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Design of the Low-order Controllers by the H_∞ Criterion: A Parametric Approach

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Abstract—Consideration was given to the problem of describing all stabilizing controllers of a given structure (for example, the the PID-controllers) satisfying the H_∞ criterion. Controllers of a certain family were defined by the parameters $\mathbf{k}$, and in the parameter space a domain corresponding to the desired criteria was specified. Two approaches were proposed where (i) the desired domain is represented as an intersection of the admissible sets or (ii) its boundary is determined analytically. The two-parameter case is of special importance because it allows one to make use of the graphical mathematics.

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1. INTRODUCTION

The problem of designing the low-order controllers lies in determining a stabilizing controller where the closed-loop system should satisfy some additional criteria. As a rule, the structure of the controller is defined initially, and it remains just to select appropriately the controller parameters.

Despite the development of the analytical theories of controller design such as the H_∞-theory, l_1-approach, QFT-approach, μ-design, and others [1–5], the industrial applications still are widely using simple, if not the simplest, controllers such as the proportional-integrating, proportional-integrating-differentiating, first-order, and other controllers which have simple structure and rely on the understandable physical principles. Possibly, this fact accounts for their popularity and wide acceptance in control [6–8]. Design of a controller satisfying the performance criteria still remains a challenge. The present paper considers criteria like H_∞ arising both in the design of robust controllers guaranteeing stability despite the plant uncertainty and in the problem of system stabilization under a given performance criterion.

The analytical theory of design of the H_∞-optimal controllers has been developed in detail, but the resulting controllers may be of a very high order, sometimes even exceeding that of the original system [9]. Additionally, stability of the closed-loop system is very sensitive to the controller parameters, their small variation often resulting in instability [10]. Since the H_∞ theory does not allow one to constrain the order of the designed controller, its direct application to the design of controllers of a given structure faces substantial difficulties.

This paper presents an alternative approach to the design of controllers of a given structure, that is, the order of the controller is fixed and it is only the choice of its parameters that remains free. In what follows, we mostly assume that there are only two adjustable controller parameters, which allows us to use widely the graphic methods. Therefore, the problem lies in finding in the parameter space a domain such that the corresponding controllers (i) stabilize the system and (ii) guarantee satisfaction of the H_∞-criterion.

In the general case, the low-order stabilizing controllers can determined using the Neimark D-decomposition technique [11] whose state-of-the-art is described in [12]. Some papers [13–15] demonstrated that the H_∞-criterion is representable as a constraint on the behavior of the Nyquist plot which should lie outside some circle. Relying on this fact, the authors formulate the so-called "sensitivity constraint" defining an admissible domain in the parameter space and seek PI-controllers and PID-controllers with the highest possible coefficient of the integrating unit.

The recently proposed approach to the problem with the H_∞ criterion [16–18] relies on the fact that satisfaction of the H_∞-criterion amounts to stability of the one-parameter family of polynomials with complex coefficients. The problem of stabilization is solved for each value of the parameter, and the desired domain is represented as the intersection of the stability domains. An alternative is represented by the random methods [19]. In this case, the problem is solved effectively for part of the parameters, the rest of the parameters being chosen randomly.

The present paper proposes two approaches. In the first case, the desired domain is characterized by the intersection of the admissible sets defining the values of parameters. In the second case, the boundary of this domain is established using the concepts of D-decomposition. Both methods enable one to determine the desired domain and are applicable to a wide class of controllers including the PI, PID, first-order, and other controllers.

The paper is arranged as follows. Section 2 formulates and substantiates the problem. The following section presents the main results concerning the design of the H_∞-controllers and describes two methods of determining a domain in the parameter space where the closed-loop system is stable and the requirements for robustness or performance are satisfied. Section 4 presents examples demonstrating effectiveness of both methods. Finally, the last section draws conclusions and discusses some aspects of the proposed methods.

2. FORMULATION OF THE PROBLEM

Let us consider a linear stationary one-dimensional system where the plant is defined by the scalar transfer function $G(s)$. Let the system be closed by the controller $C(s, \mathbf{k})$, (Fig. 1). The parameters $\mathbf{k} \in \mathbb{R}^m$, $m \leqslant 3$ define the set of admissible low-order controllers of known structure. Among the admissible controllers, consideration is given below to the PI-controller $k_p + \dfrac{k_i}{s}$, PID-controller $k_p + \dfrac{k_i}{s} + k_d s$, and the first-order controller $\dfrac{k_1 s + k_2}{s + k_3}$.

