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REVIEWS

D-decomposition Technique State-of-the-art[1]

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Abstract—It is a survey of recent extensions and new applications for the classical D-decomposition technique. We investigate the structure of the parameter space decomposition into root invariant regions for single-input single-output systems linear depending on the parameters. The D-decomposition for uncertain polynomials is considered as well as the problem of describing all stabilizing controllers of the certain structure (for instance, PID-controllers) that satisfy given H_∞-criterion. It is shown that the D-decomposition technique can be naturally linked with $M-\Delta$ framework (a general scheme for analysis of uncertain systems) and it is applicable for describing feasible sets for linear matrix inequalities. The problem of robust synthesis for linear systems can be also treated via D-decomposition technique.

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1. INTRODUCTION

Consider a linear system depending on a vector parameter k with a characteristic polynomial $a(s,k)$. The boundary of a stability domain (in the space k) is given by the equation

$$a(j\omega,k)=0,\quad \omega\in(-\infty,+\infty), \tag{1}$$

that is the imaginary axis (the boundary of instability in the root plane) is mapped into the parameter space. If $k\in\mathbb{R}^2$ (or $k\in\mathbb{C}$) then we have two equations (real and imaginary part of (1)) in two variables and (in general) can define the parametric curve $k(\omega), -\infty<\omega<\infty$ defining a boundary of the stability domain. Moreover, the curve $k(\omega)$ divides the plane into root invariant regions (i.e., regions with a fixed number of stable and unstable roots of $a(s,k)$). This is the basic idea of D-decomposition approach. The idea can be traced to Vishnegradsky [85] who reduced a cubic polynomial to the form $a(s,k)=s^3+k_1s^2+k_2s+1$ and treated the coefficients $k_1,\ k_2$ as parameters. Then Eq. (1) yields $k_1\omega^2=1,\ \omega(k_2-\omega^2)=0$. Eliminating ω we get that D-decomposition is given by the hyperbola $k_1k_2=1$. The stability domain is the set $k_1k_2>1$.

For the general case, similar ideas were exploited in [1, 27, 51] (the latter two papers deal with time-delay systems). Moreover, Nyquist plot can be considered as the realization of the same idea. But it was Yu. Neimark [13, 14] who developed the rigorous algorithm (and coined the name "D-decomposition"). In the Western literature the technique is described first in [68]; the mapping of contours other that imaginary axis was also proposed. This line of research was significantly developed by Siljak [80–82]. He extended the approach for nonlinear systems and for the case of nonlinear parameter dependence. In his works D-decomposition (which he calls the parameter plane method) was broadened to become a useful tool for design purposes. Neimark's method is also

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investigated in [63], where some results on multi-parameter cases can be found. The D-decomposition technique is described in Soviet textbooks on automatic control, in the western literature it can be found in [32, 38]. The papers [33, 39, 52, 61] show its high efficiency for the problems of low-order controllers with H_∞-specifications. The analysis of the geometrical properties of the D-decomposition is given in [17–19].

In the 80s within the appearance of the problems with uncertainty (robust control) the D-decomposition seems to be efficient for analysis of the families of linear systems [15]. Within the stability investigation there is a question of interest: if the system remains stable since its parameters are unknown constants belonging to a particular set. These parameters are called "uncertainties." The most famous results in this field are the Kharitonov theorem and its analogs concerning the robust stability of interval polynomials as well as Tsypkin–Polyak locus for interval and ellipsoidal restrictions on the polynomial coefficients; see [21, 24, 30].

The D-decomposition technique can be applied to the systems in state space form. More specifically, given a class $\mathcal{K}$ of $r \times m$ matrices K, find all the matrices $K \in \mathcal{K}$ such that $A + BKC$ is stable:

$$D = \{K \in \mathcal{K} : \ A + BKC \quad \text{is stable}\}. \tag{2}$$

Here A, B, C are given real matrices of dimensions $n \times n, \ n \times r, \ m \times n$ respectively; stability is understood either in continuous-time sense (all eigenvalues are in the open LHP) or discrete-time sense (all eigenvalues are in the open unit disc). Class $\mathcal{K}$ may be different; below we analyze in detail the simplest cases:

$$K = k \in \mathbb{R}^n \ \text{ or } \ K = k^T, \quad k \in \mathbb{R}^n \quad (\ m = 1 \text{ or } r = 1), \tag{3}$$

$$K = kI, \quad k \in \mathbb{R} \quad \text{or} \quad k \in \mathbb{C}, \quad m = r, \tag{4}$$

$$K \in \mathbb{R}^{2\times 2}, \tag{5}$$

where all calculations can be performed explicitly in the graphical form. Case (3) is equivalent to the polynomial framework, two others are essentially matrix ones. Nevertheless we present general description of D-decomposition. It is closely related to the standard $M - \Delta$ setting.

Problem (2) arises in the design or robustness studies. For instance, to find all stabilizing static output controllers for the system

$$\dot{x} = Ax + Bu, \ y = Cx \tag{6}$$

one can construct the set D (2) with $\mathcal{K} = \mathbb{R}^{r\times m}$; here K plays the role of the feedback gain. On the other hand, if A is a nominal stable matrix and it is perturbed as $A + BKC$, where K is a constant $r \times m$ matrix; then (2) provides all admissible perturbations which preserve stability. It is obvious that if we know the boundary of the stability domain ∂D, then we can find the distance to it:

$$\rho = \min_{K \in \partial D} \|K\|. \tag{7}$$

The quantity ρ^{-1} is closely related to μ (structured singular value) [86]. If $\mathcal{K}$ is the set of all $\mathbb{C}^{r\times m}$ ($\mathbb{R}^{r\times m}$) matrices, then ρ is complex (real) stability radius [55, 75]. Of course, the knowledge of the entire set D provides much more information than the value of ρ. For instance, for design purposes a designer can solve performance or specification problems on the set of the stabilizing controllers D.

Instead of direct optimization on stability domain D we can get more information on the roots of the characteristic polynomial inside D by similar graphical tools. For instance, if one is interested in the stability degree $\sigma > 0$, then the regions of interest can be obtained replacing $j\omega$ by $-\sigma + j\omega$ in the D-decomposition equation. This approach has been proposed by Neimark [14] and it was significantly developed by Mitrovic [68] and Siljak [80–82]. Of course, we can deal with contours other than shifted imaginary axis, for instance, with a sector or the other root placement. This information on the root placement inside the stability domain is very helpful for the design purposes. However we will not develop this line of research and deal with the simplest situations of the root placement—the left half-plane (for continuous-time systems) and the unit disc (for discrete-time systems).

Additional detailed decomposition can be a subject of future research. Several examples below illustrate how the technique can be extended to become a practical tool for a designer. On the other hand, we restrict ourselves with a presentation of basic results of the proposed approach and do not consider practical applications.

The natural question arises: what is the use of finding all root invariant domains, while in practice we are interested in the stability domain only? The first answer to the question is very simple—we are unable to construct the stability domain separately, the only way suggested by the proposed technique is to provide the complete decomposition of the parameter space and then to pick out the stability regions (if they exist). On the other hand there are some situations when eigenvalue invariant domains are also of interest. For instance, dealing with Nyquist diagram under uncertainty one should guarantee that all transfer functions under consideration have the same number of stable poles. In some cases, we can propose the method for distinguishing stability regions among all the root invariant regions; see Section 2.2 for details.

In 1991 the problem of robust D-decomposition was formulated and partially solved in [20]. In general, the problem is as follows: first we select two real (or one complex) parameters, the rest of the parameters are assumed to be uncertain but bounded by norm. The problem is to describe the region in the plane of the selected parameters such that the polynomial is stable for all feasible parameter values. This division for two parameter classes is very typical for the problem of robust fixed-structure controllers. The paper [28] contains analytical expressions of the robust D-decomposition for two real parameters for an affine polynomial family with l_p-bounded parametric uncertainties.

Another new application of the D-decomposition technique is the theory of linear matrix inequalities [2, 41]. For this problem statements, the technique allows defining all the regions in the parameter space such that the affine family of symmetric matrices has a fixed number of like-sign eigenvalues inside a region [25].

Finally, the randomization technique inside the stability domain seems promising for optimal controller synthesis [74].

The rest of the paper is organized as follows. In Section 2 we consider the rank-one case (i.e., single-input or single-output systems), where the stability is determined by the location of the roots of the characteristic polynomial.

Section 2.1 is addressed for the classical D-decomposition for polynomials with one or two real parameters. The main contribution here is the study of the D-decomposition geometry. In particular, we estimate the number of all root invariant regions as well as the number of simply connected stability regions. Section 2.2 is refereed to the problem of aperiodic stability.

Section 3 is devoted to characteristic polynomials with affine uncertain of a certain structure and the arbitrary number of parameters. The stability domain for this polynomials is a union of polytopes in the parameter space. We suggest the method to describe the stability components and to calculate the stability radius for various norm of uncertainty for continuous and discrete systems.

Section 4 contains the results on the robust D-decomposition.

Section 5 describes the problem of the fixed-order controller synthesis with H_∞ specifications. We consider linear time-invariant system with a plant given by scalar transfer function $G(s)$ and feedback controller $C(s,k)$. The controller parameters $k \in \mathbb{R}^m$, $m \leqslant 3$ define the feasible set of low-order controllers. The structure of the controller is fixed. The problem is to describe the set of the parameters corresponding to all the controllers such that $\|H(s,k)\|_\infty < \gamma$, where $H(s,k)$ is a transfer function for the closed-loop system.

In Section 6 we extend the D-decomposition idea to the matrix case. Namely, we consider the systems with matrix transfer functions. The first subsection relates to the case, where a graphical representation is possible. The number of parameters is restricted by two.

Section 7 is devoted to the application of the D-decomposition in the theory of linear matrix inequities. For a small number of parameters (1 or 2) on the line or on the plane we plot the regions such that the affine family of symmetric matrices has a fixed number of like-sign eigenvalues inside a region.

In Section 8 it is shown that the D-decomposition can be applied in the randomized algorithms for hard problems of synthesis. Combined with Hit-and-Run algorithm it allows us to generate points asymptotically uniformly distributed on a multidimensional domain (generally speaking nonconvex and not simply connected). In particular, it may be a stability domain.

Section 9 contains conclusive remarks.

2. D-DECOMPOSITION FOR POLYNOMIALS

Consider a linear system with the characteristic polynomial of the form:

$$a(s,k) = a_0(s) + \sum_{i=1}^{m} k_i a_i(s), \tag{8}$$

where $a_i(s)$, $i = 0, \ldots, m$ are polynomials of degree no more than n and the parameters $k \in \mathbb{R}^m$.

Definition 1. The decomposition of the parameter space k into the regions $D_l = \{k : a(s,k)$ has l stable roots$\}$, $l = 0, \ldots, n$, is called the D-decomposition. The equation describing the boundary of regions D_l is called D-decomposition equations. Thus D_n is the stability domain D.

We parametrize the boundary of the stability domain on the root plane: for continuous-time systems it is the imaginary axis $j\omega$, $\omega \in (-\infty, +\infty)$, for discrete-time systems it is the unit circumference $e^{j\omega}$, $\omega \in [0, 2\pi)$. Then the number of stable roots may change in one of the following cases: the degree of polynomials changes, one real or two complex roots cross the stability boundary on the root plane. For the discrete-time systems the difference is that the decreasing of the degree does not lead to the change of the stable root number.

Thus the D-decomposition is given by the equations:

$$a(j\omega, k) = 0, \quad \omega \in (-\infty, +\infty) \quad \text{or} \quad a^{(n)}(k) = 0; \tag{9}$$

$$a(e^{j\omega}, k) = 0, \quad \omega \in [0, 2\pi), \tag{10}$$

where $a^{(n)}$ is the leading coefficient of the polynomial $a(s,k)$. Further, we deal with polynomials with real coefficients. Since for continuous-time systems $a(j\omega, k) = 0$ leads to $a(-j\omega, k) = 0$ (similarly for discrete-time systems $a(e^{j\omega}, k) = 0$ leads to $a(e^{-j\omega}, k) = 0$), the interval of ω may be reduced to $[0, \infty)$ (correspondingly $[0, \pi]$).

2.1. Development of Graphical Algorithms

In the section we consider cases when the number of parameters does not exceed two. Parameter space is the plane and the root invariant regions can be graphically represented.

Linearity of D-decomposition equation allows one to find easily the stability radius (7) for various l^p norms of the vector k. This link between robust stability and D-decomposition has been emphasized in [15]. On the other hand, the connection between μ-analysis and the known results on parametric robustness [21,32,35,38,62] has been pointed out by Chen [43]. Scalar constant gain problem has been solved semi-analytically in [71], the result was extended to compute stabilizing controllers of a given order [76]. We will not highlight these links but will focus on the simplest cases when Eq. (8) provides graphical tools to describe D-decomposition in the space of parameters k.

The case of the single-input single-output systems with small number of parameters is the classical framework of the D-decomposition.

In many cases we can state that the stability domain is simply connected. But there exist examples [14, 18], where it is not true. Remind that the set D is simply connected if for any $K_0 \in D$, $K_1 \in D$ there exist the continuous parametrization $K(t)$ such that $K(t) \in D$, $0 \leqslant t \leqslant 1$, $K(0) = K_0$, $K(1) = K_1$. It is not clear what is the maximal number of connected stability regions. Moreover, the total number of the root invariant regions is also unknown.

