following example confirms.

Example 6

D-decomposition for the polynomial $s^n + k$, where $n = 2m$, $k \in \mathbb{C}$, consists of one ray and there are m stable roots for any k except this ray. Indeed, D-decomposition is given by $k(\omega) = -(j\omega)^n = -(-1)^m \omega^n$, $\omega \in (-\infty,\ \infty)$, i.e. ray $(-\infty, 0]$ for m even and $[0, \infty)$ for m odd.

Using the mapping $s = \frac{z+1}{z-1}$, we can proceed from the continuous-time case to the discrete one. Thus the discrete analog of Example 6 is $(z+1)^n + k(z-1)^n$ and it also has one root invariant region.

3.3 TWO REAL PARAMETERS

This is the case of single-input double-output ($r = 1,\ m = 2$) or double-input single-output ($r = 2,\ m = 1$) systems. Parameters $k_1,\ k_2$ are assumed to be real and equation of D-decomposition (8) reads $1 + k_1 M_1(j\omega) + k_2 M_2(j\omega) = 0$. For transfer functions $M_1(s) = \frac{b(s)}{a(s)},\ M_2(s) = \frac{c(s)}{a(s)}$ the characteristic polynomial is

$$p(s,k) = a(s) + k_1 b(s) + k_2 c(s) \qquad (13)$$

and above equation is reduced to

$$p(j\omega,k) = a(j\omega) + k_1 b(j\omega) + k_2 c(j\omega). \qquad (14)$$

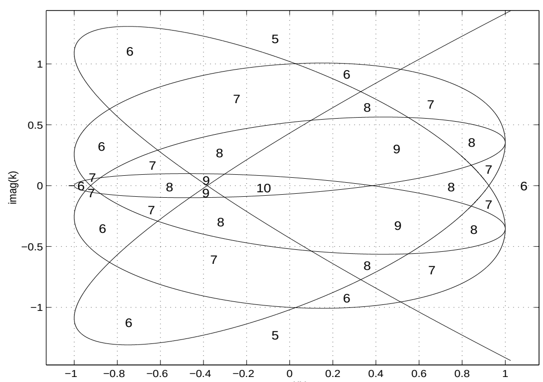

Fig. 8. Root invariant regions in Example 5

This is the classical setting of D-decomposition (Neimark, 1948 and 1949), where polynomials $a(s)$, $b(s)$, $c(s)$ have the form $a(s) = a_0 + a_1 s + ... + a_n s^n$, $a_n \neq 0$, $b(s) = b_0 + b_1 s + ... + b_n s^n$, $c(s) = c_0 + c_1 s + ... + c_n s^n$. Solving (14) with respect to $k_1,\ k_2$ for a fixed $\omega \in \mathbb{R}$ we get for the nonsingular case ($\Delta \neq 0$):

$$k_1 = -\frac{\Delta_1}{\Delta},\ k_2 = -\frac{\Delta_2}{\Delta}, \qquad (15)$$

where $\Delta = \begin{vmatrix} U_b & U_c \\ V_b & V_c \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} U_a & U_c \\ V_a & V_c \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} U_b & U_a \\ V_b & V_a \end{vmatrix}$; $a(j\omega) = U_a + j\omega V_a$, $b(j\omega) = U_b + j\omega V_b$, $c(j\omega) = U_c + j\omega V_c$.

For $\Delta \neq 0$ the above formulae define the parametric curve $k_1(\omega),\ k_2(\omega),\ \omega \in \mathbb{R}$. It is symmetric in ω: $k_i(-\omega) = k_i(\omega),\ i = 1, 2$, thus it suffices to take $0 \neq \omega < \infty$. There are two straight lines for $\omega = 0$, $L_1:\ a_0 + k_1 b_0 + k_2 c_0 = 0$, and for $\omega = \infty$, $L_2:\ a_n + k_1 b_n + k_2 c_n = 0$.

For $\omega = 0$ we have only real part of $p(j\omega, k)$ nonvanishing and (14) reduces to L_1. The line L_2 relates to the case of degree dropping, then the number of stable roots can change. If both $c_n = b_n = 0$, there is no degree dropping, and L_2 is lacking. The curve $k(\omega)$ (15) starts at a point on L_1 (for $\omega = 0$) and terminates at a point on L_2. If $\Delta = 0$ but $\Delta_1 \neq 0$ or $\Delta_2 \neq 0$, then the curve (15) goes to infinity; it can have discontinuity at these points. If this situation is met for all ω, the curve (15) is lacking, this means that the entire plane k_1, k_2 is the root invariant region (see Example 9 below). Finally, if $\Delta = \Delta_1 = \Delta_2 = 0$, then equation (15) reduces to one equation in two variables, i.e. it defines a straight line. These lines (including $L_1,\ L_2$) we call *singular*, as well as the corresponding values of ω – *singular frequencies* (they include $\omega = 0,\ \omega = \infty$). Thus D-decomposition of the plane $\{k_1,\ k_2\}$ is generated by the curve $k(\omega)$, $0 \neq \omega < \infty$ (15) and singular lines. The number of stable roots in each region defined by this decomposition changes at two with crossing the curve (15) and at one with crossing a singular line.