Needed is to describe the set of parameters common to all controllers and satisfying the performance criterion

$$\|H(s, \mathbf{k})\|_\infty < \gamma. \tag{1}$$

We recall that the H_∞-norm is finite only for the functions with stable denominator and is equal to $\|H(s)\|_\infty = \sup\limits_{\omega} |H(j\omega)|$. Such an H_∞-criterion arises in several formulations of the design problems of which we present the most important examples.

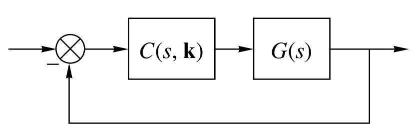

Fig. 1. Block diagram of the closed-loop system.

• Determination of the controller $C(s)$ guaranteeing the H_∞-performance of the closed-loop system,

$$\|W_1(s)T(s)\|_\infty \leqslant \gamma, \tag{2}$$

where $W_1(s)$ is assumed to be a stable rational function ($W_1 \in \mathbf{RH}_\infty$), $T(s) = \dfrac{C(s)G(s)}{1 + C(s)G(s)}$ is the transfer function of the closed-loop system (additional sensitivity), and γ is the given level of performance.

• Determination of a controller stabilizing robustly the family of plants with additive uncertainty $G(s) = G_0(s) + \Delta G(s)$, where $G_0(s)$ is the transfer function of the nominal plant and the frequency (nonparametric [1]) uncertainty $\Delta G(s)$ is bounded in the weighted norm $\|W_2(s)\Delta G(s)\|_\infty \leqslant 1$. The problem comes to satisfying the criterion

$$\|W_2^{-1}(s)U(s)\|_\infty \leqslant 1, \tag{3}$$

where

$$U(s) = \frac{C(s)}{1 + C(s)G(s)}$$

is the error transfer function, $W_2 \in \mathbf{RH}_\infty$. The criterion $\|W_2^{-1}(s)T(s)\|_\infty \leqslant 1$ corresponds to the multiplicative uncertainty $G(s) = G_0(s)(1 + \Delta G(s))$.

• In the quantitative feedback theory (QFT) [20], the closed-loop system must satisfy the following constraints:

$$\begin{gathered} m_1(\omega) < |W(j\omega)T(j\omega)| < m_2(\omega), \omega \in [0, \omega_1], \\ |S(j\omega)| < l_1(\omega), \omega \in [0, \omega_1], \\ |T(j\omega)| < l_2(\omega), \omega \in [\omega_2, \infty), \end{gathered} \tag{4}$$

where $S(s) = \dfrac{1}{1 + C(s)G(s)}$ is sensitivity and m_1, m_2, l_1, and l_2 are the limitative functions. Appropriate choice of a weight function allows one to pass from the finite intervals of frequencies to the interval $[0, \infty)$, but the methods proposed in the paper enable one to handle the intervals directly (see Example 2).

We note that the majority of approaches such as the H_∞ theory and robust stabilization make use of the Yule parametrization and its counterparts which provide functional analogs of the controller family that not only satisfy the problem conditions, but are criterion-optimal as well [4, 5]. The disadvantages of these approaches lie in complexity of the resulting family, unobvious parametrization constraining inclusion of the given optimal controller into the family of given controllers, as well as in "fragility" of the resulting controllers [10].

Let us focus on the properties of the parametric families of controllers, assume the kind of the controller, and seek a subset where criterion (1) is satisfied. The optimal controller of a given structure can be determined by decreasing γ until the set of suitable controllers becomes empty.

As follows from the definition of the H_∞-norm, it makes sense only for the functions from $\mathbf{RH}_\infty$ that are analytical in the right complex half-plane. Additionally, we are interested only in the stabilizing controllers. Therefore, the following requirements must be met:

(1) the closed-loop system is stable, that is, its characteristic polynomial $\delta(s, \mathbf{k})$ is Hurwitzian (all roots lie in the left complex half-plane);

(2) the function H also is stable ($H(s) = W_1(s)T(s)$ in the problem of H_∞-performance (2), $H(s) = W_2(s)^{-1}U(s)$ in the problem of H_∞-robustness (3), and so on): $H(s, \mathbf{k}) \in \mathbf{RH}_\infty$;

(3) for the given number $\gamma > 0$, the inequality $|H(j\omega, \mathbf{k})| < \gamma$, $\omega \in [0, \infty)$ is satisfied.