For single-input single-output systems with a scalar transfer function $H(s) = \frac{b(s)}{a(s)}$, where $a(s)$, $b(s)$ are polynomials of degree n, with the feedback gain k and characteristic polynomial $p(s) = a(s) + kb(s)$ the D-decomposition is given by

$$a(j\omega) + kb(j\omega) = 0, \quad \omega \in [0, \infty). \tag{11}$$

We avoid the situations when $a(s)$, $b(s)$ have a common imaginary (or zero) root. Thus (39) is equivalent to $-1/k = H(j\omega)$ or to the standard Nyquist diagram: the critical values of the gain k (which correspond to a change of the stable roots number for the characteristic polynomial) are defined by the intersections of the Nyquist plot $H(j\omega)$ and the real axis. Note that the alternative way to analyze the situation is the root locus technique [3, 16, 49]. However, D-decomposition approach allows one to find critical points analytically and to estimate their number.

Theorem 1 [52]. *For polynomial family* (11) *the real axis may be divided by points* $-\infty < k_1 < k_2 < \ldots < k_m < k_{m-1} < +\infty$ *into no more than* $m \leqslant n + 2$ *intervals* $(-\infty, k_1)$, $(k_1, k_2), \ldots,$ $(k_{m-1}, +\infty)$ *such that for every interval* $k_i < k < k_{i+1}$ *polynomial* $a(s) + kb(s)$ *has a constant number of stable roots* ν_i. *Moreover the number of stability intervals* (*i.e., intervals* (k_i, k_{i+1}) *with* $\nu_i = n$) *does not exceed* $\lfloor \frac{n}{2} \rfloor + 2$ ($\lfloor \alpha \rfloor$ *is the maximal integer less than or equal to* α).

Note that the maximal number of stability intervals may be attained for the case of odd n when both $(-\infty, k_1)$ and $(k_{m-1}, +\infty)$ intervals are stability intervals.

The algorithm for finding the critical values k_i and the numbers ν_i is based on this theorem. We formulate it here under assumption that polynomials $a(s)$ and $b(s)$ do not have imaginary roots and all critical values ω are simple roots of the equation $\operatorname{Im} H(j\omega) = 0$. The case of multiple ω requires additional investigations and we omit it here.

Algorithm.

Step 1. Solve the polynomial equation $\operatorname{Im} H(j\omega) = 0$ in $0 \leqslant \omega < \infty$ (due to symmetry $\operatorname{Im} H(j\omega) = -\operatorname{Im} H(-j\omega)$). Denote the obtained solutions $0 = \omega_0 < \omega_1 < \omega_2 < \ldots < \omega_{m-2}$ and calculate $u_i = \operatorname{Re} H(j\omega_i)$, $i = 0, \ldots, m-2$, $u_{m-1} = b^{(n)}/a^{(n)}$.

Step 2. Order quantities $-\frac{1}{u_i}$, $i = 0, \ldots, m-1$ and denote them $k_1 < \ldots < k_{m-1}$ such that $k_s = -\frac{1}{u_{i_s}}$.

Step 3. Find the interval containing zero: $0 \in (k_\ell, k_{\ell+1})$ and equate ν_ℓ with the number of stable roots of the polynomial $a(s)$.

Step 4. The numbers ν_s $(s = 1, \ldots, m)$ are calculated successfully:

$$\nu_{s\pm1} = \begin{cases} \nu_s \mp 2\mathrm{sgn}\,\varphi(\omega_{i_s}), & \omega_{i_s} \neq 0, \quad \varphi(\omega_{i_s}) \neq 0 \\ \nu_s \mp \mathrm{sgn}\,\varphi(0), & \omega_{i_s} = 0, \quad \varphi(\omega_{i_s}) \neq 0 \\ \nu_s \mp \mathrm{sgn}\,\varphi(\Omega), & k_s = -a^{(n)}/b^{(n)},\ \Omega \ \text{ being large enough,} \end{cases}$$

where $\varphi(\omega_{i_s}) = \frac{d}{d\omega}\mathrm{Im}\ H(j\omega_{i_s})$.

For the case of two real parameters, the D-decomposition for the polynomial family

$$a(s,k) = a_0(s) + k_1 a_1(s) + k_2 a_2(s), \quad k_1, k_2 \in \mathbb{R}, \tag{12}$$

is given by $a(j\omega, k_1, k_2) = 0$.

Resolving this equation with the respect to the parameters we obtain the curve

$$k_1 = -\frac{\Delta_1}{\Delta}, \quad k_2 = -\frac{\Delta_2}{\Delta}, \tag{13}$$

where

$$\Delta = \begin{vmatrix} \mathrm{Re}\,a_1(j\omega) & \mathrm{Re}\,a_2(j\omega) \\ \mathrm{Im}\,a_1(j\omega) & \mathrm{Im}\,a_2(j\omega) \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} \mathrm{Re}\,a_0(j\omega) & \mathrm{Re}\,a_2(j\omega) \\ \mathrm{Im}\,a_0(j\omega) & \mathrm{Im}\,a_2(j\omega) \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} \mathrm{Re}\,a_1(j\omega) & \mathrm{Re}\,a_0(j\omega) \\ \mathrm{Im}\,a_1(j\omega) & \mathrm{Im}\,a_0(j\omega) \end{vmatrix}.$$

For $\Delta \neq 0$ the formulae (13) define the parametric curve $k_1(\omega)$, $k_2(\omega)$, and two roots cross the stability boundary with crossing this curve. It is symmetric in ω: $k_i(-\omega) = k_i(\omega)$, $i = 1, 2$, thus it suffices to take for continuous-time systems $\omega \in (0, \infty)$. Moreover, there are two straight lines

$$\omega = 0: \quad a_0(0) + k_1 a_1(0) + k_2 a_2(0) = 0,$$
$$\omega = \infty: \quad a_0^{(n)} + k_1 a_1^{(n)} + k_2 a_2^{(n)} = 0.$$

Curve (13) starts at a point on the first line (for $\omega = 0$) and terminates at a point on the other line. The intersection of these lines specifies one root (0 or ∞) to cross the stability boundary. We call these lines *boundary lines*.

If $\Delta = 0$ but $\Delta_1 \neq 0$ or $\Delta_2 \neq 0$, then curve (13) goes to infinity; it may have discontinuity at these points. The value of ω such that $\Delta = \Delta_1 = \Delta_2 = 0$ is called *singular frequency*. In this case, the D-decomposition reduces into a straight line. We call these lines *singular lines*.

Note that the singular ω such that $\Delta = \Delta_1 = \Delta_2 = 0$ may fill a particular interval. For instance, for the polynomial $a(s,k) = k_1 s^2 + k_2 + 1$ (proposed by E.M. Solnechnyi) the system of the D-decomposition equations reduces to one equation in two variables because $\mathrm{Im}\,a(j\omega, k) \equiv 0$. Thus the line $-k_1\omega^2 + k_2 + 1 = 0$ is singular for all ω, and for any point from the region $k_1(k_2+1) \geqslant 0$ the polynomial has a pair of imaginary roots.

Theorem 2 [6, 53]**.** *For family* (11) *with one complex parameter, the number of D-decomposition regions does not exceed $N \leqslant (n-1)^2 + 2$, and for family* (12) *with two real parameter, $N \leqslant 2n(n-1) + 3$.*

The above mentioned consideration is valid for the continuous-time systems. But using fractional transformation

$$s = \frac{z+1}{z-1}$$

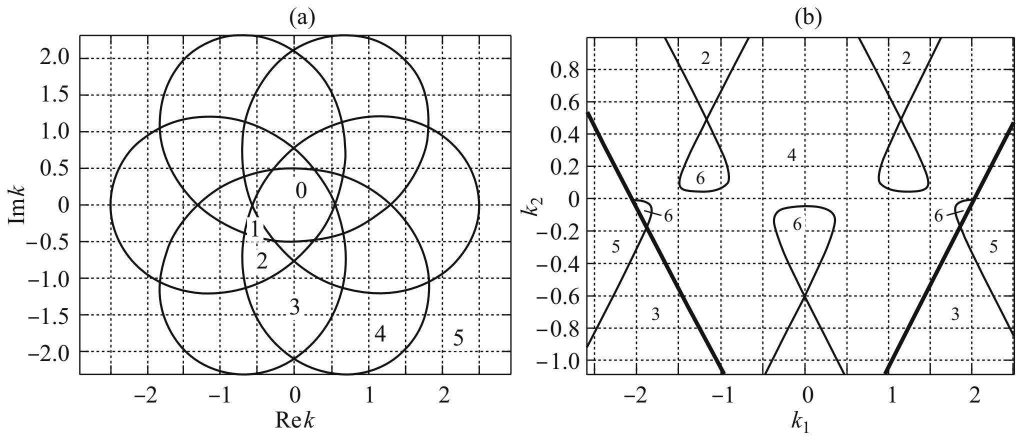

Fig. 1. D-decomposition in (a) Example 1 and (b) Example 2.

allows us to proceed to the discrete-time systems and vice versa. Thus due to conformity the results remain valid for the discrete-time systems also. Most of the examples are demonstrative in the discrete-time version but the demonstrated features are preserved also for the continuous-time systems. We prefer discrete-time systems because it is easier to make the D-decomposition analysis for the finite frequency interval $\omega \in [0, 2\pi)$ rather than infinite interval for the continuous-time systems. Here and elsewhere we denote by $a(s, k)$ the continuous-time polynomial, i.e., the polynomials is stable when all its roots are in the left half plane, and $a(z, k)$ being the discrete-time polynomial and it is stable when all its roots are inside the unit circumference.

Example 1 [6]. The D-decomposition for the polynomial $a(z, k) = z^n + kz^{n-1} + \alpha$, where $k \in \mathbb{C}$, has $(n-1)^2 + 1$ root invariant regions for $\alpha > 1$, and two root invariant regions for $\alpha < 1/(n-1)$. The D-decomposition is given by the parametric curve $k(\omega) = -e^{j\omega} - \alpha e^{-j\omega(n-1)}$, $0 \le \omega < 2\pi$, depicted in Fig. 1a. In this figure (as well as in the further figures) the digit inside the region marks the number of stable roots. This curve is generated by a moving point on the complex plane. This motion is a superposition of two rotations. First rotation has radius 1 and frequency 2π while the second one has radius α and frequency $(n-1)2\pi$. For $\alpha > 1$, this curve consists of n arcs, each arc intersects any other twice. Thus the calculation of the number of intersections gives $N = n^2 - 2n + 2$; this is just one region less the upper bound given by Theorem 2.

Example 2 [18]. For the polynomial $a(z, k) = z^n + k_1 z^{n-1} + \alpha z^{n-2} + k_2$, $1 < \alpha < \frac{n}{n-2}$ there are $n - 1$ simply connected stability regions in the $\{k_1,\ k_2\}$ plane. The structure of the regions for $n = 6$ can be seen in Fig. 1b; three regions (marked by the digit 6) are in the loops of the boundary curve and the other two regions are formed by the intersection of the curve and boundary lines given by the equations $a(e^{j0}, k) = 0$, $a(e^{j\pi}, k) = 0$ and plotted by solid lines.

2.2. D-decomposition for Aperiodicity

Remind that the polynomial $a(s)$ is called aperiodic if all its roots are simple real and negative [9]. For the discrete-time system with the polynomial $a(z)$ the definition is similar but "negative" should be replaced by "absolute value less than one." The aperiodic property of the linear system (i.e., the system with aperiodic characteristic polynomial) guarantees that the transient process is not oscillating. This property is desirable in many cases so it is natural to formulate the problem of describing aperiodic domain in the parameter space.

We consider the same affine polynomial family (8). Definition 1 is modified as follows:

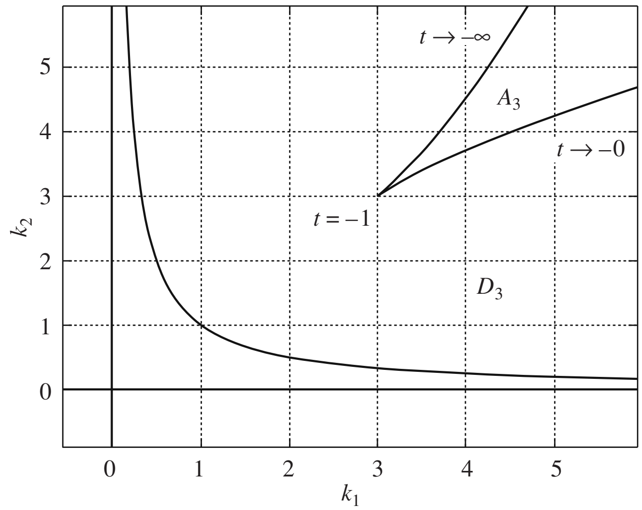

Fig. 2. Stability and aperiodicity regions for Visgnegradsky exmaple.

Definition 2. The decomposition of the parameter space k into the regions $A_l = \{k : a(s,k)$ has l real simple negative roots$\}$, $l = 0, 1, \ldots, n$, is called A-decomposition.

Obviously, the transition from one A_l to the other occurs only when a multiple real or zero root appears or the degree of polynomial changes. Thus the boundary of the A-decomposition is defined by the system of equations

$$\begin{aligned} a(t,k) &= 0, \\ a'(t,k) &= 0, \end{aligned} \tag{14}$$

where $t \in (-\infty, 0)$ is the value of the real multiple root, $'$ denotes differentiation in t, or the equation

$$a(0,k) = 0,$$

or the condition

$$a^{(n)}(k) = 0.$$

(The notations are the same as in the beginning of the section.)

Specifically, for the cubic polynomial in Vishnegradsky form $a(s,k) = s^3 + k_1 s^2 + k_2 s + 1$ the boundary is defined just by Eq. (14): $t^3 + k_1 t^2 + k_2 t + 1 = 0$, $3t^2 + 2k_1 t + k_2 = 0$. Expressing k_1, k_2 in terms of t obtain:

$$k_1(t) = \frac{1}{t^2} - 2t, \quad k_2(t) = t^2 - \frac{2}{t}.$$

Remind that $t < 0$, thus the whole curve lies in the positive ortant and it is the boundary of the aperiodic stability region (Fig. 2).