Some minor changes should be done for discrete-time case. Equation (14) becomes

$$a(e^{j\omega}) + k_1 b(e^{j\omega}) + k_2 c(e^{j\omega}) = 0,\ 0 \neq \omega < 2\pi \qquad (16)$$

and the curve of D-decomposition is given by (15), where $a(e^{j\omega}) = U_a + jV_a$, $b(e^{j\omega}) = U_b + jV_b$, $c(e^{j\omega}) = U_c + jV_c$. Singular frequencies are $\omega = 0,\ \omega = \pi$ and singular straight lines $L_1,\ L_2$ due to (16) are replaced with $L_1:\ a(1) + k_1 b(1) + k_2 c(1) = 0$, $L_2:\ a(-1) + k_1 b(-1) + k_2 c(-1) = 0$. (Here $a(1)$, $b(1)$ etc. are values of the polynomials $a(z)$, $b(z)$ for $z = 1$.) Other singular frequencies are defined by the condition $\Delta = \Delta_1 = \Delta_2 = 0$. The curve separates the regions with ± 2 difference in the number of stable roots and singular lines separate the regions with ± 1 difference in the number of stable roots.

Theorem 4
The number N of root invariant regions for the polynomial (13) on the $\{k_1, k_2\}$ plane has the following upper bound: $N \leq 2n(n-1) + 3$.

The smallest number of root invariant regions is one, see the example below.

Example 7
Let $n = 4m$, $p(s, k) = s^n + k_1 s^3 + k_2 s + 1$. Then equation $p(j\omega, k) = 0$ has no solutions for arbitrary ω (because Re $p(j\omega, k) \neq 0$) and R^2 plane is root invariant region: for any k the polynomial $p(s, k)$ has $2m$ stable and $2m$ unstable roots.

This example can be easily extended for any arbitrary number of parameters. Let $p(s, k) = a(s^{4m}) + \sum_{i=1}^{r} k_i b_i(s)$, where $a(t) > 0$ for $t \geq 0$ and $b_i(s)$ are odd polynomials: $b_i(-s) = -b_i(s)$, then $p(s, k)$ has the same number of stable/unstable roots for all $k \in \mathbb{R}^r$. For instance, the polynomial $p(s, k) = 0.1s^{24} + 7.4s^{16} - 13.1s^8 + k_1 s^7 + k_2 s^5 + k_3 s^3 + k_4 s + 15.6$ (suggested as a test example by a reviewer) has entire $\mathbb{R}^4$ as a root invariant region because $a(t) = 0.1t^6 + 7.4t^4 - 13.1t^2 + 15.6 \geq 7.4t^4 - 13.1t^2 + 15.6 > 0$ for $t \geq 0$.

Example 8
For the polynomial $s^n + k_1 s + k_2$, where n is odd, D-decomposition consists of one line and there exist two root invariant regions. Indeed, for $\omega = 0$ we have the line $k_2 = 0$ and for $\omega \neq 0$ we have $k_1 = -(-1)^{\frac{n-1}{2}} \omega^{n-1}$, $k_2 = 0$, i.e. a part of the line $k_2 = 0$.

Example 9
The following example demonstrates that the number of root invariant regions N can achieve $O(n^2)$. Let $p(s, k) = a(s^2) + s(k_1 b(s^2) + k_2 c(s^2) + \alpha)$, where $a(t), b(t), c(t)$ are polynomials of degree $m, m-1, m-1$ correspondingly (thus $p(s, k)$ has degree $n = 2m$), $a(t)$ has m negative real roots $-\tau_i^2,\ i = 1, \dots, m$. Then D-decomposition equation is $p(j\omega, k) = U(\omega^2) + j\omega V(\omega^2) = 0$ and we get two equations $U(\omega^2) = a(-\omega^2) = 0, \omega V(\omega^2) = \omega(k_1 b(-\omega^2) + k_2 c(-\omega^2) + \alpha) = 0$. The first equation does not depend on k, it has n real roots $\omega_i = \pm\tau_i$. Hence D-decomposition is generated by singular straight lines $k_1 b(\omega_i^2) + k_2 c(\omega_i^2) + \alpha = 0$, their total number equals m. The plane is divided into $(m^2 + m)/2 + 1$ regions by m straight lines of generic position (this well-known fact can be confirmed by induction), thus $N = n^2/4 + o(n^2)$.

For instance, let $p(s, k) = Q(s) + k_1 s^2 + k_2$, where $Q(s)$ is a stable polynomial of degree n from Example 3. D-decomposition consists of $\xi = \lceil \frac{n}{2} \rceil$ lines of generic position. Thus, the total region number is $(\xi^2 + \xi)/2 + 1$.

In the particular example below we do not intend to achieve the largest number of root invariant regions, but our goal is to demonstrate, how extraordinary D-decomposition can look for the polynomials of this form.

Let $m = 4$, $a(t) = (1 + t)(2 + t)\dots(8 + t)$, $b(t) = (1+t)(3+t)(5+t)(7+t)$, $c(t) = (2+t)(4+t)\,(6+t)(8+t)$, $\alpha = 105$. Then we have six orthogonal lines $k_1 = 7$, $k_1 = -1$, $k_1 = -11\frac{2}{3}$, $k_2 = 7$, $k_2 = -1$, $k_2 = -11\frac{2}{3}$ depicted in Fig. 9.