The desired set of parameters and, consequently, controllers is the intersection of the three domains in which these conditions are satisfied taken separately. The controllers for which all three conditions are met will be called for brevity the H_∞-controllers. In actual practice, the first two requirements often coincide because the denominator of the function H is equal to the characteristic polynomial of the closed-loop system. This is true for all aforementioned problems (2), (3), and (4). In what follows, we take this case as the basic one.

Therefore, the main task lies in describing the set of H_∞-controllers which is constructed in two stages. First, in the space of controller parameters we specify the domains satisfying the condition $|H(j\omega, \mathbf{k})| < \gamma$, $\forall\omega \in [0, \infty)$ (the same condition defines indirectly the parameters for which the function $H(j\omega, \mathbf{k})$ is not defined). Then, it remains to specify among these domains those where the function $H(s, \mathbf{k})$ is stable.

3. DESIGN OF H_∞-CONTROLLERS

Let us assume that the controller stabilizes the system, that is, the denominator $H(s, \mathbf{k})$ is stable. Then, $\|H(s, \mathbf{k})\|_\infty = \sup\limits_{\omega\in[0,\infty)} |H(j\omega, \mathbf{k})|$, where

$$H(s, \mathbf{k}) = \frac{H_n(s, \mathbf{k})}{H_d(s, \mathbf{k})}$$

is the common rational transfer function. We assume in what follows that the numerator H_n and the denominator H_d depend linearly on the parameters $\mathbf{k}$, which is the case for all considered problems (2)–(4), provided that the numerator and denominator of the controller also linearly depend on the parameters, which is true for the PI, PID, first-order, and other controllers. For the PID-controller and the sensitivity function $H_n(s, \mathbf{k}) = sD(s)$, for example, $H_d(s, \mathbf{k}) = sD(s) + (k_i + k_p s + k_d s^2)N(s)$, where $N(s)$ is the numerator of the transfer function of the plant $G(s)$ and $D(s)$ is its denominator.

Let's consider the set in the parameter space

$$\mathcal{K} = \{\mathbf{k} : |H(j\omega, \mathbf{k})| < \gamma, \forall\omega \in [0, \infty)\} \tag{5}$$

whose the intersection with the stability domain defines all H_∞-controllers. Together with definition (5), we use the notation

$$\mathcal{K} = \{\mathbf{k} : |H_n(j\omega, \mathbf{k})| < \gamma|H_d(j\omega, \mathbf{k})|, \forall\omega \in [0, \infty)\} \tag{6}$$

which is equivalent in terms of the set $\mathcal{K}$ and removes the requirement that there should be the function $H(j\omega, \mathbf{k})$ for all ω and $\mathbf{k}$. It turns out that there is no need to determine the stability domain because the following assertion is valid ($\mathcal{K}$ usually is a multiply connected set).

Assertion 1. *If the point $\mathbf{k}^* \in \mathcal{K}$ corresponds to the H_∞-controller, then all points of the simply connected component $\mathcal{K}$ where $\mathbf{k}^*$ lies also define H_∞-controllers.*

The proof follows immediately from the fact that the boundary of the stability domain is included in the boundaries of the D-decomposition for the characteristic polynomial [11] which obey the condition $H_d(j\omega, \mathbf{k}) = 0$; by definition (6), for example, these boundaries do not belong to the set $\mathcal{K}$.

Since the present paper makes use of the graphic methods which are effective only on the plane, the number of free (controllable) parameters must be small. If the dimensionality of the vector $\mathbf{k}$ exceeds two, then one should take two parameters to seek the H_∞-controller and fix the rest of them. In the problem of design of controllers of a given structure, the dimensionality $\mathbf{k}$ does not exceed as a rule three (PID, first order). See Example 2.

3.1. Admissible Sets

Let us consider in more detail the inequality $|H_n(j\omega, \mathbf{k})| < \gamma |H_d(j\omega, \mathbf{k})|$ from the definition of the set $\mathcal{K}$. In the parameter space $\mathbf{k}$, it defines for each ω the set $\mathcal{K}^\omega$ which will be called the admissible set. By definition, $\mathcal{K}$ is the intersection in ω of the admissible sets.