This result was obtained by Vishnegradsky [85] and it is published in old textbooks on automatic control.

Now consider the case of one real parameter, i.e., the polynomial of the form

$$p(s,k) = a(s) + kb(s), \quad k \in \mathbb{R}.$$

Equations (14) for this case can be rewritten as:

$$\begin{aligned} a(t) + kb(t) &= 0, \\ a'(t) + kb'(t) &= 0. \end{aligned}$$

Suppose that $a(s)$ and $b(s)$ do not have a common negative real root. Then we can eliminate k, and obtain

$$q(t) = a'(t)b(t) - a(t)b'(t) = 0.$$

Let $a(s)$, $b(s)$ be the polynomials of degree n. Then $q(t)$ is a polynomial of degree no more than $2n-1$ and it has no more than $2n-1$ real negative roots $t_1 < t_2 < \cdots < t_m < 0$, $m \leqslant 2n-1$. Thus the ray $(-\infty, 0)$ can be divided into no more than $2n$ intervals. This fact allows us to obtain the analog of Theorem 1 and an algorithm for construction of the A-decomposition of the parameter space; we don't deep into details here.

The problem of the robust aperiodicity can be solved using the technique described in [21, 23].

3. CONSTRUCTION OF THE MULTIDIMENSIONAL STABILITY DOMAIN

In this section we proceed to the consideration of the single-input or output systems with the characteristic polynomial linearly depending on the parameters. In some special cases, the stability domain can be described constructively in the multidimensional (not only in two dimensional) parameter space. For the continuous-time systems, we consider the families of the form:

$$a(s,k) = a_0(s) + f(s)\sum_{i=1}^{v} k_i a_i(s^2), \quad k = \{k_1, \ldots, k_v\} \in \mathbb{R}^v, \tag{15}$$

where a_i, $i = 0, 1, \ldots, v$, f are the polynomials with real coefficients, $deg(a_0) = n_0$, $deg(f) = n_1$, $\max_i\{deg(f) + deg(a_i), n_0\} = n$. Note that all the polynomials a_i, $i = 1, \ldots, v$ contain only even degrees of s.

The results of this section are valid also for the discrete-time systems of the form

$$a(z,k) = a_0(z) + f(z)\sum_{i=1}^{v} k_i a_i(z), \quad k = \{k_1, \ldots, k_v\} \in \mathbb{R}^v, \tag{16}$$

where $deg(a_0) = n_0$, $deg(f) = n_1$, and the polynomials $a_i(z)$, $i = 1, 2, \ldots, v$ are simultaneously symmetric or antisymmetric as the polynomials of degree $n \leqslant n_0 - n_1$. We recall that the polynomial $a_i(z)$ is called the symmetric polynomial of degree n if $a_i(z) \equiv z^n a_i(z^{-1})$, and antisymmetric polynomial if $a_i(z) \equiv -z^n a_i(z^{-1})$ (for instance, $z^3 + z$ is a symmetric polynomial of the fourth degree). Here we consider only continuous-time systems, for more details see [8].

It seems that the works [64, 47] were the first studies in this direction which considered the continuous-time case with $f(s) \equiv 1$ and the stability domain described by a system of linear inequalities. A similar result for the discrete-time systems [48] is mentioned in the survey [10] among other non-Kharitonov approaches to robustness of the discrete-time systems. It is also worth mentioning the works of J. Ackermann [33], N. Munro [84], S.P. Bhattacharyya [38, 57], where the stability domain was constructed for the PID-controller. Further, we show that an algebraic method for detecting the stability domain discussed in the previous sections works for the multidimensional parameter space.

We note that in the pure form the polynomials of this kind are encountered rarely, but the problem often may be reduced to the considered family, for example, by fixing the parameters at the odd degrees. This approach recently proved well in the combined probabilistic/deterministic methods [52, 45].

We apply the D-decomposition idea for continuous-time system (15). The boundaries of the regions with the constant number of stable roots according to (9) are described by the system of equations:

$$\operatorname{Re} a_0(j\omega) + \operatorname{Re} f(j\omega) \sum_{i=1}^{v} k_i a_i(-\omega^2) = 0, \tag{17}$$

$$\operatorname{Im} a_0(j\omega) + \operatorname{Im} f(j\omega) \sum_{i=1}^{v} k_i a_i(-\omega^2) = 0, \quad \omega \in [0, \infty]. \tag{18}$$

For $\omega = \infty$, the equations contain coefficients with the degree n only.

Expressing $\sum_{i=1}^{v} k_i a_i(-\omega^2)$ from one equation and substituting in into another, we obtain an equation without parameters:

$$\operatorname{Re} a_0(j\omega) \operatorname{Im} f(j\omega) - \operatorname{Re} f(j\omega) \operatorname{Im} a_0(j\omega) = 0. \tag{19}$$

The simple real positive roots of this equation $0 = \omega_0 < \omega_1 < \ldots < \omega_{m-1}$ such that $f(\pm j\omega) \neq 0$ are called *critical frequencies*. Moreover, if the equality $deg(a_0) = \max_i \{deg(f) + deg(a_i)\}$ holds, we denote $\omega_m = \infty$, this value corresponds to a change on the degree. For each critical frequency Eqs. (17) and (18) are linearly dependent and describe the hyperplane in the parameter space which is the boundary of the D-decomposition:

$$p(\omega_i) + |f(j\omega_i)|^2 \sum_{\ell=1}^{v} k_\ell a_\ell(-\omega_i^2) = 0, \quad i = 0, 1, \ldots, m, \tag{20}$$
$$\text{where} \quad p(\omega) = \operatorname{Re} a_0(j\omega) \operatorname{Re} f(j\omega) + \operatorname{Im} a_0(j\omega) \operatorname{Im} f(j\omega).$$

Equation (20) is the real part of the polynomial $a^*(j\omega, k) = a(j\omega, k) f(-j\omega)$. At the points ω_i, it is equivalent to the system of Eqs. (17) and (18). Note that Eq. (19) without parameters corresponds to the imaginary part of the polynomial $a^*(j\omega, k)$.

These hyperplanes divide the parameter space $\mathbb{R}^v$ into the regions with constant number of stable roots, any such region being a convex polytope. In turn, every region is a solution of the system of linear inequalities with the respect to the parameters:

$$\xi_i \left(p(\omega_i) + |f(j\omega_i)|^2 \sum_{\ell=1}^{v} k_\ell a_\ell(-\omega_i^2) \right) > 0, \quad \xi_i \in \{-1; 1\}, \quad i = 0, 1, \ldots, m. \tag{21}$$

Assertion 1 [8]. *Any root invariant region D_l of polynomial* (15) *is a union of nonintersecting polytopes.*

Since the space of parameters is divided into domains with constant number of stable roots, then there are several methods of determining the number of stable roots in each domain. The simplest method is to take a point in every region and to calculate the roots of the corresponding polynomials. The classical D-decomposition technique prescribes to use shading that shows the extent of changes in the number of stable roots at the crossing of the curves of D-decomposition. Efficient use of shading in general form is possible only for a low dimensionality of the parameter space, already for $\mathbb{R}^3$ this technique becomes hindered. For the one-parameter family, an algebraic method of determining the number of stable roots in various domains was proposed in [6, 53]. The generalization of the Hermite–Biehler theorem [58] enables one to exploit the idea of shading for the multidimensional case for the polynomials of a certain form (15).

Let some region in the parameter space be described by the system of linear inequalities (21) for a fixed set $\{\xi_i\}$, $i = 0, 1, \ldots, m$, where ω_i are the zeros of the imaginary part of the original

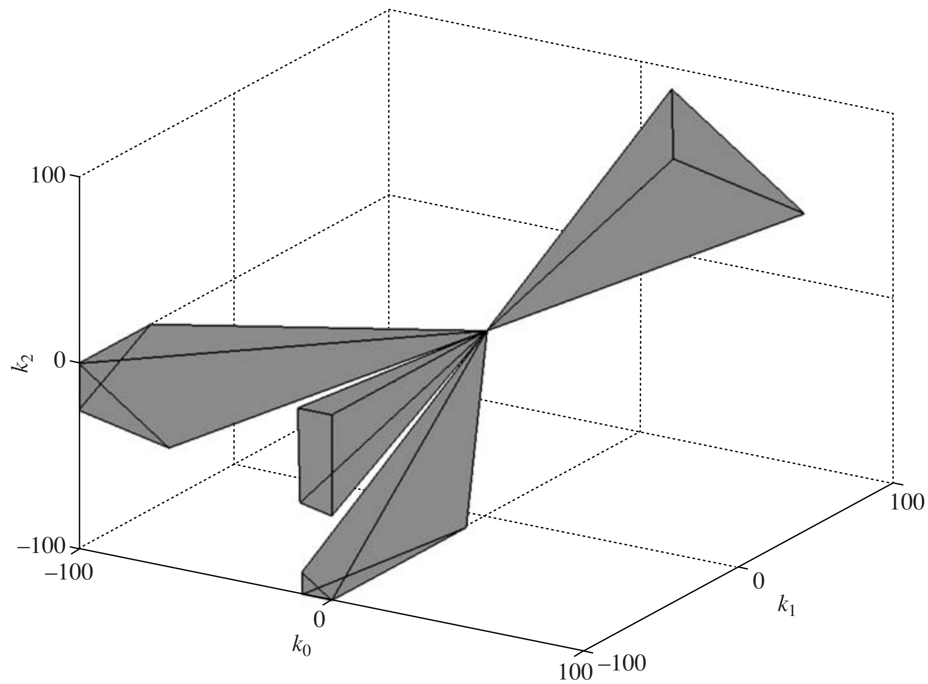

Fig. 3. Stability domain in the parameter space in Example 3.

polynomial which contains no parameters. Since by construction the region boundaries satisfy the equations of D-decomposition, within this region the polynomial $a(s,k)$ has the certain number of l stable and r unstable roots. The difference $l-r$ (that is called *the signature of the polynomial* $\sigma(a)$) in this region [58] is defined by the set of signs of the real part at the points ω_i, which corresponds to $\xi_i = \mathrm{sgnRe}\, a(j\omega_i)$, $i = 0, 1, \ldots, m$, that is,

$$\sigma(a) = \begin{cases} (\xi_0 - 2\xi_1 + \ldots + (-1)^{m-1}2\xi_{m-1} + (-1)^m\xi_m)(-1)^{m-1}\mathrm{sgn}(\mathrm{Im}\, a(j\infty)), & \text{if } n \text{ is even} \\ (\xi_0 - 2\xi_1 + \ldots + (-1)^{m-1}2\xi_{m-1})(-1)^{m-1}\mathrm{sgn}(\mathrm{Im}\, a(j\infty)), & \text{if } n \text{ is odd.} \end{cases}$$

Let $f(s)$ has l_1 roots to the left of the imaginary axis and r_1 roots to its right. Then, the polynomial $a(s,k)$ has l stable and $n-l$ unstable roots if $a^*(s,k) = a(s,k)f(-s)$ has $l+r_1$ stable roots; at that, $\sigma(a^*) = l + r_1 - (n-l) - l_1 = 2l - n + r_1 - l_1$. The generalization of the Hermite–Biehler theorem allows one to establish what relation $\mathrm{sgn}\,\mathrm{Re}\, a^*(j\omega_i)$, $i = 0, \ldots, m$ provides given signature, and these signs specify the choice of ξ_i for the system of inequalities (21).

The advantages of the considered method manifest themselves with increase in the number of parameters because solution of systems of linear inequalities still remains the main operations.

Example 3. Consider the polynomial $a(s,k) = 0.1 + f(s)(k_0 + k_1 s^2 + k_2 s^4)$, where $f(s)$ is a stable polynomial; at that, $f(j\omega) = T_m(1-\omega^2) + j\omega T_{m-1}(1-\omega^2)$, where $T_i(t)$ are the Chebyshev polynomials. These polynomials are defined for all $s \in \mathbb{C}$, but their root are located in the segment $[-1; 1]$. Thus without loss of generality we use the representation for the Chebyshev polynomials of the form $T_i(t) = \cos(i \arccos t)$, $1 - \omega^2 = t \in [-1; 1]$.

The imaginary part $f(j\omega)$ has m different simple positive roots, $deg(f) = 2m$, $deg(a) = 2m + 4$, and due to combinatorial estimations there exist $C_m^{(m/2)+1}$ sets $\{\xi_i\}$ each defining a system of inequalities describing a component of the stability domain.

For $m = 4$, the number of permissible sets $\{\xi_i\}$ is four, and as shown in Fig. 3, solution of inequalities for all such $\{\xi_i\}$ turns out to be nonempty.

If the polynomial $a_0(s)$ is stable the theorem holds:

Theorem 3 [8]. *The component of the stability domain of family* (15) *containing the nominal stable polynomial* $a_0(s)$ *obeys the system of inequalities:*

$$\operatorname{sgn}(p(\omega_i))\left(p(\omega_i)+|f(j\omega_i)|^2\sum_{\ell=1}^{v}k_\ell a_\ell(-\omega_i^2)\right)>0,\quad i=0,1,\ldots,m,$$

where $p(\omega)=\operatorname{Re}a_0(j\omega)\operatorname{Re}f(j\omega)+\operatorname{Im}a_0(j\omega)\operatorname{Im}f(j\omega)$.