Consider the characteristic polynomial with the structure $p(s, k_I, k_P, k_D) = a(s)(k_I + k_P s + k_D s^2) + b(s)$, which correspond to a system with PID controller. For any fixed k_P D-decomposition consists of straight lines. These lines divide $(k_I,\ k_D)$-plane into a finite number of convex polygons. An approach for the calculation of root invariant regions in the $(k_I,\ k_P,\ k_D)$-space is to grid k_P and use a tomographic representation of the result. This idea was proposed by Ackermann and Kaesbauer (2003); Keel and Bhattacharyya (2005) use the same approach for model-free controller synthesis and suggest an algebraic algorithm to distinguish the stability domain.

What is the largest number of stability regions is an open problem. The following example (originated in (Nikolayev, 2002)) demonstrates that this number can achieve $n - 1$.

Example 10

Suppose $p(z, k) = z^n + k_1 z^{n-1} + \alpha z^{n-2} + k_2$, $1 < \alpha < \frac{n}{n-2}$. Then there are $n - 1$ simply connected stability regions in $\{k_1,\ k_2\}$ plane. The structure of the regions for $n = 5$ (in the left) and for $n = 6$ (in the right) can be seen in Fig. 10 ($\alpha = 1.05$ for both cases); $n - 3$ regions are the loops of the D-decomposition curve while two other regions are generated by intersection of the curve with two singular lines L_1 and L_2.

4 SCALAR GAIN

In this section we address systems with a scalar gain, i.e. $m = r$ and $K = kI$, $k \in \mathbb{R}$ or $k \in \mathbb{C}$. By the terminology of μ-analysis this is class $\mathcal{K}$ with one scalar block. Then the matrix $A + BKC$ is equal to $A + kBC$ and the problem is reduced to the simplest one: given $n \times n$ real matrices A and F, find $D(l) = \{k \in \mathbb{C}$ (or $k \in \mathbb{R}$) : $A + kF$ has l stable eigenvalues$\}$.

Equation (7.a) or (7.b) now reads

$$\det(I + kM(j\omega)) = 0, \quad -\infty < \omega < \infty \qquad (17.a)$$

or

$$\det(I + kM(e^{j\omega})) = 0, \quad 0 \le \omega < 2\pi. \qquad (17.b)$$

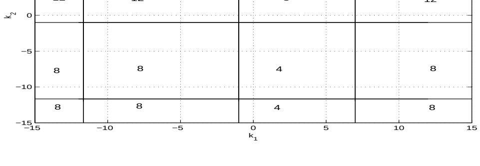

Fig. 9. Root invariant regions in Example 9

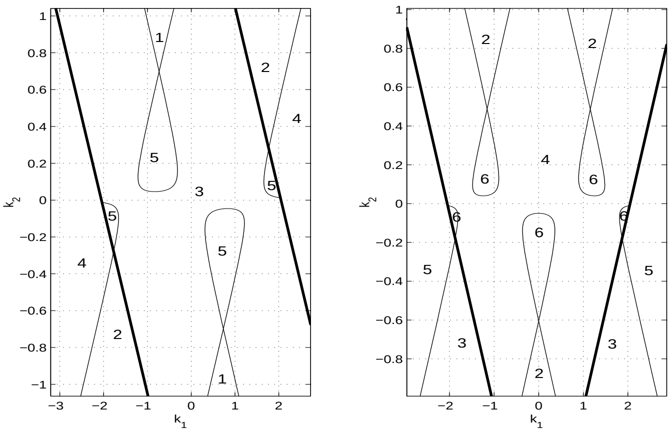

Fig. 10. Root invariant regions in Example 10

If we denote the eigenvalues of $M(j\omega)$ or $M(e^{j\omega})$ as $\lambda_i(\omega)$, $i = 1, ..., n$, equations (17.a)-(17.b) split into $1 + k\lambda_i(\omega) = 0$, $i = 1, ..., m$ and D-decomposition boundary consists of n branches $k^{(i)}(\omega)$:

$$k^{(i)}(\omega) = -\frac{1}{\lambda_i(\omega)},\ i = 1, ..., n. \qquad (18)$$

Equation of D-decomposition (18) can be obtained in a different form with no use of the transfer function. If $A + kF$ ($F = BC$) has an imaginary eigenvalue then the matrix $A + kF - j\omega I$ is singular for some $\omega \in \mathbb{R}$, that is $(A + kF - j\omega I)x = 0$ for $x \in \mathbb{C}^n$ or $(A - j\omega I)x = -kFx$. Thus we conclude that k is a generalized eigenvalue for the matrix pair $A - j\omega I$ and $-F$:

$$k(\omega) = \mathrm{eig}(A - j\omega I, -F). \qquad (19)$$

Similarly for discrete-time case

$$k(\omega) = \mathrm{eig}(A - e^{j\omega} I, -F). \qquad (20)$$

In contrast with (18), (19)-(20) can be used when $A - j\omega I$ (or $A - e^{j\omega} I$) is singular, however the total number of the generalized eigenvalues in this case can be less than n.

Note that the eigenvalues are complex numbers, thus $k(\omega)$ provided by (18) or (19) are complex as well. For the case $k \in \mathbb{C}$ these equations generate the boundary of eigenvalue invariant regions $D(l)$. There are some special cases, when $\mathrm{eig}(M)$ or $\mathrm{eig}(A - j\omega I, -F)$ can be calculated explicitly. However in most situations we construct the boundary numerically as follows.