Let us rearrange the inequality at hand in

$$|H_n(j\omega, \mathbf{k})|^2 < \gamma^2 |H_d(j\omega, \mathbf{k})|^2. \qquad (7)$$

As was already noted, consideration is given to the case of two free parameters $\mathbf{k} \doteq (k_1, k_2)$, the controller numerator and denominator (and, consequently, the functions H) depending on them linearly, the rest of the parameters being fixed. Then, (7) goes over to

$$|H_n^0(j\omega) + H_n^1(j\omega)k_1 + H_n^2(j\omega)k_2|^2 < \gamma^2 |H_d^0(j\omega) + H_d^1(j\omega)k_1 + H_d^2(j\omega)k_2|^2,$$

and the admissible set obeys an equation like

$$\mathcal{K}^\omega = \left\{(k_1, k_2) : a(\omega)k_1^2 + b(\omega)k_1k_2 + c(\omega)k_2^2 + d(\omega)k_1 + e(\omega)k_2 + f(\omega) < 0\right\}, \qquad (8)$$

where $a(\omega)$, $b(\omega)$, $c(\omega)$, $d(\omega)$, $e(\omega)$, and $f(\omega)$ are the polynomials with real coefficients (the coefficients of the weight and transfer functions of the plant and controller are assumed to be real as well). For each ω, the boundary of the admissible set is a second-order curve with respect to the two selected parameters k_1 and k_2. Classification of the controller parametrization characteristics provides the following result.

Theorem 1. *The set $\mathcal{K}$ is the intersection of the quadratic sets $\mathcal{K}^\omega$, $\omega \in [0, \infty)$. At that,*

(a) *if k_1 and k_2 occur in the controller numerator and denominator as $k_1 + k_2 s^2$ (the case of PID-controller with fixed k_p), then $\mathcal{K}^\omega$ is the exterior or interior of the band;*

(b) *if $C(s) = \dfrac{k_1 s + k_2}{s + a}$ (first-order controller) or $C(s) = \dfrac{k_1 + k_2 s + a s^2}{s}$ (PI-controller), then $\mathcal{K}^\omega$ is the exterior or interior of the ellipse.*

An assertion more general than (a) and (b) is valid: the admissible set will be a band if the parameters k_1 and k_2 appear in the numerator and denominator of the controller only in the linear combination $k_1 p(s^2) + k_2 q(s^2)$. Moreover, in this case $\mathcal{K}$ consists of a finite number of convex sets.

For a first-order controller with another fixed coefficient, $C(s) = \dfrac{k_1 s + a}{s + k_2}$ for example, as well as for the PD-controller, the boundaries of the admissible sets will be ellipses as in Item (b).

For a special family of controllers of the form $C(s, k_1, k_3) = \dfrac{k_1 + k_2 s + k_3 s^2}{dk_1 + s + dk_3 s^2}$ (d and k_2 are fixed), this method was first proposed by Blanchini et al. [21]. In this case, the set $\mathcal{K}$ needs not to be simply connected; the same paper estimates the maximum number of its simply connected components. Construction of $\mathcal{K}$ via its complement $\overline{\mathcal{K}}$ has much in common with the robust D-decomposition for the elliptical uncertainties [22].

3.2. Analog of the D-decomposition

Another method of determining the set of H_∞-controllers is based on calculating the explicit boundary of the set $\mathcal{K} = \{\mathbf{k} : |H_n(j\omega, \mathbf{k})| < \gamma |H_d(j\omega, \mathbf{k})|, \forall \omega \in [0, \infty)\}$. We follow the approach of D-decomposition in considering the passage from the point $\mathcal{K}$ to the point $\overline{\mathcal{K}}$. There may be two essentially different cases: either the function $|H(j\omega, \mathbf{k})|$ reaches the value γ or its numerator and denominator vanish simultaneously. The points where only the denominator is zero do not belong to the boundary because their small neighborhood has not a single point where the inequality $|H_n(j\omega, \mathbf{k})| < \gamma |H_d(j\omega, \mathbf{k})|$ is satisfied. Wherever H is defined, it is possible to set down the necessary condition for extremum with respect to ω. As the result, we get an easy-to-use theorem.

Theorem 2. *The boundary of the set $\mathcal{K}$ is contained in the solutions of the systems*

$$\begin{cases} H_n(j\omega, \mathbf{k}) = 0 \\ H_d(j\omega, \mathbf{k}) = 0, \end{cases} \quad \omega \in [0, \infty)\ , \tag{9}$$

$$\begin{cases} |H(j\omega, \mathbf{k})|^2 = \gamma^2 \\ \dfrac{\partial |H(j\omega, \mathbf{k})|^2}{\partial \omega} = 0, \end{cases} \quad \omega \in [0, \infty)\ , \tag{10}$$