3.1. Stability Radius

Let in the original affine family (15) which is certainly stable for $k=0$, the parameters arbitrarily remaining bounded in some weighted p-norm:

$$a(s,k)=a_0(s)+f(s)\sum_{i=1}^{v}k_ia_i(s^2),\quad \|k\|_p^\alpha\leqslant\gamma, \tag{22}$$

where $\alpha=\{\alpha_1,\ldots,\alpha_v\}$, $\alpha_i>0$ are some weights $\|k\|_p^\alpha=\left(\sum\limits_{i=1}^{v}|\alpha_ik_i|^p\right)^{\frac{1}{p}}$.

With this formulation, for $p=\infty$ the parameters can vary independently, each within its interval $\alpha_i|k_i|\leqslant\gamma$; for $p=1$, the parameters can assume values from the simplex $\sum\limits_{i=1}^{v}\alpha_i|k_i|\leqslant\gamma$, that is, be bounded in the so-called octahedral norm, and for $p=2$, the elliptic constrains like $\sum\limits_{i=1}^{v}\alpha_i^2k_i^2\leqslant\gamma^2$ are imposed on the parameters. the proposed description of the stability domain allows to handle also an arbitrary p-norm.

We set down the inequalities specifying the stability region as:

$$\varphi_0^i+\sum_{\ell=1}^{v}k_\ell\varphi_\ell^i>0,\quad i=0,\ldots,m, \tag{23}$$

where φ_ℓ^i are the values of the corresponding polynomials on the critical frequencies ω_i, namely: $\varphi_0^i=\operatorname{sgn}(p(\omega_i))p(\omega_i)=|p(\omega_i)|$, $\varphi_\ell^i=\operatorname{sgn}(p(\omega_i))|f(j\omega_i)|^2a_\ell(-\omega_i^2)$.

Now we are interested in the stability radius, that is, need to determine the maximum value γ^* such that family (22) is stable for all $\gamma<\gamma^*$. Stated differently, there exists a polyhedron defined by the system of linear inequalities (23), and we have to inscribe in it a sphere in the corresponding norm of the maximal radius.

The dual-norm lemma enables us to answer this question for each hyperplane:

Lemma 1. *Let in* $\mathbb{R}^v$ *the half-space* $\mathcal{K}=\{k:1+(k,\varphi)=0\}$ *be given. Then*

$$\gamma^*=\inf\{\|k\|_p^\alpha:k\in\mathcal{K}\}=\frac{1}{\|\varphi\|_q^{1/\alpha}}=\frac{1}{\left(\sum\limits_{i=1}^{v}\left|\frac{\varphi_i}{\alpha_i}\right|^q\right)^{1/q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

For the family under consideration, the boundary of the stability domain is defined by several hyperplanes. Therefore it suffices to take the minimum for all γ_i^* corresponding to each hyperplane in order to answer the question of the stability radius. The aforementioned in summed up by the following theorem.

Theorem 4 [8]. *The stability radius for the affine families* (22) *obeys* $\gamma^* = \min_i \frac{\varphi_0^i}{\|\varphi^i\|_q^{1/\alpha}}$, *where* $\varphi_0^i + (k, \varphi^i) > 0$, $i = 0, \ldots, m$ *is the system of inequalities describing the stability domain.*

This result is exceptionally simple in terms of computations. At first, by solving Eq. (19) we have to determine the critical frequencies and then set down the system of inequalities (23) defining the stability domain. Now, for each hyperplane we have to calculate the corresponding norm of the normal vector φ^i and the corresponding γ_i^*. The final answer is obtained by taking the minimum from the finite set of the elements γ_i^* whose number corresponds to the number of critical frequencies.

Note that there are other classes of multi-parametric polynomials that allows one to calculate stability radius easily. For instance, for interval polynomials, the answer is obtained using Tsypkin–Polyak locus [21, 22].

4. ROBUST *D*-DECOMPOSITION FOR POLYNOMIALS

Consider systems with uncertainty. Suppose that a characteristic polynomial depends on two types of parameters k and x. The number of parameters of the first type does not exceed two while for the others $x \in \mathbb{R}^\ell$ the domain of variation $\mathcal{X}$ is given. This problem statement is natural for the problem of controller synthesis under uncertainty, k being controller parameters.

A system is robustly stable (in x), if for a given k it is stable for all $x \in \mathcal{X}$ [24]. In this section we discuss the methods for describing the stability domain in the parameter space k.

Further we consider continuous-time systems and characteristic polynomials with real coefficients, for the discrete-time systems (see remark).

The D-decomposition idea is also efficient in this case [20, 15]:

Definition 3. For the polynomial $a(s, k, x)$ of the degree n with parameters k, robust D-decomposition is the decomposition of the parameter space k into regions D_l ($l = 0, \ldots, n$) in each of which the number of stable roots is the same for all admissible uncertainties:

$$D_l = \{k : a(s, k, x) \quad \text{has} \quad l \quad \text{stable roots}, \quad \forall x \in \mathcal{X}\}.$$

Thus D_n is the robust stability domain. First we define the set Υ that separates the regions D_l. In contrast to the classical D-decomposition it consists not of the curve and several lines but it is a set given by two conditions from *zero exclusion principe* (compare to (9)):

$$\Upsilon_s = \left\{k : a^{(n)}(k, x) = 0,\ \exists x \in \mathcal{X}\right\}; \tag{24}$$

$$\Upsilon_\Omega = \{a(j\omega, k, x) = 0,\ \exists x \in \mathcal{X},\ \exists \omega \in [0, \infty)\} = \bigcup_{\omega \in [0,\infty)} \Upsilon(\omega), \tag{25}$$

where $a^{(n)}$ is the leading coefficient, $\Upsilon(\omega) = \{k : a(j\,\omega, k,\ x) = 0, \exists\, x \in \mathcal{X}\}$. The set Υ_Ω is a sort of smear of the classical D-decomposition curve into the strip, while Υ_s is smear of the singular line defining by the order reduction condition. Within the stripes polynomial $a(s, k, x)$ may have various number of stable roots depending in the value of $x \in \mathcal{X}$.

We consider robust D-decomposition for two real or one complex parameters, i.e., $k \in \mathbb{R}^2$ or $k \in \mathbb{C}$. In this case $\Upsilon(\omega)$ is a parametric family of the figures in the plane, Υ_Ω is a swept area, and a boundary is the envelope of the family $\Upsilon(\omega)$.

4.1. Robust D-decomposition for Two Real Parameters

Consider the linear system with the characteristic polynomial linearly dependent on all the parameters with uncertainty x bounded in l_p-norm (or weighted l_p-norm):

$$a(s,k,x) = a_0(s) + k_1 a_1(s) + k_2 a_2(s) + \sum_{i=1}^{\ell} x_i b_i(s),\ \|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p} \leqslant 1. \tag{26}$$

The maximal degree of polynomials a_i, b_i is n. In this case the set Υ_s is bounded by two lines:

$$a_0^{(n)} + k_1 a_1^{(n)} + k_2 a_2^{(n)} \pm \|b^{(n)}\|_q = 0, \tag{27}$$

where index $^{(n)}$ denotes the s^n held coefficients of the corresponding polynomials, $b^{(n)} \in \mathbb{R}^\ell$, $q = p/(p-1)$ is the dual norm index.

Set $\Upsilon(\omega)$ for polynomial (26) in the nonsingular case is

$$\Upsilon(\omega) = -T^{-1}(\omega)\mathcal{B}_{\mathcal{X}}(\omega), \tag{28}$$

where

$$T(\omega) \doteq \begin{bmatrix} \operatorname{Re} a_1(j\omega) & \operatorname{Re} a_2(j\omega) \\ \operatorname{Im} a_1(j\omega) & \operatorname{Im} a_2(j\omega) \end{bmatrix},$$
$$\mathcal{B}_{\mathcal{X}}(\omega) \doteq \left\{ \begin{bmatrix} \operatorname{Re}\left(a_0(j\omega) + \sum\limits_{i=1}^{\ell} x_i b_i(j\omega)\right) \\ \operatorname{Im}\left(a_0(j\omega) + \sum\limits_{i=1}^{\ell} x_i b_i(j\omega)\right) \end{bmatrix}, \|x\|_p \leqslant 1 \right\}. \tag{29}$$

Set $\mathcal{B}_{\mathcal{X}}$ depending on the uncertainty may be [28, 20]:

- a 2ℓ-polygon if $x \in \mathbb{R}^\ell$, $p = \{1, \infty\}$;
- an ellipse if $x \in \mathbb{R}^\ell$, $p = 2$;
- a circle if $x \in \mathbb{C}^\ell$, $p \in [1, \infty)$.

In the nonsingular case $\Upsilon(\omega)$ is obtained by linear transformation $\mathcal{B}_{\mathcal{X}}(\omega)$ (28), thus it is either a polygon or an ellipse. For the polygon the problem of description the envelope is hard. For the ellipse $\Upsilon(\omega)$ is defined by

$$\Upsilon(\omega) = \left\{k : (T(\omega)k - b_0(\omega))^{\mathrm{T}} M(\omega)^{-1} (T(\omega)k - b_0(\omega)) \leqslant \rho(\omega)^2\right\},$$

where $k = (k_1, k_2)^{\mathrm{T}}$, $b_0 = -(\operatorname{Re} a_0, \operatorname{Im} a_0)^T$ and the particular from of M and ρ depend on the uncertainty.

If one of the matrices T or M is singular then $\Upsilon(\omega)$ may be an empty set, a line, s strip or the whole plane. If both matrices are nonsingular the envelope of the elliptic family $\Upsilon(\omega)$ is obtained analytically as the solution of the system of two second-order equations [12, 28].

Remark. In the case of discrete-time systems, the substitution $s = e^{j\omega}$, $\omega \in [0, 2\pi)$ is used and variation of the degree of the polynomial does not lead to the change of the number of stable roots.

4.2. Robust D-decomposition for One Complex Parameter

For the robust D-decomposition for one complex parameter k of the affine polynomial

$$a(s,k,x) = a_0(s) + k a_1(s) + \sum_{i=1}^{\ell} x_i b_i(s), \quad x \in \mathbb{C}^\ell, \quad \|x\|_p \leqslant 1, \tag{30}$$

boundaries $\partial\Upsilon_s$ and $\partial\Upsilon_\Omega$ of sets Υ_s and Υ_Ω can be obtained explicitly:

$$\partial\Upsilon_s = \left\{k : \left|a_0^{(n)} + k a_1^{(n)}\right| = \left\|b^{(n)}\right\|_q\right\},$$
$$\partial\Upsilon_\Omega \subset \Upsilon_\Omega^{reg} \cup \Upsilon_\Omega^{spec},$$

where

$$\Upsilon_\Omega^{reg} = \left\{ k_0(\omega) - \rho(\omega)\frac{k_0(\omega)'}{|k_0(\omega)'|}\left(\frac{\rho(\omega)'}{|k_0(\omega)'|} \pm j\sqrt{1-\left(\frac{\rho(\omega)'}{|k_0(\omega)'|}\right)^2}\right) : \begin{array}{l} |k_0(\omega)'| \geqslant |\rho(\omega)'| > 0, \\ a_1(j\omega) \neq 0, \ \omega \in \mathbb{R} \end{array} \right\},$$
$$\Upsilon_\Omega^{spec} = \left\{ k : \ |a_0(j\omega) + k a_1(j\omega)| = \|b(j\omega)\|_q, \ \begin{array}{l} \text{if } \ \tau_0(\omega)' = |\rho(\omega)'| = 0, \ \omega \in \mathbb{R} \\ \text{if these derivatives are not defined} \end{array} \right\},$$
$$k_0(\omega) \doteq -a_0(j\omega)/a_1(j\omega), \quad \rho(\omega) \doteq \|b(j\omega)\|_q/|a_1(j\omega)|, \quad q = p/(p-1).$$

Example 4. Let us consider the robust D-decomposition for one complex parameter for the discrete-time system with polynomial (robust version of Example 1):

$$\begin{gathered} z^6(1+0.05x_1) + kz^5 + 1.5 + 0.05(x_1+2x_2) = 0; \quad z = e^{j\omega}, \\ |x_i| \leqslant 1, \quad x_i \in \mathbb{C}, \quad i = 1,2, \quad k \in \mathbb{C}. \end{gathered} \tag{31}$$

The figure of D-decomposition is shown in Fig. 4; regions D_l are denoted by digits l. D-decomposition without uncertainty is plotted by dotted curve; see also Fig. 1a.

5. SYNTHESIS OF LOW-ORDER CONTROLLERS WITH H_∞ SPECIFICATIONS

The problem of designing the low-order controllers is to determine a stabilizing controller such that the closed-loop system is stable and some addition quality criterion holds. Here we consider H_∞ criteria. We recall that the H_∞-norm is finite only for rational functions with stable denominator ($H(s) \in \mathbf{RH}_\infty$) and equals to $\|H(s)\|_\infty = \sup\limits_{\omega} |H(j\omega)|$.

The analytical theory of the design of the H_∞-optimal controllers has been developed in detail in [5, 50, 86] but the resulting controllers may be of a very high order, sometimes even exceeding that of the original system [11]. Additionally, stability of the closed-loop system is very sensitive to the controller parameters, their small variation often results in instability [59]. Since the H_∞ theory does not allow one to constrain the order of the designed controller, its direct application to the design of controllers of a given structure faces substantial difficulties. The other approaches to the fixed order controller synthesis with H_∞ specifications are discussed in [52, 56, 60, 77, 78].

We present an alternative approach to the design of a given structure controllers, i.e., the order of the controller is fixed and it is the only choice of its parameters remains free. The results are published in [7, 83]. Further we assume that there are only two or three adjustable controller parameters that allows us to use widely the graphic methods. Thus the problem is to find the regions in the parameter space such that the corresponding controllers (i) stabilize the system and (ii) satisfy the H_∞-criterion. It is important that our approach describes the whole set of controllers with given specifications, if any. For some particular controllers the description of the whole domain of admissible parameters in the parameter space is proposed in [40, 54, 70].