Algorithm 2.

a. Choose a grid $\omega \in \mathbb{R}$ (or $\omega \in [0, 2\pi]$ for discrete-time case).

b. Calculate $A - j\omega I$ (or $A - e^{j\omega} I$) for all ω in the grid.

c. Calculate $k(\omega) = \mathrm{eig}(A - j\omega I, -BC)$ (or $k(\omega) = \mathrm{eig}(A - e^{j\omega} I, -BC)$).

d. Plot $k(\omega)$ on the complex plane.

Note that generally $k(\omega)$ consists of n branches, however the resulting curve can split into a smaller number of disconnected arcs. For instance, it may happen that $k(\omega)$ is the single Jordan curve (see Example 11 and Fig. 11).

Example 11

Let $A = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 1 & 1 & \dots & 1 & 0 \end{bmatrix}$, $F = \beta I + \alpha A^T$, where A, I are $n \times n$ matrices. D-decomposition for discrete-time case, $n = 4, \alpha = 0.01, \beta = 1$ is depicted in Fig. 11 and consists of one Jordan curve. It is not clear, what is the largest number N of eigenvalue invariant regions. We conjecture that $N = O(n^3)$ (compare with the estimates $O(n^2)$ from the previous section). Also the largest number of simply connected stability regions is not known yet. The smallest number of eigenvalue invariant regions is one. Indeed, set $\alpha = -1, \beta = 0$ in the Example 11, then the characteristic polynomial of $A + kF$ can be calculated explicitly: $p(z, k) = (z+1)^n k + (z-1)^n$, and after standard change of variables we are in the framework of Example 6.

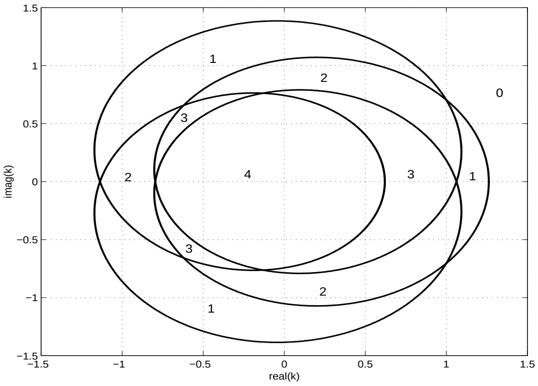

Fig. 11. Single curve D-decomposition in Example 11

For real k D-decomposition technique should be modified as follows. First, we draw the curves $k(\omega)$ as in the complex case (Algorithm 2). Second, we find the intersections k_i of $k(\omega)$ with the real axis. If we order these points such that $k_1 < k_2 < ... < k_N$, then intervals $(-\infty, k_1), (k_1, k_2), ...(k_N, \infty)$ are eigenvalue invariant regions. The number of such intervals can be estimated.

Theorem 5

For real k the number of intervals preserving the same number of stable eigenvalues of $A+kBC$ does not exceed $n(n+1)+1$.

For $n = 2, 3$ the estimate is not conservative, as illustrated below. Unfortunately we do not know the extension of the examples for the arbitrary n.

Example 12

a. $n = 2, A = \begin{bmatrix} 0 & 0.9 \\ 0.9 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$, $C = I$. There are $N = 7$ eigenvalue invariant intervals, 3 of them are stability intervals.

b. $n = 3, A = \begin{bmatrix} 0.95 & 1 & 0 \\ 0 & 0 & 0.6 \\ 0 & 0 & -0.95 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & -0.22 \\ 0 & -0.3 & 0 \\ 0.4 & 0 & 0 \end{bmatrix}$, $C = I$. Here (Fig. 12) there are $N = 13$ eigenvalue invariant intervals and 5 stability intervals.

5 DOUBLE-INPUT DOUBLE-OUTPUT SYSTEMS

We consider the case $r = m = 2$ and K real. Then for $M = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}$, $K = \begin{bmatrix} k_1 & k_3 \\ k_2 & k_4 \end{bmatrix}$, $m_i \in \mathbb{C},\ k_i \in \mathbb{R},\ i = 1, ..., 4$ the equation (7.a) has the form

$$0 = \det(I + MK) = 1 + \sum_{i=1}^{4} k_i m_i + \det M \det K. \qquad (21)$$

This quadratic in K equation defines D-decomposition in $4D$ space $K \in \mathbb{R}^{2\times 2}$. To take the opportunity of the graphical representation we restrict ourselves by situations with K depending on two parameters only. Then equation (21) contains one quadratic term $k_1 k_2$. Siljak (1966, 1969) was the first who considered such D-decomposition equations.

5.1 *CASE 1.* $K = \mathrm{diag}(k_1,\ k_2)$

For $K = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$ equation (21) becomes $0 = 1 + k_1 m_1 + k_2 m_4 + k_1 k_2 (m_1 m_4 - m_2 m_3)$. Substituting $m_i = u_i + jv_i,\ i = 1, ..., 4$ we get two quadratic equation in two variables $k_1,\ k_2$:

$$\begin{aligned} &1 + k_1 u_1 + k_2 u_4 + \alpha k_1 k_2 = 0, \\ &k_1 v_1 + k_2 v_4 + \beta k_1 k_2 = 0, \end{aligned} \qquad (22)$$

where $\alpha = u_1 u_4 - v_1 v_4 - u_2 u_3 + v_2 v_3$, $\beta = u_1 v_4 + v_1 u_4 - u_2 v_3 - v_2 u_3$. The quantities $u_i,\ v_i,\ \alpha,\ \beta$ depend on ω. For $\omega = 0$ the matrix $M(j\omega) = C(A - j\omega I)^{-1} B$ is real and $v_i(0) = 0,\ i = 1, ..., 4$ as well as $\beta(0) = 0$. Hence the second equation vanishes and the first equation $1 + k_1 u_1(0) + k_2 u_4(0) + \alpha(0) k_1 k_2 = 0$ defines the singular curve (a hyperbola).