and equation

$$|H(j\infty, \mathbf{k})| = \gamma. \tag{11}$$

Proof. The boundary points where the function H is not defined are spanned by the solutions of system (9). This system and Eq. (11) comprise all points of the boundary of the set $\mathcal{K}$ for which $\sup\limits_{\omega \in [0,\infty)} |H(j\omega, \mathbf{k})| = \gamma$. Taken is the square of the absolute value of H because it is the rational function of ω and its derivative is defined. Then, since the finite values of ω where $|H(j\omega, \mathbf{k})|^2$ takes the values γ^2 must be the maximum points, the necessary condition for extremum—equality to zero of the derivative with respect to ω—is satisfied. All points $\mathbf{k}$ such that the maximum of $|H(j\omega, \mathbf{k})|$ is equal to γ are contained in the solutions of system (10). Equation (11) describes attainment of the exact upper face of the function $|H(j\omega, \mathbf{k})|$ on infinity, which establishes the theorem.

For the two variables (k_1, k_2), it is the aggregate of points that will be the solution of system (10). The first equation defines for each ω either a one-dimensional curve or an empty set. The second equation of the system enables one to specify those points $\mathbf{k}$ on the curve at which the function $|H|^2$ reaches its extremum for the given ω. Since the condition for equality to zero of the derivative keeps one from discriminating whether it is the maximum or minimum that is reached at this point, points not belonging to the desired boundary may occur among the solutions. Such $\mathbf{k}$, however, lie outside the admissible domains.

Since we are interested only in the connected components of $\mathcal{K}$ which lie inside the stability domain, it suffices according to Assertion 1 to take one point in each component and verify whether the H_∞-criterion is satisfied.

In some cases, system (10) admits an explicit solution which we present for the case of the PID-controller with fixed k_p where the closed-loop system should satisfy criterion (2).

The function $|H(j\omega, k_i, k_d)|^2 = |W(j\omega) T(j\omega, k_i, k_d)|^2$ is as follows:

$$|W(j\omega)|^2 \frac{|(k_i + j\omega k_p - \omega^2 k_d) N(j\omega)|^2}{|j\omega D(j\omega) + (k_i + j\omega k_p - \omega^2 k_d) N(j\omega)|^2}$$

$$= |W(j\omega)|^2 \frac{((k_i - \omega^2 k_d)^2 + \omega^2 k_p^2)|N(j\omega)|^2}{(-\omega \mathrm{Im}[D(j\omega) + k_p N(j\omega)] + (k_i - \omega^2 k_d)\mathrm{Re} N(j\omega))^2 + (\omega \mathrm{Re}[D(j\omega) + k_p N(j\omega)] + (k_i - \omega^2 k_d)\mathrm{Im} N(j\omega))^2}.$$

After simple conversions and reduction to a compact form, system (10) takes on the form

$$\begin{cases} t^2 P(\omega) + 2t Q(\omega) + R(\omega) = 0 \\ t^2 \dfrac{dP(\omega)}{d\omega} + 2t \dfrac{dQ(\omega)}{d\omega} + \dfrac{dR(\omega)}{d\omega} + 2\left(t^2 P(\omega) + Q(\omega)\right) \dfrac{\partial t(\omega, k_i, k_d)}{\partial \omega} = 0, \end{cases} \quad \omega \in [0, \infty),$$

where

$$t(\omega, k_i, k_d) = k_i - \omega^2 k_d,$$
$$P(\omega) = (|W(j\omega)|^2 - \gamma^2)|N(j\omega)|^2,$$
$$Q(\omega) = -\omega\gamma^2 \mathrm{Im}\,[D(-j\omega)N(j\omega)]\,,$$
$$R(\omega) = \omega^2(|W(j\omega)|^2|k_pN(j\omega)|^2 - \gamma^2|D(j\omega) + k_pN(j\omega)|^2).$$

The real solutions of the first equation

$$t_{1,2}(\omega) = \frac{-Q(\omega) \pm \sqrt{Q^2(\omega) - P(\omega)R(\omega)}}{P(\omega)} \tag{12}$$

exist only for a nonnegative discriminant. The condition $Q^2(\omega) \geqslant P(\omega)R(\omega)$ allows one to specify the corresponding intervals of ω.