Let us consider a linear stationary one-dimensional system, where the plane is given by the scalar transfer function $G(s)$. Let the system be closed by the controller $C(s,k)$; the block diagram of the system is shown in Fig. 5. Parameters $k \in \mathbb{R}^m$, $m \leqslant 3$ define the set of admissible low-order controllers. For instance, it may be the PI-controller $k_p + \frac{k_i}{s}$, the PID-controller $k_p + \frac{k_i}{s} + k_d s$, and

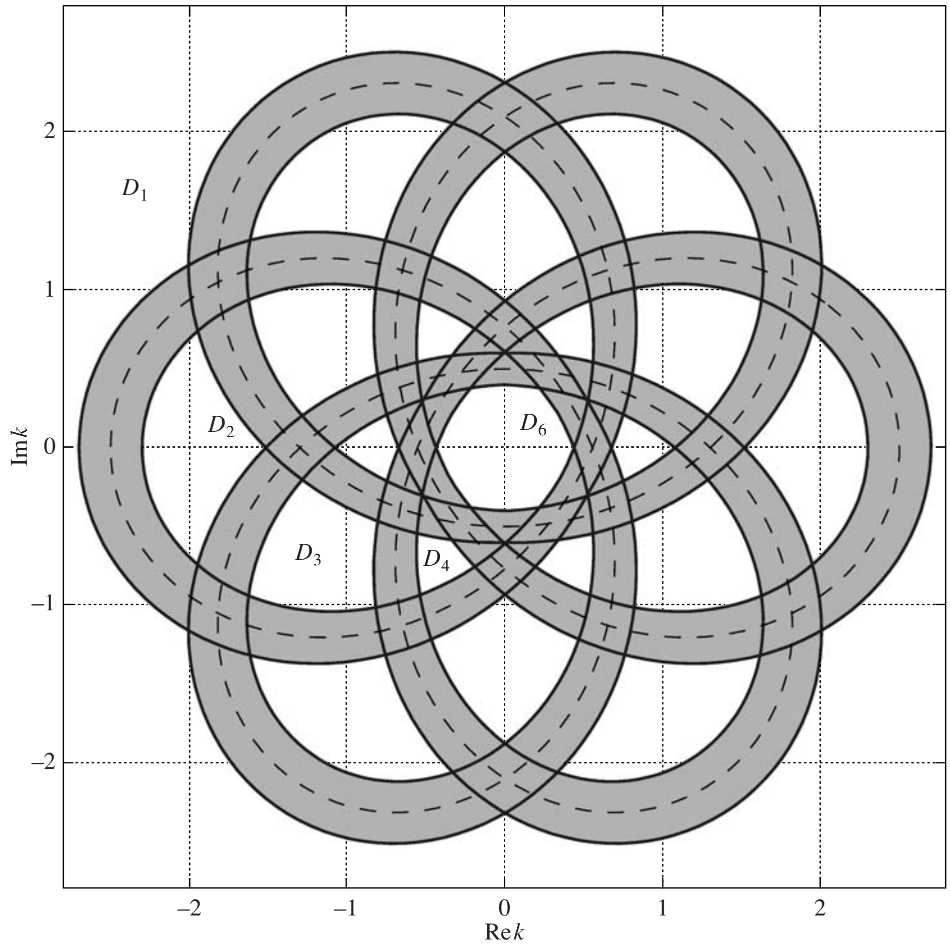

Fig. 4. Robust D-decomposition in Example 4.

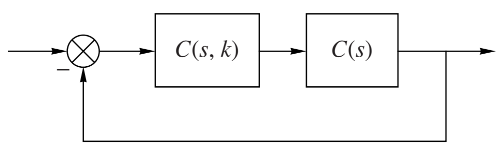

Fig. 5. Block diagram of the control system.

the first-order controller $\frac{k_1 s+k_2}{s+k_3}$. The problem is to describe the set of the parameters common to all controllers satisfying the performance criterion

$$\|H(s,k)\|_\infty < \gamma. \tag{32}$$

We call these controllers the H_∞-controllers. The particular form of the transfer function $H(s,k)$ may be different. H_∞-criterion arises in several formulations of the design problems of which we present the most important examples as well as the corresponding functions $H(s,k)$.

• Determination of the controller $C(s)$ guaranteeing the H_∞-performance of the closed-loop system,

$$\|W_1(s)T(s)\|_\infty \leqslant \gamma,$$

where $W_1(s)$ is a weight function ($W_1^{-1} \in \mathbf{RH}_\infty$), $T(s) = \frac{C(s)G(s)}{1+C(s)G(s)}$ is the transfer function of the closed-loop system (additional sensitivity), and γ is the given level of performance.

• Determination of a controller robustly stabilizing the plant family with additive uncertainty $G(s) = G_0(s) + \Delta G(s)$, where $G_0(s)$ is the transfer function of the nominal plant and the frequency (nonparametric [38]) uncertainty $\Delta G(s)$ is bounded in the weighted norm $\|W_2(s)\Delta G(s)\|_\infty \leqslant 1$. The problem comes to satisfying the criterion $\|W_2^{-1}(s)U(s)\|_\infty \leqslant 1$, where $U(s) = \frac{C(s)}{1+C(s)G(s)}$, $W_2 \in \mathbf{RH}_\infty$. The criterion $\|W_2^{-1}(s)T(s)\|_\infty \leqslant 1$ corresponds to the multiplicative uncertainty $G(s) = G_0(s)(1 + \Delta G(s))$; see [50, 86].

• In the quantitative feedback theory (QFT) [5], the closed-loop system must satisfy the following constraints:

$$\begin{gathered} m_1(\omega) < |W(j\omega)T(j\omega)| < m_2(\omega), \quad \omega \in [0, \omega_1], \\ |S(j\omega)| < l_1(\omega), \quad \omega \in [0, \omega_1], \\ |T(j\omega)| < l_2(\omega), \quad \omega \in [\omega_2, \infty), \end{gathered}$$

where $S(s) = \frac{1}{1+C(s)G(s)}$ is sensitivity and m_1, m_2, l_1 and l_2 are the limitative functions.

Assume that the controller stabilized the system, i.e., the denominator $H(s)$ is stable. Then $\|H(s,k)\|_\infty = \sup\limits_{\omega\in[0,\infty)} |H(j\omega,k)|$, where $H(s,k) = \frac{H_n(s,k)}{H_d(s,k)}$ is the common rational transfer function. Further we assume that the numerator H_n and denominator H_d depend linearly on the parameters k, which is the case for all considered types of the controllers (PI, PID, first-order controller). For instance, for the PID-controller and the sensitivity function $H_n(s,k) = sD(s), H_d(s,k) = sD(s) + (k_i + k_p s + k_d s^2)N(s)$, where $N(s)$ is the numerator of the transfer function of the plant $G(s)$ and $D(s)$ is its denominator.

5.1. D-decomposition for H_∞ Criterion

One of the approaches to define the admissible set of H_∞-controllers is based on the description of the boundary of the set $\mathcal{K} = \{k : |H_n(j\omega,k)| < \gamma|H_d(j\omega,k)|, \forall\omega \in [0,\infty)\}$ in the explicit form. The set of parameters k corresponding to H_∞-controllers is an intersection of $\mathcal{K}$ and the stability domain for the characteristic polynomial. Similarly to the classical D-decomposition approach, we consider the crossing the boundary of $\mathcal{K}$. There may be three essentially different cases: either the function $|H(j\omega,k)|$ reaches the value γ, its numerator and denominator vanish simultaneously, or the degree of a denominator will decrease. The points where only the denominator is zero do not belong to the boundary because their small neighborhood has not a single point where the inequality $|H_n(j\omega,k)| < \gamma|H_d(j\omega,k)|$ holds.

Whenever $H(j\omega,k)$ is defined we use sufficient extremum condition with the respect to ω to obtain an easy-to-use theorem.

Theorem 5 [7]. *The boundary of the set $\mathcal{K}$ is contained in the solutions of the systems*

$$\begin{cases} H_n(j\omega,k) = 0 \\ H_d(j\omega,k) = 0, \end{cases} \quad \omega \in [0,\infty), \tag{33}$$

$$\begin{cases} |H(j\omega,k)|^2 = \gamma^2 \\ \dfrac{\partial |H(j\omega,k)|^2}{\partial\omega} = 0, \end{cases} \quad \omega \in [0,\infty), \tag{34}$$

and equations

$$|H(j\infty,k)| = \gamma, \quad |H(0,k)| = \gamma. \tag{35}$$

$$h_d(k) = 0, \tag{36}$$

where h_d is coefficient at the higher order of a polynom H_d.

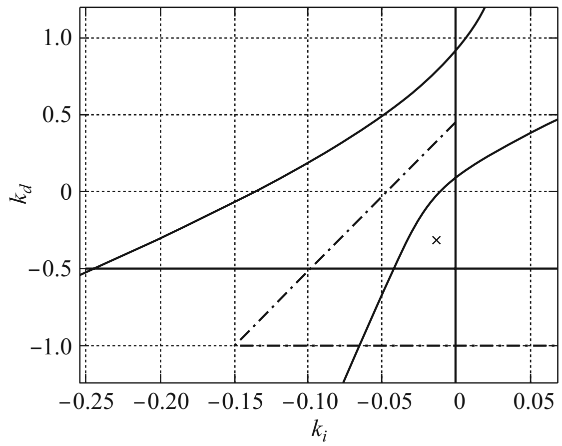

Fig. 6. H_∞ region for PID-controllers.

For the fixed k function $|H(j\omega)|$ may be at its maximum either at the solution of system (33) or system (34) or at the solution of Eqs. (35). Equation (36) describes an opportunity of loss of stability of the closed-loop system.

For the two variables $k = (k_1, k_2)$, it is the aggregate of points that are the solution of system (34). The first equation defines for each ω either a one-dimensional curve or an empty set. The second equation of the system enables one to specify such k on the curve the function $|H|^2$ reaches its extremum for the given ω. Since the condition for equality to zero of the derivative keeps one from discrimination whether it is the maximum or minimum that is reached at this point, points not belonging to the desired boundary may occur among the solution.

Example 5. Consider the application of the proposed method for designing a fixed-order PID-controller [46]. The plant is defined by the transfer function $G(s) = \frac{s-1}{s^2+0.8s-0.2}$, the controller, by $C(s, k_i, k_d) = \frac{k_i+k_ps+k_ds^2}{s}$, $k_p = -0.35$. It is required to determine all the controllers satisfying the H_∞-criterion of performance $\|W(s)T(s)\|_\infty \leqslant 1$ with high-frequency filter $W(s) = \frac{s+0.1}{s+1}$.

The boundary of the domain $\mathcal{K}$ in the parameter space is specified by Theorem 5. Equation (33) gives the vertical line $k_i = 0$ (a part of this line is also the stability domain boundary), Eq. (35) gives the horizontal line $k_d = -0.5$, and system (34) gives the curves depicted in Fig. 6. Checking each of the regions we find out the desirable one marked with "×." Indeed, the whole this region is inside the stability domain that is plotted with chain line.

6. D-DECOMPOSITION FOR MATRICES

Consider real matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, $C \in \mathbb{R}^{m\times n}$, and let $\mathcal{K}$ be a connected set of real or complex matrices $r \times m$. Assume that the matrix A does not have zero or imaginary eigenvalues for the continuous-time case, and does not have eigenvalues on the unit circumference in the discrete-time systems.

Define the matrix transfer function:

$$M(s) = C(A - sI)^{-1}B \tag{37}$$

for the continuous-time systems and

$$M(z) = C(A - zI)^{-1}B \tag{38}$$

for the discrete-time systems, where the variables s and z are using to distinguish between these two cases.

Definition 4. The decomposition of the parameter space into the regions $D_l = \{K \in \mathcal{K} : A + BKC$ has l stable eigenvalues$\}$, $l = 0, \dots, n$, is called D-decomposition. The equation describing the boundary of the regions D_l is called the D-decomposition equation. Thus D_n is the set of stabilizing matrices K.

The D-decomposition technique is based on the following theorem.

Theorem 6 [53]. *The equation*

$$\det(I + M(j\omega)K) = 0, \quad \omega \in (-\infty, +\infty) \tag{39}$$

or

$$\det(I + M(e^{j\omega})K) = 0, \quad \omega \in [0, 2\pi) \tag{40}$$

defines the D-decomposition for the class $\mathcal{K}$, i.e., if $Q \subset \mathcal{K}$ is a connected set and $\det(I + M(j\omega)K) \neq 0$, *$\omega \in (-\infty, +\infty)$, $\forall K \in Q$ (continuous-time case) or* $\det(I + M(e^{j\omega})K) \neq 0$, *$\omega \in [0, 2\pi)$, $\forall K \in Q$ (discrete-time case), then $A + BKC$ has the same number of stable eigenvalues for all matrices K from Q.*

Equations (39) and (40) define the D-decomposition in the implicit from. Here we consider some particular cases when we can write the equations of the boundary of the D-decomposition regions explicitly. Note that this is in contrast with μ-analysis, where the problem

$$\min_{K \in \mathcal{K},\ \det(I+M(j\omega)K)=0} \|K\|$$

is under consideration (i.e., one is seeking for the largest ball contained in D). Better approximation of D was proposed in [34]; it was the ellipsoid of the largest volume, inscribed in D.