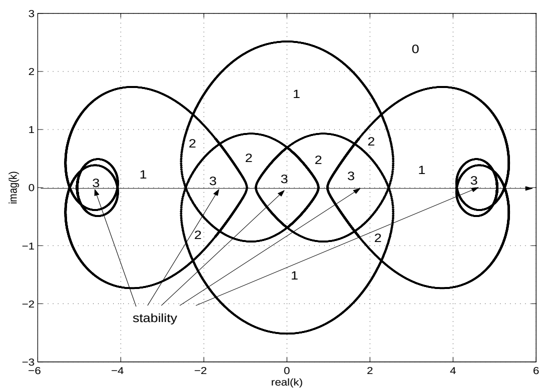

Fig. 12. Stability intervals in Example 12.b

For $\omega \neq 0$ we solve system (22). If for some ω the solution is complex we ignore it because it does not belong to D-decomposition. It means that there is no parameter value K^* such that $A + BK^*C$ has eigenvalues $\pm j\omega$.

Example 13

Discrete-time system $A + B \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} C$, where $A = \begin{bmatrix} -0.8848 & 0.4457 \\ -0.8733 & -0.9326 \end{bmatrix}$, $B = \begin{bmatrix} 0.3914 & 0.2508 \\ -0.5576 & 0.0266 \end{bmatrix}$, $C = \begin{bmatrix} 0.1514 & 0.7854 \\ -0.4255 & -0.8148 \end{bmatrix}$ has a typical D-decomposition structure. In Fig. 13 one can see two singular hyperbolas (subtle lines) and two branches of the nonsingular curve (solid lines).

5.2 CASE 2. $K = [\ k_1\ k_2\ ;\ -k_2\ k_1\ ]$

This is a real 2×2 analog of a complex scalar (note that the eigenvalues of such K are $k_1 \pm jk_2$). For such K equation (21) reads $0 = 1 + k_1(m_1 + m_4) - k_2(m_2 - m_3) + (k_1^2 + k_2^2)(m_1 m_4 - m_2 m_3)$ and (22) is replaced with

$$\begin{aligned} 1 + k_1(u_1 + u_4) - k_2(u_2 - u_3) + \alpha(k_1^2 + k_2^2) &= 0, \\ k_1(v_1 + v_4) - k_2(v_2 - v_3) + \beta(k_1^2 + k_2^2) &= 0, \end{aligned} \quad (23)$$

where $u_i(\omega),\ v_i(\omega),\ \alpha(\omega),\ \beta(\omega)$ are the same as above. For $\omega = 0$ we get $v_i(0) = 0,\ \beta(0) = 0$ and the second equation vanishes while the first equation $1 + k_1(u_1(0) + u_4(0)) - k_2(u_2(0) - u_3(0)) + \alpha(0)(k_1^2 + k_2^2) = 0$ is the equation of the circumference. For $\omega \neq 0$ we can solve (23) and define $k_1(\omega),\ k_2(\omega)$ (provided that $k_1,\ k_2$ are real and (23) is nonsingular). Thus D-decomposition consists of the components of this curve and singular circumference.

Example 14

This discrete-time example is borrowed from (Qiu *et al.*, 1995), p. 889, Example 3: $n = 3$, $m = r = 2$, $A = \begin{bmatrix} 0.4753 & 0.7579 & 7.9939 \\ -0.0415 & 0.8905 & 0.7579 \\ -0.0758 & -0.0415 & 0.4753 \end{bmatrix}$, $B = \begin{bmatrix} 0.0801 & 0.0430 \\ -0.0015 & 0.0948 \\ -0.0043 & -0.0015 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. The optimal solution for (5) is supplied with $K^* = \begin{bmatrix} 0.8483 & 0.5971 \\ -0.5971 & 0.8483 \end{bmatrix}$; it has the form $\begin{bmatrix} k_1 & k_2 \\ -k_2 & k_1 \end{bmatrix}$. Thus we restrict ourselves with matrices K of this form and construct D-decomposition for such matrices (see Fig. 14). It is generated by two singular circumferences (subtle lines) and one parametric curve (solid line). Note that the origin is close enough to the boundary of the stability domain and the distance to it, in accordance with (Qiu *et al.*, 1995), equals 1.0374. However, other directions allow larger values of perturbations preserving stability. For instance, if $K = \lambda \begin{bmatrix} -0.9680 & -0.2508 \\ 0.2508 & -0.9680 \end{bmatrix}$, then $A + BKC$ remains stable for $0 \leq \lambda \leq 19.6932$, and the admissible perturbation has the norm 19.6932, that is 18.9832 times larger than the real stability radius. The detailed D-decomposition of the stability