From the second equation one can explicitly express k_d and then $k_i = t + \omega^2 k_d$. As the result, we have two parametric curves

$$k_d^{1,2}(\omega) = \frac{t_{1,2}^2(\omega)P'(\omega) + 2t_{1,2}(\omega)Q'(\omega) + R'(\omega)}{4\omega(t_{1,2}(\omega)P(\omega) + Q(\omega))}, \tag{13}$$

$$k_i^{1,2}(\omega) = t_{1,2}(\omega) + \omega\frac{t_{1,2}^2(\omega)P'(\omega) + 2t_{1,2}(\omega)Q'(\omega) + R'(\omega)}{4(t_{1,2}(\omega)P(\omega) + Q(\omega))} \tag{14}$$

that are parametric in ω or, in an expanded form,

$$k_d^{1,2}(\omega) = \frac{Q'(\omega)P(\omega) - P'(\omega)Q(\omega)}{2\omega P^2(\omega)}$$
$$\pm \frac{2Q(\omega)(Q(\omega)P'(\omega) - Q'(\omega)P(\omega)) + P(\omega)(R'(\omega)P(\omega) - P'(\omega)R(\omega))}{4\omega P^2(\omega)\sqrt{(Q^2(\omega) - P(\omega)R(\omega))}},$$

$$k_i^{1,2}(\omega) = \frac{-Q(\omega) \pm \sqrt{Q^2(\omega) - P(\omega)R(\omega)}}{P(\omega)} + \frac{1}{4\omega P^2(\omega)}\Bigg(2(Q'(\omega)P(\omega) - P'(\omega)Q(\omega))$$
$$\pm \frac{P(\omega)(R'(\omega)P(\omega) - P'(\omega)R(\omega)) - 2Q(\omega)(Q'(\omega)P(\omega) - P'(\omega)Q(\omega))}{\sqrt{(Q^2(\omega) - P(\omega)R(\omega))}}\Bigg).$$

We note that system (10) is equivalent to the equation of the envelope of a family of curves defined by the equation $|H(j\omega, k_1, k_2)|^2 - \gamma^2 = 0$ in the space (k_1, k_2).

System (9) also can be expressed explicitly for the PID-controller. For example, if for the problem (2) of H_∞-performance the numerator and denominator of the weight function $W(s)$ and the transfer function of the plant $G(s)$ have no purely imaginary roots (including zero), then the plane $k_i = 0$ is the solution of system (9) for $\omega = 0$.

4. EXAMPLES

Example 1 ([8]). Let us consider application of the proposed methods by the example of designing a fixed-gain PID-controller. The plane is defined by the transfer function $G(s) = \dfrac{s-1}{s^2 + 0.8s - 0.2}$, the controller, by $C(s, k_i, k_d) = \dfrac{k_i + k_ps + k_ds^2}{s}$, $k_p = -0.35$. Needed is to determine all controllers satisfying the H_∞-criterion of performance (2) with $\gamma = 1$ and high-frequency filter $W(s) = \dfrac{s + 0.1}{s + 1}$.

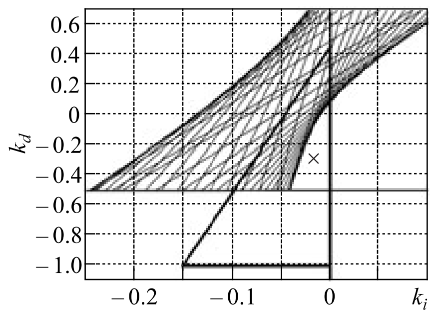

Fig. 2. Example 1. Construction of $\mathcal{K}$ using the admissible sets.

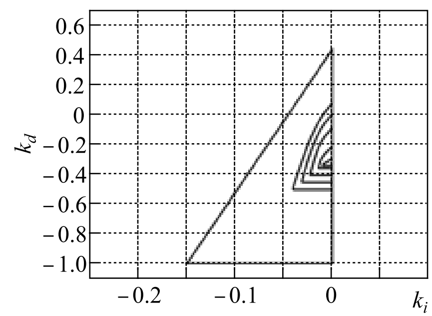

Fig. 3. Example 1. The boundaries of the H_∞-controllers for different levels of γ.

The stability domain on the plane (k_i, k_d) is the triangular with vertices at the points $(0; 0.4507)$, $(0; -1)$, $(-0.15; -1)$. We seek in the parameter space a domain $\mathcal{K}$ such that for all points inside it the condition $|W(j\omega)T(j\omega, k_i, k_d)| < 1$, $\omega \in [0, \infty)$, is satisfied. We begin by constructing it using the admissible sets.