Note that the system of Eqs. (39) and (40) defines the D-decomposition in general form and contains the polynomial case that is discussed in details in Section 2. Indeed, consider single-output system; i.e., $r = 1$. Then K is a row vector: $K = [k_1, \dots, k_m]$ while M is a column vector $M = [M_1, \dots, M_m]^T \in \mathbb{C}^m$. According to the formula $\det(I + ab^T) = 1 + \sum\limits_{i=1}^{m} a_i b_i$ with $a, b \in \mathbb{C}^m$ the general D-decomposition Eq. (39) is reduced to:

$$1 + \sum_{i=1}^{m} k_i M_i(j\omega) = 0 \quad \text{or} \quad a_0(j\omega) + \sum_{i=1}^{m} k_i a_i(j\omega) = 0,$$

where $a_0(s) + \sum k_i a_i(s)$ is the characteristic polynomial of the degree n, $M_i(s) = \dfrac{a_i(s)}{a_0(s)}$. Thus the Eq. (39) is linear in K. Similarly, for the single-input systems $m = 1$, K is a column vector $K = [k_1, \dots, k_r]^T$, and with have the same D-decomposition Eq. (8).

However in general case Eqs. (39) and (40) are not linear in K, and the analysis requires the extension of the D-decomposition technique.

6.1. Graphical Algorithms

Here we consider classes of $\mathcal{K}$ such that the transfer function is not the ration of the polynomials and classical D-decomposition is not applicable. Though the modified D-decomposition technique allows us to describe the regions with constant number of stable eigenvalues.

We start with the class (4). In the terms of μ-analysis this is class $\mathcal{K}$ with one scalar block. The matrix K has the form $K = kI$, where $k \in \mathbb{C}$ or $k \in \mathbb{R}$, I is a unit matrix $m \times m$. Then the matrix $A + BKC$ is equal to $A + kBC$ and the problem is reduced to the simplest one: given $n \times n$ real matrices A and F, find $D(l) = \{k \in \mathbb{C}$ (or $k \in \mathbb{R}) : A + kF$ has l stable eigenvalues$\}$. The general D-decomposition Eq. (39) now reads

$$\det(I + kM(j\omega)) = 0, \quad \omega \in (-\infty, +\infty). \tag{41}$$

Denote the eigenvalues of $M(j\omega)$ as $\lambda_i(\omega)$, $i = 1, \ldots, n$, Eqs. (41) split into $1 + k\lambda_i(\omega) = 0$, $i = 1, \ldots, n$, and D-decomposition boundary consists of n branches $k_{(i)}(\omega)$:

$$k_{(i)}(\omega) = -\frac{1}{\lambda_i(\omega)}, \quad i = 1, \ldots, n. \tag{42}$$

This equations can be obtained in a different form with no use of the transfer function (37). If $A + kF$ $(F = BC)$ has an imaginary eigenvalue then the matrix $A + kF - j\omega I$ is singular for some $\omega \in \mathbb{R}$, that is $(A + kF - j\omega I)x = 0$ for $x \in \mathbb{C}^n$ or $(A - j\omega I)x = -kFx$. Thus we conclude that parametric D-decomposition boundary $k(\omega)$ is a generalized eigenvalue for the matrix pair $A - j\omega I$ and $-F$:

$$k(\omega) = \mathrm{eig}\,(A - j\omega I, -F). \tag{43}$$

For the case of real k we should take into consideration only the intersection of the obtained D-decomposition picture and the real axis.

Theorem 7 [53]. *For the real k the number of intervals preserving the same number of stable eigenvalues of $A + kBC$ does not exceed $n(n + 1) + 1$.*

Now let us consider systems with two inputs and two outputs corresponding to class (5). The transfer function now is the 2×2 matrix $M = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}$, the matrix K contains in general 4 parameters $K = \begin{bmatrix} k_1 & k_3 \\ k_2 & k_4 \end{bmatrix}$, and the general D-decomposition Eq. (39) reduces to:

$$0 = \det(I + MK) = 1 + \sum_{i=1}^{4} k_i m_i + \det M \det K. \tag{44}$$

To take the opportunity of the graphical representation we restrict ourselves by situations with K depending on two parameters only. Consider the cases when it is possible.

(1) $K = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$.

In terms of μ-analysis this structure of K corresponds to two scalar blocks. Equation (44) has the form $0 = 1 + k_1m_1 + k_2m_4 + k_1k_2(m_1m_4 - m_2m_3)$. Denote $m_i = u_i + jv_i$, $i = 1, \ldots, 4$ and write separately real and imaginary parts:

$$\begin{cases} 1 + k_1u_1 + k_2u_4 + \alpha k_1k_2 = 0 \\ k_1v_1 + k_2v_4 + \beta k_1k_2 = 0, \end{cases} \tag{45}$$

where $\alpha = u_1u_4 - v_1v_4 - u_2u_3 + v_2v_3$, $\beta = u_1v_4 + v_1u_4 - u_2v_3 - v_2u_3$. Thus we get two quadratic equation in two variables, the variables u_i, v_i, α, β depend on ω. For $\omega = 0$ matrix $M(j\omega) =$

$C(A - j\omega I)^{-1}B$ is real and $v_i(0) = 0$, $i = 1, \dots, 4$ as well as $\beta(0) = 0$. Hence the second equation vanishes and the first equation: $1 + k_1u_1(0) + k_2u_4(0) + \alpha(0)k_1k_2 = 0$. It is a hyperbola.

For every $\omega \neq 0$, solution of system (45) specifies the point of the D-decomposition boundary. If for some ω^* the solution is complex we ignore it because it does not belong to the D-decomposition. It means that there is no parameter value K^* such that $A + BKC$ has eigenvalues $\pm j\omega^*$.

(2) $K = \begin{bmatrix} k_1 & k_2 \\ -k_2 & k_1 \end{bmatrix}$.

In terms of μ-analysis this is a real 2×2 analog of a complex scalar (note that the eigenvalues of such K are $k_1 \pm jk_2$). For such K Eq. (44) reads $0 = 1 + k_1(m_1 + m_4) - k_2(m_2 - m_3) + (k_1^2 + k_2^2) \times (m_1m_4 - m_2m_3)$ and (45) is replaced with

$$\begin{cases} 1 + k_1(u_1 + u_4) - k_2(u_2 - u_3) + \alpha(k_1^2 + k_2^2) = 0 \\ \quad k_1(v_1 + v_4) - k_2(v_2 - v_3) + \beta(k_1^2 + k_2^2) = 0, \end{cases} \tag{46}$$

where $u_i(\omega)$, $v_i(\omega)$, $\alpha(\omega)$, $\beta(\omega)$ are the same as above. For $\omega = 0$, we get $v_i(0) = 0$, $\beta(0) = 0$, and the second equation vanishes while the first equation defined the second order curve $1 + k_1(u_1(0) + u_4(0)) - k_2(u_2(0) - u_3(0)) + \alpha(0)(k_1^2 + k_2^2) = 0$. For this case it is a circumference. Solving Eq. (46) for $\omega \neq 0$, we obtain $k_1(\omega)$, $k_2(\omega)$ corresponding to the boundary of the D-decomposition. Thus the D-decomposition consists of the components of this curve and singular circumference.

(3) $K = \begin{bmatrix} -k_1 & k_2 \\ k_2 & k_1 \end{bmatrix}$.

Similarly to the previous case, Eq. (44) is reduces to:

$$\begin{cases} 1 - k_1(u_1 - u_4) + k_2(u_2 + u_3) - \alpha(k_1^2 + k_2^2) = 0 \\ \quad -k_1(v_1 - v_4) + k_2(v_2 + v_3) - \beta(k_1^2 + k_2^2) = 0. \end{cases} \tag{47}$$

The D-decomposition consists of the parametric curve $k_1(\omega)$, $k_2(\omega)$, $\omega \neq 0$ and the second order curve corresponding to $\omega = 0$. Again it the circumference $1 + k_1(u_1(0) - u_4(0)) + k_2(u_2(0) + u_3(0)) - \alpha(0)(k_1^2 + k_2^2) = 0$.

6.2. Examples

Example 6. This example demonstrates the attainability of the estimation of Theorem 7 for $n = 3$. Let

$$A = \begin{bmatrix} 095 & 1 & 0 \\ 0 & 0 & 0.6 \\ 0 & 0 & -0.95 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & -0.22 \\ 0 & -03 & 0 \\ 04 & 0 & 0 \end{bmatrix}, \quad C = I.$$

In Fig. 7a the D-decomposition is depicted. It consists of three branches (parametric curves corresponding to the different generalized eigenvalues). In the intersection with real axis k there are 13 intervals with constant number of stable eigenvalues, among them there 5 intervals corresponding to stable matrices $A + kB$.

Example 7. Consider the discrete-time system $A + B \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} C$, where

$$A = \begin{bmatrix} -0.8848 & 0.4457 \\ -0.8733 & -0.9326 \end{bmatrix}, \quad B = \begin{bmatrix} 0.3914 & 0.2508 \\ -0.5576 & 0.0266 \end{bmatrix}, \quad C = \begin{bmatrix} 0.1514 & 0.7854 \\ -0.4255 & -0.8148 \end{bmatrix},$$

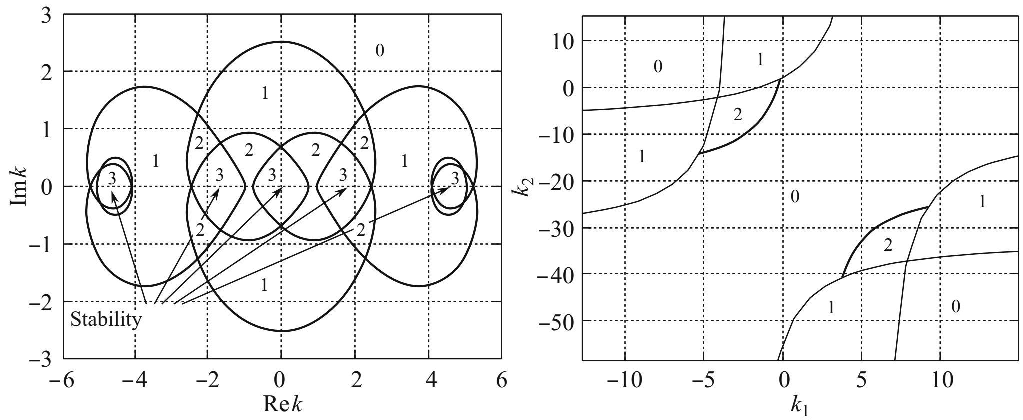

Fig. 7. D-decomposition in Examples 6 (left) and 7 (right).

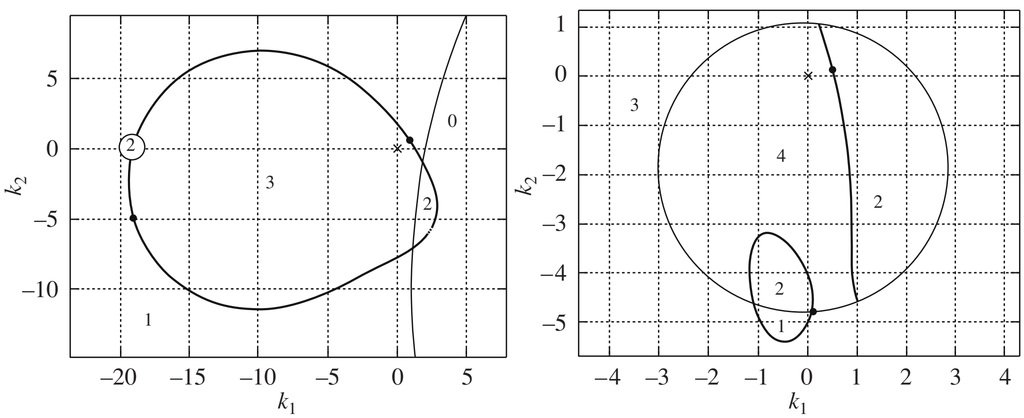

Fig. 8. D-decomposition in Examples 8 (left) and 9 (right).

has a typical D-decomposition structure. In Fig. 7b one can see two singular hyperbolas (subtle lines) and two branches of the nonsingular curve (solid lines).

Example 8. This discrete-time example is borrowed from [75, c. 889, Example 3]; $n = 3$, $m = r = 2$. The optimal solution for (7) is supplied with $K^* = \begin{bmatrix} 0.8483 & 0.5971 \\ -0.5971 & 0.8483 \end{bmatrix}$; it has the form $\begin{bmatrix} k_1 & k_2 \\ -k_2 & k_1 \end{bmatrix}$. Thus we restrict ourselves with matrices K of this form and construct D-decomposition for such matrices (see Fig. 8a). It is generated by two singular circumferences (subtle lines) and one parametric curve (solid line). Note that the nominal system (the origin marked with "×") is close enough to the boundary of the stability domain and the distance to it, in accordance with [75], equals 1.0374. However, other directions allow larger values of perturbations preserving stability. For instance, if $K = \lambda \begin{bmatrix} -0.9680 & -0.2508 \\ 0.2508 & -0.9680 \end{bmatrix}$, then $A + BKC$ remains stable for $0 \leq \lambda \leq 19.6932$.

Example 9. This continuous-time example is again originated in [75, c. 889, Example 2], and the same problem of the stability radius is considered, $n = 4$, $m = r = 2$. The optimal solution found in [75] is the matrix of the form $K = \begin{bmatrix} -k_1 & k_2 \\ k_2 & k_1 \end{bmatrix}$. The D-decomposition of (k_1, k_2) plane for the given class of matrices is shown in Fig. 8b. There are two disconnected components of the parametric curve (solid lines) and one singular curve—the circumference (subtle line). Similar to the previous example, for many directions preserve stability for perturbations larger than 0.5141 (stability radius).