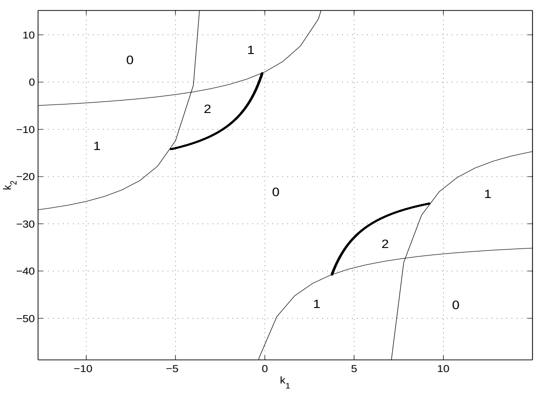

Fig. 13. D-decomposition in Example 13

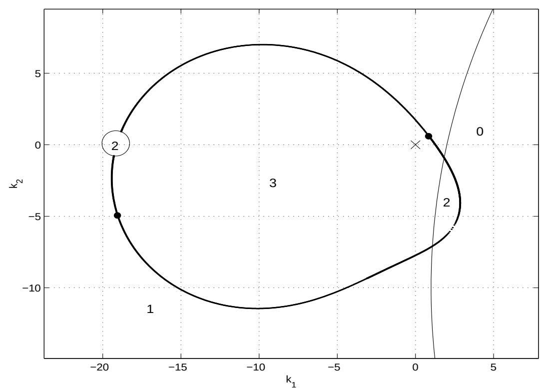

Fig. 14. D-decomposition in Example 14

domain (Fig. 15) depicts regions with fixed stability degree, i.e. regions where all the eigenvalues of matrix $A + BKC$ are inside the r-disc, $r < 1$. This extension of D-decomposition can be obtained by replacing $e^{j\omega}$ term in the equations with $re^{j\omega}$. This additional information is helpful for design purposes (when K is treated as a design parameter but not uncertainty).

5.3 CASE 3. $K = [\ -k_1\ k_2\ ;\ k_2\ k_1\ ]$

The calculations are similar to the ones for Case 2 and (23) becomes

$$\begin{aligned} 1 - k_1(u_1 - u_4) + k_2(u_2 - u_3) - \alpha(k_1^2 + k_2^2) &= 0, \\ -k_1(v_1 - v_4) + k_2(v_2 - v_3) - \beta(k_1^2 + k_2^2) &= 0. \end{aligned} \quad (24)$$

D-decomposition curves are the parametric curve $k_1(\omega),\ k_2(\omega),\ \omega \neq 0$ and the singular curve – the circumference $1 + k_1(u_1 - u_4) + k_2(u_2 - u_3) - \alpha(k_1^2 + k_2^2) = 0$, $\omega = 0$.

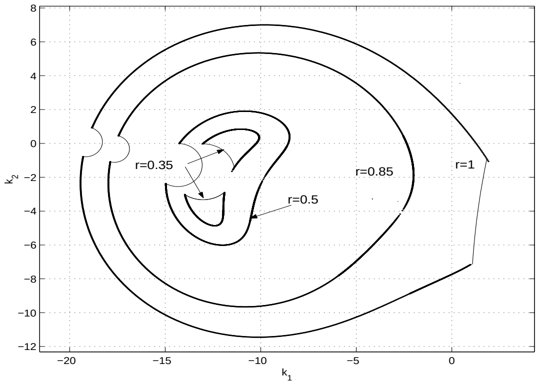

Fig. 15. Detailed D-decomposition in Example 14

Example 15

This continuous-time example is again originated in (Qiu *et al.*, 1995), p. 889, Example 2. Here $n = 4,\ m = r = 2$,

$$A = \begin{bmatrix} 79 & 20 & -30 & -20 \\ -41 & -12 & 17 & 13 \\ 167 & 40 & -60 & -38 \\ 33.5 & 9 & -14.5 & -11 \end{bmatrix},\ B = \begin{bmatrix} .219 & .9346 \\ .047 & .3835 \\ .6789 & .5194 \\ .6793 & .831 \end{bmatrix},$$

$C = \begin{bmatrix} .0346 & .5297 & .0077 & .0668 \\ .0535 & .6711 & .3834 & .4175 \end{bmatrix}$. The smallest norm perturbation destroying the stability of $A + BKC$ is $K^* = \begin{bmatrix} -0.4996 & 0.1214 \\ 0.1214 & 0.4996 \end{bmatrix}$, that is it has the form considered in the present subsection. D-decomposition of (k_1, k_2) plane for matrices $K = \begin{bmatrix} -k_1 & k_2 \\ k_2 & k_1 \end{bmatrix}$ is shown in Fig.16. There are two disconnected components of the

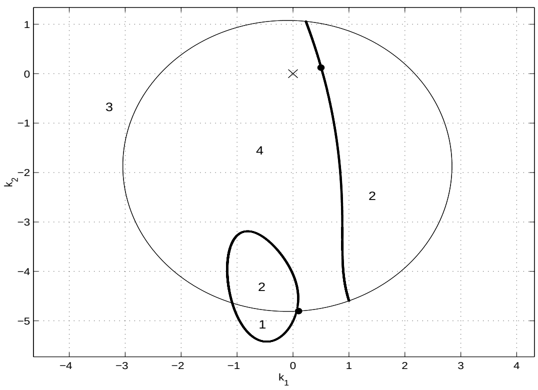

Fig. 16. D-decomposition in Example 15

parametric curve (solid lines) and one singular curve - the circumference (subtle line). The nearest to the origin point on the boundary of stability domain is $k_1^* = 0.4996,\ k_2^* = 0.1214$; it corresponds to the matrix K^* above. Other directions preserve stability for larger perturbations. For instance, $A + BKC$ is stable for $K = \lambda \begin{bmatrix} -0.0211 & -0.9998 \\ -0.9998 & 0.0211 \end{bmatrix},\ 0 \le \lambda \le 4.8032$, in particular for $K_1 = \begin{bmatrix} -0.1013 & -4.802 \\ -4.8021 & 0.1013 \end{bmatrix}$, $||K_1|| = 4.8$ (0.5141 being the real stability radius).