The admissible sets $\mathcal{K}^\omega$ are the external parts of the bands for $\omega \in \{0\} \cup [0.2085, 0.506] \cup [1.375, \infty)$; for the rest of the frequencies, they coincide with the entire plane and, therefore, are disregarded. The last interval is of interest only at the limiting value $\omega = \infty$ where inequality (7) defines the admissible set as the half-plane $k_d > -0.5$ where construction of other admissible sets is constrained.

For the first frequency interval, Fig. 2 depicts pairs of straight lines bounding the bands. The set $\mathcal{K}$ consists of four disjoint components, "×" marking that corresponding to the H_∞-controller.

Figure 3 shows the boundaries $\mathcal{K}$ obtained using the second method for different levels of $\gamma = \{1,\ 0.82,\ 0.68,\ 0.56,\ 0.52\}$. By using the first positiveness condition for discriminant (12), it is possible to calculate for $\gamma = 1$ the ends of the two aforementioned intervals as the roots of the equation

$$Q^2(\omega) - P(\omega)R(\omega) = \omega^8 - 2.19\,\omega^6 + 0.5778\,\omega^4 - 0.02106225\,\omega^2 = 0.$$

Example 2. This example was for the first time considered in [4]. The problem lies in determining a stabilizing controller such that at lower frequencies the sensitivity is small. The value of the criterion $F = \max\limits_{\omega<0.01} |S(j\omega, \mathbf{k})|$ should not exceed 0.1. The plant is defined by its transfer function $G(s) = \dfrac{(s-1)(s-2)}{(s+1)(s^2+s+1)}$. The H_∞-theory suggests to take for solution a weight function $W(s) \in \mathbf{RH}_\infty$, which by its sense is the low-frequency filter, and carry out the H_∞-optimization. In [4], the function $W(s) = \dfrac{s+1}{10s+1}$ was chosen. For the fourth-order controller obtained by the Nevanlinna–Pick method, the criterion was equal to $F = 0.1202$. Later on a PI-controller where $F = 0.0373$ is attained was determined in [9] by selection. This example clearly demonstrates that the procedure of H_∞-optimal design together with the weight function used instead of the direct criterion may provide unsatisfactory results. Therefore, the design of a low-order controller of a given structure can be quite competitive with more complicated schemes of design.

Let us consider only the PID-controllers. We state right away that no PID-controller satisfies the H_∞-criterion $\|W(s)S(s, k_i, k_p, k_d)\|_\infty < 0.1$ for the aforementioned weight function. For another weight function $W(s) = \dfrac{s+1}{100s+1}$, Fig. 4 shows a section of the H_∞-PID-controllers for $k_p = 0.16$.

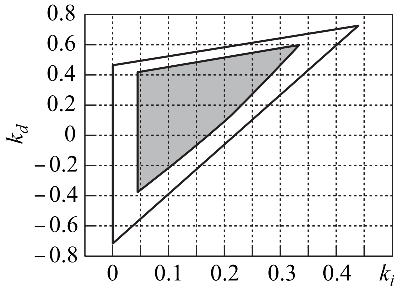

Fig. 4. Example 2. H_∞-PID-controllers using the weight function.

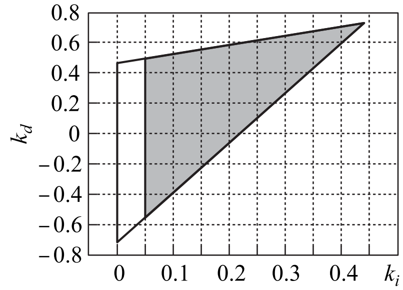

Fig. 5. Example 2. PID-controllers satisfying the direct criterion $F < 0.1$.

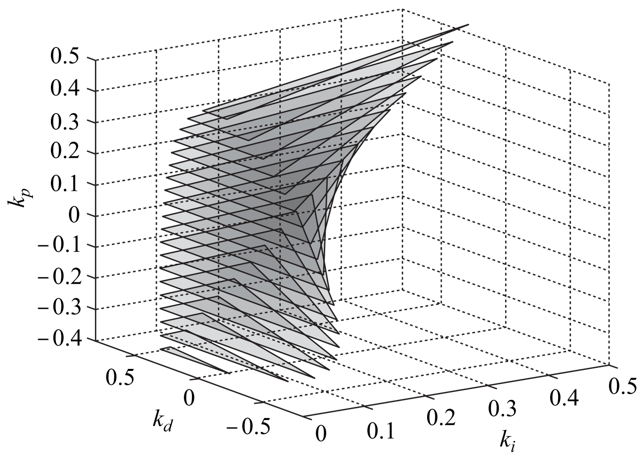

Fig. 6. The set of H_∞ PID-controllers.