7. D-DECOMPOSITION FOR LINEAR MATRIX INEQUALITIES

This section is devoted to the following problem: In the space of parameters $x \in \mathbb{R}^\ell$, find the regions such that

$$A(x) = A_0 + \sum_{i=1}^{\ell} x_i A_i, \quad A_i \in \mathbb{S}^{n\times n}, \tag{48}$$

has a fixed number of negative eigenvalues. Here $\mathbb{S}^{n\times n}$ denotes the space of real, symmetric $n \times n$ matrices. In this section we follow standard for linear matrix inequalities notations for parameters x, rather than k as it is in previous sections. The difference from the problem of the previous section is the requirement for all matrices in (48) to be symmetric. It makes the problem significantly easier.

A special case of this problem is to find a point x such that linear combination of symmetric matrices is negative definite. This is the well-known feasibility problem for linear matrix inequalities (the common abbreviation LMI [2], [41]) $A(x) < 0$. By med-nineties, the theory and methods for linear matrix inequalities had shaped into a self-contained discipline. In particular, optimization problems under constrains in LMI form were called semidefinite programming. However the LMI theory is not aimed at describing the whole feasible set, not to mention other signature invariant regions; various problems involving model uncertainty often defy efficient solution.

The primary goal here is not only to describe the whole feasible set $\{x \in \mathbb{R}^\ell \colon A(x) < 0\}$ but also to characterize all the regions in the parameter space such that the signature of the matrix $A(x)$ remains unaltered. It is accomplished using the D-decomposition technique modified for the case of symmetric matrices. But as it tool place in previous sections, for a small number of parameters, $\ell = 2$, the regions are easy to depict graphically on the plane. A complete description of the feasible set gives a clue to new statements and solutions for the semidefinite programming problems, and the determination of the other eigenvalue invariant regions is highly important for checking the applicability conditions of the generalized S-procedure; see, [36]. Moreover, this potentially leads to relatively simple solution methods for systems of matrix inequalities with uncertainty in matrix coefficients.

7.1. One Parameter Problems

We start with the case of one scalar parameter:

$$A(x) = A + xB, \quad A, B \in \mathbb{S}^{n\times n}, \quad x \in \mathbb{R}.$$

The goal is to define the intervals on the parameter axis x such that the signature of the matrix $A(x)$ is constant:

$$D_l = \Big\{ x \in \mathbb{R} \colon \;\; A(x) \text{ has } l \text{ negative eigenvalues and } n-l \text{ positive eigenvalues } \Big\}.$$

All basic ideas of the proposed approach become apparent in this simplest scalar case.

Assume that for a certain real x^*, symmetric matrix $A(x^*) = A + x^*B$ is nonsingular and has l negative and $n - l$ positive eigenvalues. As x varies, the number of like-sign eigenvalues can only alter if one of them (or several at once) crosses the origin, i.e., $\det(A + xB) = 0$ for some $x \in \mathbb{R}$. This means that there exists a nonzero vector e such that $(A + xB)e = 0$, i.e., $Ae = -xBe$. In other words, x is a generalized eigenvalue of the pair of matrices A and $-B$, and e is the associated generalized eigenvector. Therefore, the determination of the boundaries (points) of the D-decomposition regions in the scalar case reduces to finding all real generalized eigenvalues: $\mathrm{eig}\,(A, -B) \in \mathbb{R}$.

It is seen that the maximal possible number of regions (intervals on the x-axis) is equal to $n+1$, since the equation $\det(A+xB) = 0$ has at most n real solutions. On the other hand, there might be no real solutions at all, in this case the D-decomposition is represented by a unique domain—the number of negative eigenvalues of the matrix $A + xB$ remains unaltered for all values of x.

Of a particular interest is the stability interval D_n. The following theorem holds.

Theorem 8 [25]. *Let $A, B \in \mathbb{S}^{n\times n}$ and B be nonsingular. Let λ_i, e_i, $i = 1, \dots, n$ be the generalized eigenvalues and the associated generalized eigenvectors of the pair $(A, -B)$. Then*

(1) *If there exist complex-valued eigenvalues λ_i then D_n is empty;*

(2) *If all λ_i are real, then denote*

$$\underline{x} = \begin{cases} \max\limits_{i\in I_-} \lambda_i, & I_- \doteq \{i\colon (Be_i, e_i) < 0\} \neq \emptyset \\ -\infty, & I_- = \emptyset; \end{cases}$$

$$\overline{x} = \begin{cases} \min\limits_{i\in I_+} \lambda_i, & I_+ \doteq \{i\colon (Be_i, e_i) > 0\} \neq \emptyset \\ +\infty, & I_+ = \emptyset. \end{cases}$$

Then

$$D_n = \begin{cases} (\underline{x}, \quad \overline{x}), & \text{if } \ \underline{x} < \overline{x} \\ \emptyset, & \text{if } \ \underline{x} \geqslant \overline{x}. \end{cases}$$

If it is known that $A < 0$, the formulas above take the form:

$$\underline{x} = \begin{cases} \max\limits_{\lambda_i<0} \lambda_i \\ -\infty, \quad \text{provided all} \quad \lambda_i > 0 \end{cases} \quad \text{and} \quad \overline{x} = \begin{cases} \min\limits_{\lambda_i>0} \lambda_i \\ +\infty, \quad \text{provided all} \quad \lambda_i < 0. \end{cases} \tag{49}$$

7.2. Problems with Two and More Parameters

We now turn to the two-parameter case: $A(x) = A_0 + x_1A_1 + x_2A_2$, $A_i \in \mathbb{S}^{n\times n}$, $i = 0, 1, 2$. Let one of the matrices A_1, A_2, say, A_2 be non singular. We fix x_1 and denote $A = \overline{A}(x_1) \doteq A_0 + x_1A_1$ and $B \doteq A_2$. Then we are in the conditions of the previous section, and the critical values of the parameter x_2 for the given x_1 are determined as the real generalized eigenvalues $x_2(x_1) = \mathrm{eig}\left(\overline{A}(x_1), -B\right)$. By varying x_1 we obtain the boundaries of the D-decomposition regions. For every x_1 equation $\det\left(\overline{A}(x_1) + x_2B\right) = 0$ has no more than n real roots, hence, the boundary of D-decomposition consists of no more than n branches.

In the general situation, the stability region

$$D_n = \{x \in \mathbb{R}^2\colon A(x) < 0\}$$

can be described separately (it is obviously convex). Namely, for every x_1, determine the interval $(\underline{x}_2(x_1), \overline{x}_2(x_1))$ according to Theorem 8 (it may happen to be empty for some or all values of x_1);

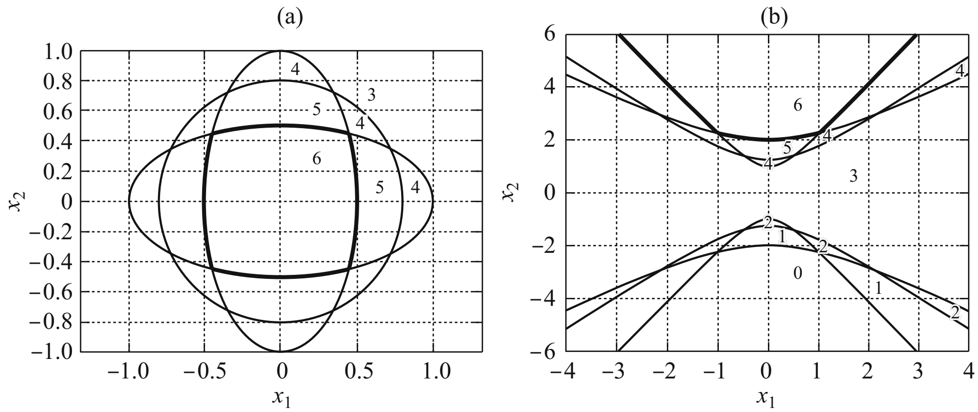

Fig. 9. D-decomposition in Example 10.

as x_1 varies, the endpoints of the interval sweep the boundary of the stability region. Note that D_n is knowingly unbounded if one of the matrices A_i, $i = 1, \dots, \ell$, in LMI (48) is sign-definite.

The marking can be performed in a standard way, i.e., upon constructing D_l regions on the plane, pick a point inside each of them and compute the eigenvalues of the corresponding matrix.

Example 10. In some special cases the description of the regions can be obtained explicitly. As an example of this sort, consider the following $n \times n, n = 2m$ matrices:

$$\begin{aligned} &A_0 = A = -I; \quad A_1 = B = \operatorname{diag}(b_1, \dots, b_m, -b_m, \dots, -b_1); \\ &A_2 = C = \overline{\operatorname{diag}}(c_1, \dots, c_m, c_m, \dots, c_1), \end{aligned} \tag{50}$$

where $\overline{\operatorname{diag}}$ denotes an antidiagonal matrix. The determinant is straightforward to compute:

$$\det(A + x_1 B + x_2 C) = (-1)^m (x_1^2 b_1^2 + x_2^2 c_1^2 - 1) \dots (x_1^2 b_m^2 + x_2^2 c_m^2 - 1),$$

so the boundaries of the D-decomposition are represented by m ellipses $x_1^2 b_i^2 + x_2 c_i^2 = 1$ (they divide the plane into $n(n-2)/2 + 2$ regions). The example of the D-decomposition for 6×6 matrices of this sort with parameters $b_1 = 1$, $b_2 = 1.25$, $b_3 = 2$; $c_1 = 2$, $c_2 = 1.25$, $c_3 = 1$ is depicted in Fig. 9a.

If we replace A and C in (50)

$$\begin{aligned} A &= \overline{diag}(c_1, \dots, c_m, c_m, \dots, c_1); \\ B &= diag(b_1, \dots, b_m, -b_m, \dots, -b_1); \quad C = -I, \end{aligned} \tag{51}$$

then the determinant is

$$\det(A + x_1 B + x_2 C) = (-1)^m (x_1^2 b_1^2 - x_2^2 + c_1^2) \dots (x_1^2 b_m^2 - x_2^2 + c_m^2).$$

The boundary of the D-decomposition is given by the family of m pairs of hyperbolas $x_1^2 b_i^2 - x_2^2 + c_i^2 = 0$, however the total number of the regions does not exceed $n^2/2 + 1$. Taking for b_i, c_i the values form the previous example we depict the regions in Fig. 9b.

7.3. Linear Matrix Inequalities with Uncertainty

One of the valuable extensions of the proposed in the previous section approach is its modification to the presence of uncertainty. There are various formulation of LMI problems and assumptions on uncertainties available in the literature; e.g., [41, 37]. Here we concentrate on the situation where uncertainty is bounded in the spectral norm.

Let

$$A_i(\Delta_i) = A_i + \Delta_i, \quad A_i, \Delta_i \in \mathbb{S}^{n\times n}, \quad \|\Delta_i\| \leqslant \varepsilon_i, \quad i = 0, \ldots, \ell, \tag{52}$$

where $\|\cdot\|$ is the *spectral* norm and $\varepsilon_i \geqslant 0$ are given numbers. Note that in order to retain the LMI structure of the problem, only symmetric perturbation Δ_i are considered. We thus arrive at the following uncertain linear function:

$$A(x,\Delta) = A_0(\Delta_0) + \sum_{i=1}^{\ell} x_i A_i(\Delta_i), \;\; \Delta \in \mathcal{D} \doteq \left\{(\Delta_0, \ldots, \Delta_\ell) : \Delta_i \in \mathbb{S}^{n\times n}, \|\Delta_i\| \leqslant \varepsilon_i\right\}. \tag{53}$$

The regions D_l of robust D-decomposition are now defined as follows:

$$D_l^{rob} = \Big\{x \in \mathbb{R}^\ell\colon\; A(x,\Delta) \;\text{ has exactly } l \text{ negative eigenvalues } \;\forall\, \Delta \in \mathcal{D}\;\Big\}; \tag{54}$$

in particular, the robustly feasible region is defined by:

$$D_n^{rob} = \Big\{x \in \mathbb{R}^\ell\colon A(x,\Delta) < 0 \quad \forall\, \Delta \in \mathcal{D}\Big\}.$$

The problem is to describe the boundaries of the regions of robust D-decomposition; these boundaries are now defined as the values of the parameters x such that the matrix $A(x,\Delta)$ becomes singular for some $\Delta \in \mathcal{D}$.

Similarly to the problem of robust D-decomposition for polynomials, the boundaries of robust D-decomposition smear into "strips" inside which the matrix $A(x,\Delta)$ may have different values of signature depending on one or another admissible value of Δ.

Similarly to the analysis of the problem without uncertainty, we first consider one-parameter families in the simplest case where only the A matrix is subjected to additive uncertainty:

$$A(x,\Delta) = (A+\Delta) + xB; \quad \Delta \in \mathcal{D} \doteq \{\Delta \in \mathbb{S}^{n\times n}\colon \|\Delta\| \leqslant \varepsilon\}. \tag{55}$$

The boundaries of the regions (intervals) of the robust D-decomposition are defined from the condition that the matrix $A + xB + \Delta$ be singular for some $\Delta \in \mathcal{D}$, i.e., the problem is to determine the radius of nonsingularity of the matrix $A + xB$. The theorem below is the main tool:

Theorem 9 [25]. *For a nonsingular matrix $M \in \mathbb{S}^{n\times n}$, its symmetric radius of nonsingularity*

$$\rho(M) \doteq \inf\{\|P\|\colon P \in \mathbb{S}^{n\times n}, \quad M + P \quad \textit{is singular}\},$$

is equal to

$$\rho(M) = 1/\|M^{-1}\| = \min_i |\lambda_i(M)|.$$

The critical value of P is given by $P = -\lambda e e^{\mathrm{T}}$, where λ is the minimal (in absolute value) eigenvalue of M, and e is the associated eigenvector.