6 CONCLUSIONS

We provided the simple and effective techniques to construct the stability domain in the parameter space. This is an extension of D-decomposition method for polynomials. Simultaneously with the stability domain we construct all the root invariant regions, i.e. simply connected regions of the domains with the invariant number of eigenvalues of the system matrix. This technique can be helpful for design of low order controllers and for detailed robustness analysis. The main limitation of the proposed approach is the low dimensionality of the parameter space (one or two).

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Appendix

A Proof of Theorem 2

To count the number of critical parameter values we analyze the equation

$$\mathrm{Im} M(j\omega) = \frac{b(j\omega)}{a(j\omega)} = 0. \tag{A.1}$$

Since $a(j\omega) = u_1 + j\omega v_1,\ b(j\omega) = u_2 + j\omega v_2$, equation (A.1) is equivalent to: either $v_1 u_2 - u_1 v_2 = 0$, or $\omega = 0$. The first one is a polynomial equation of degree $n-1$ in ω^2. This equation can have no more than $n-1$ real solutions in $t = \omega^2$. Since $\mathrm{Re} M(-j\omega) = \mathrm{Re} M(j\omega)$, every solution specifies no more than one critical value. Two more critical values are $k = -\frac{1}{M(0)} = -\frac{a(0)}{b(0)} = -\frac{a_0}{b_0}$ and $k = -\frac{1}{M(j\infty)} = -\frac{a_n}{b_n}$. Thus there is no more than $n+1$ critical parameter values. These values partition parameter axis into $\le n+2$ intervals $(-\infty, k_1), (k_1, k_2), ..., (k_m, \infty)$. Due to Theorem 1 for $k_i < k < k_{i+1}$ the polynomial $a(s) + kb(s)$ has the same number of stable roots, thus the number of root invariant intervals is no more than $n+1$. Since every two nearby intervals cannot be both stability intervals and if $(-\infty, k_1)$ is a stability interval then (k_m, ∞) is also a stability interval, there can be no more than $\lfloor \frac{n}{2} \rfloor + 1$ stability intervals.

The proof for the discrete-time case is similar. □

B Proof of Theorem 3

The number of regions depends on the number of self-crossing points of the boundary curve. A bounded curve without self-crossing points divides the parameter plane into two regions, and every simple self-crossing point appends an extra region. So the number of root invariant regions is equal to the number of self-crossing point plus two. It is easy to be convinced of the truth of this statement in terms of Euler's Polyhedral formula, which relates the number of vertices V, faces F, and edges E of a simply connected polyhedron $V + F - E = 2$. We can treat self-crossing points of curve (11) as vertices, arcs as edges, thus faces are regions in the parameter plane. Since we consider simple self-crossings only, there are four edges incident to each vertex. Suppose there are m vertices, therefore there are $2m$ edges and due to Euler's formula $m + 2$ faces.

At the core of the proof is the need to count up the number of self-crossing points of the algebraic curve (11). Self-crossing points are specified by $\lambda(\omega_1) = \lambda(\omega_2)$, $\omega_1 \neq \omega_2$. It is equivalent to the system of equations

$$\begin{aligned} \sum_{i=0}^{n-2} \sum_{l=1}^{\frac{n-i}{2}} (-1)^{i+l} c_{ik} \omega_1^i \omega_2^i (\omega_2^{2l} - \omega_1^{2l}) &= 0 \\ \sum_{i=0}^{n-1} \sum_{l=0}^{\frac{n-i-1}{2}} (-1)^{i+l+1} c_{im} \omega_1^i \omega_2^i (\omega_2^{2l+1} - \omega_1^{2l+1}) &= 0, \end{aligned} \tag{B.1}$$

where $c_{ik} = a_i b_k - a_k b_i,\ k = i + 2l,\ m = i + 2l + 1$.

Now we exploit the following result; e.g. (Walker, 1950; Ackermann, 2002).

Theorem (Bezout). Two bivariate polynomials

$P(x, y) = p_1 x^{\alpha_1} y^{\beta_1} + ... + p_k x^{\alpha_k} y^{\beta_k}$;

$\deg P(x, y) \doteq \max_i (\alpha_i + \beta_i) = n$

$Q(x, y) = q_1 x^{\gamma_1} y^{\delta_1} + ... + q_l x^{\gamma_l} y^{\delta_l}$; $\deg Q(x, y) = m$

have no more than mn common real zeros.