The neighboring Fig. 5 depicts the section of the set of controllers satisfying the direct criterion $F(k_i, k_d) < 0.1$ (gray triangle). For comparison, the external triangle is the stability domain. One can see that the weight function makes construction of the set of admissible controllers conservative.

For the direct criterion, Fig. 6 depicts in the space k_i, k_p, k_d the set of all PID-controllers for which $F(k_i, k_p, k_d) < 0.1$ is satisfied.

Example 3 ([18]). We illustrate the distinctions of constructing the stability domains by means of the admissible sets and consider the same plant $G(s) = \dfrac{s-1}{s^2+0.8s-0.2}$ as in Example 1 but embraced by the first-order controller $C(s,\mathbf{k}) = \dfrac{k_1 s + k_2}{s + k_3}$. We are interested in satisfying the criterion $\|W(s)T(s)\|_\infty < \gamma$ for the same weight function $W(s) = \dfrac{s+0.1}{s+1}$.

Let us fix $k_3 = 2.5$ and construct the stability domain using the ordinary D-decomposition (the bold line in Fig. 7). For $\gamma = 1$, the external parts of the ellipses will be the admissible sets for all ω. The resulting set $\mathcal{K}$ is depicted in Fig. 7. However, for $\gamma < 1$ (for example, $\gamma = 0.66$), the admissible sets for some ω are the internal parts of the ellipse. Analysis of Eqs. (2), (7), and (8) shows that satisfaction of the inequality $|W(j\omega)| > \gamma$ will be the necessary and sufficient condition for such ω. For this example, $|W(j\omega)| > \gamma$ for $\omega \in (0.868, \infty)$. Therefore, at changes in ω, one should first take the external and then the internal parts of the ellipse. The transient frequency or,

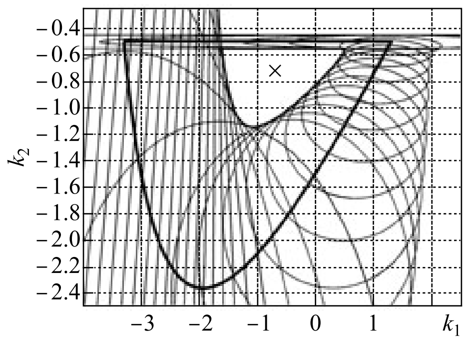

Fig. 7. Admissible sets for the first-order controller ($\gamma = 1$).

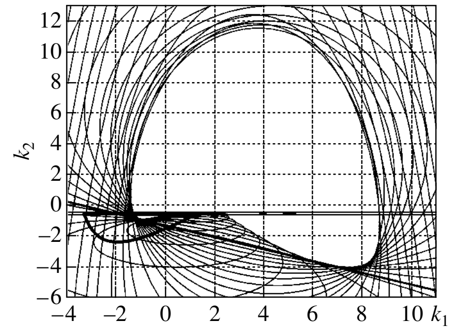

Fig. 8. Admissible sets for $\gamma = 0.66$.

in the general case, frequencies are established from the equation $|W(j\omega)| = \gamma$. The straight line in Fig. 8 for $\omega = 0.868$ corresponds to the "ellipse" degenerated into a half-plane.

If both controlled parameters of the controller are in its numerator, we obtain similar conditions for problem formulations other than (2). If an H_∞-criterion of the form (4) which includes sensitivity is used, then all admissible sets are the external parts of the ellipses (bands). In the case of robust stability where criterion (3) includes the error transfer function, $|W(j\omega)| < \gamma|G(j\omega)|$ will be the condition for the external part to be the admissible set.

5. CONCLUSIONS

The paper proposes two approaches to determination of the set of controllers of a given structure which satisfy the H_∞-criterion. The first approach represents this set in the parameter space as the intersection of the admissible sets, whereas the second approach enables determination of its boundary using the concept of D-decomposition. Both approaches provide a convenient graphic representation and need no laborious computations. The main limitation lies in the fact that consideration is given only to the single-input single-output systems. Nevertheless, these methods are applicable both to the problems with H_∞-criteria including the functions of sensitivity, additional sensitivity, or input sensitivity and to the problems of stabilization of the plants with frequency uncertainty. The possibility of simultaneous consideration of more than one criterion is a significant distinction of the proposed approaches. Since in the actual practice the low-order controllers are dominating, the above results represent an important step toward industrial use of the controllers satisfying the H_∞ criterion.

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