Due to this theorem, system (B.1) can have no more than $(2n-2)(2n-1)$ real solutions; but this is a conservative estimate. The first equation is an identity when $\omega_1 + \omega_2 = 0$. It leads to $n - 1$ self-crossing points on the real axis k. Notice that two different solutions (α, β) and (β, α) specify the same self-crossing point. To avoid this degeneracy we change the variables $\omega \Rightarrow d$: $\omega_1\omega_2 = d_1$, $\omega_1 + \omega_2 = d_2 \neq 0$.

In these variables the degrees of equation (B.1) are $n-2$ and $n - 2$ (product $\omega_1^i \omega_2^i$ of degree $2i$ becomes d_1^i of degree i). Therefore, we get no more than $(n-1)(n-2)$ solutions and each solution corresponds to a unique self-crossing point. Thus the total number of self-crossing points does not exceed $(n-1)+(n-2)(n-1) = (n-1)^2$, and hence the number of root invariant regions is less or equal $(n - 1)^2 + 2$. □

C Proof of Theorem 4

Let us estimate the number of self-crossing points. We have the parametric curve (15) and singular lines. Suppose $\Delta \neq 0$, then we have two equations in two variables $k_1(\omega_1) = k_1(\omega_2),\ k_2(\omega_1) = k_2(\omega_2)$; each of them is a polynomial equation of degree $2(n-1) - 1$. After avoiding the degeneracy (as in the previous section) we find out that it can be no more than $(n - 2)^2$ self-crossing points. Besides, there always exist two singular lines (for $\omega = 0$ and $\omega = \infty$) and the curve has no more then $n - 1$ crossings with each line. Taking into consideration a crossing point of singular lines the number of crossings of D-decomposition is no more than $(n-2)^2+2(n-1)+1$. The number of the root invariant regions in this case is no more than $(n - 2)^2 + 2(n - 1) + 3$.

If $\Delta = 0$ but $\Delta_1 \neq 0$ or $\Delta_2 \neq 0$, then the parametric curve has discontinuity and the number of root invariant regions does not exceed the number estimated above. If $\Delta = \Delta_1 = \Delta_2 = 0$, then it can exist no more than $n - 1$ singular lines (and two more singular lines for $\omega = 0$ and $\omega = \infty$). The number of crossings between these lines (for generic position) is $\frac{n(n+1)}{2}$, the number of crossings between parametric curve and lines is no more than $n^2 - 1$. Thus the total number of root invariant regions does not exceed $2n(n - 1) + 3$. □

D Proof of Theorem 5

D.1 Continuous-time case

To count the number of real critical parameter values we read equation (19) as $\det(A + kF - j\omega I) = 0,\ F = BC$. It is equivalent to

$$(j\omega)^n + ... + (j\omega)^m q_m(k) + ... + q_n(k) = 0, \tag{D.1}$$

where $q_i(k)$ is a polynomial of degree i in k. This equation is equivalent to the system of two equations (for real and imaginary parts): $q_n(k) - \omega^2 q_{n-2}(k) + \omega^4 q_{n-4}(k) + ... = 0$, $\omega(q_{n-1}(k) - \omega^2 q_{n-3}(k) + \omega^4 q_{n-5}(k) + ...) = 0$. For $\omega = 0$ we have no more than n critical values. For $\omega \neq 0$ the first equation is a polynomial equation of degree n and the second one is an equation of degree $n - 1$ in k and ω^2. Due to Bezout Theorem (see the proof of Theorem 3) we have no more than $n(n - 1)$ solutions. Since $\mathrm{Re}\ k(-\omega) = \mathrm{Re}\ k(\omega)$, $n(n-1)$ solutions of (D.1) specify $\frac{n(n-1)}{2}$ different critical values of k. Thus the total number of the critical parameter values is no more than $\frac{n(n+1)}{2}$ and the number of eigenvalue invariant intervals is no more than $\frac{n(n+1)}{2} + 1$. □

D.2 Discrete-time case

Equation (D.1) now reads $e^{nj\omega} + ... + e^{mj\omega} q_m(k) + ... + q_n(k) = 0$. It is equivalent to the system of two equations (for real and imaginary parts): $\cos n\omega + ... + \cos\omega q_{n-1}(k) + q_n(k) = 0$, $\sin n\omega + ... + \sin\omega q_{n-1}(k) + q_n(k) = 0$.

We can restrict ourselves in $\omega \in [0, \pi]$ because $\mathrm{Re}\ k(\omega) = \mathrm{Re}\ k(2\pi - \omega)$. Note that $\cos n\omega$ and $\sin n\omega$ can be represented as polynomials of degree n in $\cos\omega$: $\cos n\omega =$

$P_n(\cos\omega)$, $\sin n\omega = \sin\omega Q_{n-1}(\cos\omega)$. It can be easily proved by induction.

Thus we can pass on to polynomial equations:

$$\cos\omega = t,\ \sin\omega = \sqrt{1-t^2},\ t \in [-1, 1]$$

$$P_n(t) + ... + P_1(t)q_{n-1}(k) + q_n(k) = 0,$$

$$\sqrt{1-t^2}(Q_n(t) + ... + Q_1(t)q_{n-1}(k) + q_n(k)) = 0.$$

For $t = -1$ and $t = 1$ we have no more than n solutions. For $t \neq \pm 1$ the system has no more than $n(n-1)$ solutions due to Bezout theorem. Thus the total number of the critical parameter values is no more than $n(n-1) + 2n = n(n+1)$ and the number of eigenvalue invariant intervals is no more than $n(n+1) + 1$. □

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