Theorem 9 is a counterpart of Theorem 3 in [73] for the symmetric case; for non-symmetric matrices, the radius of nonsingularity is given by the general formula in Theorem 3 [73].

By the theorem, the matrix $(A + xB) + \Delta$ with perturbations $\|\Delta\| \leqslant \varepsilon$ remains robustly nonsingular for the values of x satisfying

$$\|(A + xB)^{-1}\| < \frac{1}{\varepsilon}, \tag{56}$$

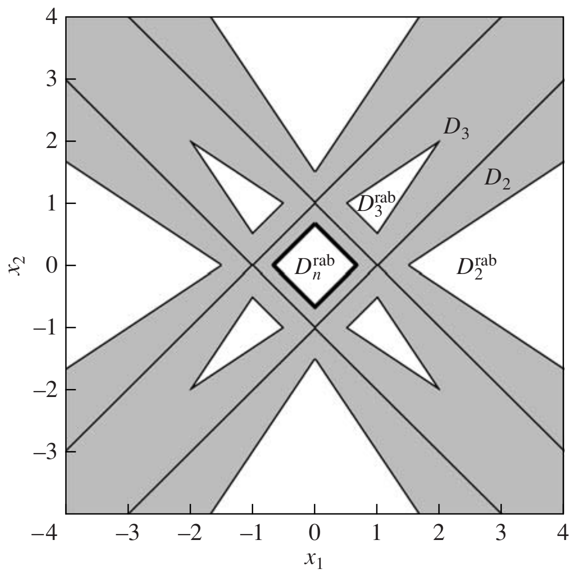

Fig. 10. Robust D-decomposition for two parameters.

hence, introducing the function $\varphi(x) \doteq \|(A + xB)^{-1}\|$, the intervals of robust nonsingularity are found numerically as $\{x\colon \varphi(x) < 1/\varepsilon\}$.

A more general problem where the uncertainty enters both matrices,

$$A(x, \Delta) = (A + \Delta_A) + x(B + \Delta_B); \quad \|\Delta_A\| \leqslant \varepsilon_A; \quad \|\Delta_B\| \leqslant \varepsilon_B, \tag{57}$$

is analyzed similarly. Represent $A(x, \Delta) = (A + xB) + (\Delta_A + x\Delta_B)$. For the perturbation $\Delta_A + x\Delta_B$ of the matrix $A + xB$ we have the estimate $\|\Delta_A + x\Delta_B\| \leqslant \varepsilon_A + |x|\varepsilon_B$, which is sharp (the equality is attainable), since Δ_A and Δ_B are chosen independently. Introduce the function $\varphi(x) = \|(A + xB)^{-1}\|$; then by Theorem 9, the intervals of robust nonsingularity of equivalently, the regions D_l^{rob} are defined by the condition

$$\varphi(x) < \frac{1}{\varepsilon_A + |x|\varepsilon_B}.$$

Slightly changing the notation, we now address the case of two-parameter families:

$$A(x, \Delta) = (A_0 + \Delta_0) + x_1(A_1 + \Delta_1) + x_2(A_2 + \Delta_2); \ \|\Delta_i\| \leqslant \varepsilon_i, \ i = 0, 1, 2.$$

We fix x_1 and denote $A = A_0 + x_1 A_1$ and $B = A_2$ and also $\varepsilon_A \doteq \varepsilon_0 + |x_1|\varepsilon_1$ and $\varepsilon_B = \varepsilon_2$. This leads us to the setup of problem (57). Indeed, to determine the boundaries, we compose the functions $\varphi(x_2) \doteq \|(A + x_2 B)^{-1}\|$ and $\varepsilon(x_2) \doteq 1/(\varepsilon_A + |x_2|\varepsilon_B)$ and check the condition $\varphi(x_2) < \varepsilon(x_2)$, which defined the segment of robustness in x_2 for a given value of x_1. As x_1 varies, these intervals fill up two-dimensional regions of robust D-decomposition. On the other hand, the values of x_2 that violate the condition $\varphi(x_2) < \varepsilon(x_2)$ correspond to the absence of robustness; as x_1 varies, these intervals of violation sweep certain two-dimensional regions which are considered the boundaries of robust D-decomposition.

We also note that similarly to the uncertainty-free case, the region D_n^{rob} can be described separately from the rest D_l^{rob} by combining the results of Theorems 8 and 9.

Example 11. For the two-parameter family in $\mathbb{S}^{4\times 4}$ with matrices $A_0 = -I$, $A_1 = \mathrm{diag}(-1, 1, 1, -1)$ and $A_2 = \mathrm{diag}(1, -1, 1, -1)$, the boundaries of D-decomposition in the absence of uncertainty are given by the two pairs of parallel lines in Fig 10.

Introducing uncertainty smears the boundaries into stripes and the D_l regions shrink. In the example considered, for equal relative levels of uncertainty $\|\Delta_i\| \leqslant \varepsilon_i = 0.2\|A_i\|$, $i = 0, 1, 2$, the boundaries of robust D-decomposition are given by the shaded region and the uncolored regions represent the regions of robustness.

Within this framework we consider more general classes of uncertainties. The analysis is based on the following result of Peterson.

Theorem 10 [72]. *Let $G = G^T$, $M \neq 0$, $N \neq 0$ be matrices of the size $n \times n$, $n \times r$, $m \times n$ correspondingly. Then inequality*

$$G + M\Delta N + N^T \Delta^T M^T < 0$$

hold for all matrices Δ of size $r \times m$, $\|\Delta\| < 1$ iff there exist $\varepsilon > 0$ such that

$$G + \varepsilon M M^T + \frac{1}{\varepsilon} N^T N < 0.$$

Various generalizations and applications of this result are addressed in [31]. For our case, if we change the family (55) by

$$A(x, \Delta) = A + \Delta_A + xB, \quad \Delta_A = M\Delta N + N^T \Delta^T M^T, \quad \|\Delta\| \leqslant 1,$$

then $A(x, \Delta) < 0$ for all $\|\Delta\| \leqslant 1$ if $A + xB < -\varepsilon M M^T - \dfrac{1}{\varepsilon} N^T N$ for some $\varepsilon > 0$. Then we can find the intervals in x that preserve robust sign-definiteness of $A(x, \Delta)$.

8. RANDOMIZED ALGORITHMS FOR SYNTHESIS USING D-DECOMPOSITION

As it was shown in the previous sections the D-decomposition idea in its classical form is applied only in one or two-dimensional space. There are only a few cases when the stability domain ban be explicitly described for higher dimensional parameter space. One of these cases (a spacial polynomial family (15)) is described in Section 3. However the general D-decomposition Eq. (9) or (10) for polynomials and (39) or (40) for matrices is valid also for the multidimensional case. The problem is to propose the detailed enough description of the multidimensional stability domain (or its boundary) in general situation.

One of the opportunities to achieve this goal is to generate point randomly (uniform generating is preferable) inside the stability domain or on its boundary. It can be done using modern schemes of Monte Carlo method. One of the methods (known as Hit-and-Run) [29, 79] generates approximately uniformly distributed points inside the given domain, generally speaking nonconvex and not simply connected. The second method (so-called Shake-and-Bake) [4, 44] allows one to generate point on the boundary of the domain that is again non convex but connected. First we briefly describe these method in general, then demonstrate how they work for the description of the stability domain (both for polynomials and matrices) and a domain specified by linear matrix inequalities.

For all the methods of this section the main tool is *boundary oracle* procedure. Similar to membership oracle and separation oracle that are exploited in modern convex optimization theory [36], for any $x, y \in \mathbb{R}^n$ boundary oracle either provides intersection points of a ray $x + \lambda y$ and the boundary of the set $\mathcal{K}$ defined implicitly or reports that the intersection is empty. Let $\mathcal{K} \subset \mathbb{R}^n$ be an open bounded, generally speaking nonconvex and not simply connected (but consisting of a finite number of connected components) set such that any intersection of the set $\mathcal{K}$ and a line consists of a finite number of intervals. For a certain point $k \in \mathcal{K}$ and a direction $d \in \mathbb{R}^n$, the boundary oracle provides the intersection of the line $k + \lambda d$, $-\infty < \lambda < +\infty$ and the set $\mathcal{K}$, i.e., the set

$$S = \{\lambda \in \mathbb{R} : k + \lambda d \in \mathcal{K}\}.$$

This set, by assumption on $\mathcal{K}$, consists of a finite number of intervals; suppose that the boundary oracle is available (i.e., for any k, d the set S can be computed). Then Hit-and-Run method works as follows.

Step 1. Find a starting point $k^0 \in \mathcal{K}$.

Step 2. At the point $k^i \in \mathcal{K}$ generate a random direction $d^i \in \mathbb{R}^n$ uniformly distributed on the unit sphere (i.e., $d^i = \xi/\|\xi\|$, $\xi = \texttt{randn}\,(n, 1)$ the n-dimensional vector with normally distributed components).

Step 3. Apply boundary oracle procedure, i.e., define the set

$$S_i = \{\lambda \in \mathbb{R} : k^i + \lambda d^i \in \mathcal{K}\}.$$

Step 4. Generate a point λ_i uniformly distributed in S_i (we recall that S_i is a finite set of intervals), and compute a new point

$$k^{i+1} = k^i + \lambda_i d^i.$$

Step 5. Go to step 2 and increase i.

The general result on Hit-and-Run method is due to Smith [79].

Theorem 11. *Under the taken assumptions the distribution of the point k^i tends to the uniform one in $\mathcal{K}$ as $i \to \infty$.*

There is also the convergence estimate in the original paper [79, Theorem 3]; more precise estimate for convex sets can be found in [65, 66].

We address the method for the polynomial stabilization problem first. We consider the affine family of polynomials (8), i.e., the polynomials of the form

$$a(s, k) = a_0(s) + \sum_{i=1}^{m} k_i a_i(s).$$

Our goal is to describe somehow the stability domain in the parameter space k, i.e.,

$$\mathcal{K} = D_n = \{k \in \mathbb{R}^m : a(s, k) \text{ is stable}\}.$$

This set satisfy the above assumption, namely, $\mathcal{K}$ is open and consists of a finite number of components. Moreover, the boundary oracle is given by Algorithm from the Section 2.1. Suppose that $k^0 \in \mathcal{K}$ is known (for instance, if $a_0(s)$ is stable then h^0 can be chosen as $k^0 = 0$), then we may apply Hit-and-Run to generate points k^i approximately uniformly filling the stability domain. Thus it is the way to solve various synthesis problems. For instance, we have a performance index $J(k)$ (gain or phase margin, overshoot or other time-response characteristics, robustness margin, H_2 or H_∞ norms). Calculating $J(k^i)$ for admissible point we find the optimal one. In the neighborhood of k^* we can repeat the process, taking the intersection of $\mathcal{K}$ with some trust region (say a ball centered at k^* of radius ε) as new admissible set and k^* as the initial point. A number of examples for various problems of synthesis are addressed in [74].

Hit-and-Run is applicable not only for stabilization problem of polynomials but also for LMI's $A(x) < 0$ (48). The difficulty sufficiently reduces because the boundary oracle of $\mathcal{K}$ provides an interval $(\underline{\lambda}, \overline{\lambda})$ that can be computed as follows. We consider a point $x \in \mathbb{R}^\ell$ and a direction $y \in \mathbb{R}^\ell$. For $\lambda \in \mathbb{R}$, we have

$$A(x + \lambda y) = A_0 + \sum_{i=1}^{\ell} x_i A_i + \lambda \sum_{i=1}^{\ell} y_i A_i,$$

and denoting

$$A \doteq A_0 + \sum_{i=1}^{\ell} x_i A_i, \quad B \doteq \sum_{i=1}^{\ell} y_i A_i,$$

we are in the one-parameter case $A(\lambda) = A + \lambda B$. Boundary points for a chosen direction y are defined as real generalized eigenvalues λ_i for the matrix pair $(A, -B)$. Generating directions uniformly on the unit ℓ-dimensional sphere in the form $y = \eta/\|\eta\|$, where η has standard ℓ-dimensional gaussian distribution, we obtain D-decomposition boundary points $x + \lambda_i y$, i.e., the boundary oracle.

The proposed boundary oracle is especially efficient for characterization of the stability domain D_n which is always convex for LMI. Let for some $x \in \mathbb{R}^\ell$ matrix $A(x)$ (48) be negative definite, $x \in D_n$, and y is a certain direction. Then critical values $\underline{\lambda}$, $\overline{\lambda}$ such that sign-definiteness of the matrix $A(x + \lambda y)$ is preserved are given by formulae (49). This approach is easy-to-use for generating uniform distribution in D_n (compare to [42]) as well as proposing new programming methods.

Numerous application examples of this approach for optimization of linear functions subject to LMI (for semidefinite programming, SDP [41, 69]) are given in [26]. These methods appear to be quite competitive compare to the standard tools such as Yalmip [67] and LMI toolbox for Matlab.

Similarly Hit-and-Run can be applied for robust LMI's. Consider the construction of the boundary oracle for robust stability domain D_n^{rob} (54) for the family (52) and (53) for $\ell > 2$. We seek for the intersection points of a ray and the boundary of robust D-decomposition. Let $x \in D_n^{rob}$ be robustly feasible point and $y \in \mathbb{R}^\ell$ be a certain direction. Consider the straight line $x + \lambda y$ and compute $\underline{\lambda}^{rob}, \overline{\lambda}^{rob}$, the minimal and maximal values of λ, which guarantee the negative definiteness of the matrix $A(x + \lambda y, \Delta)$ for all $\Delta \in \mathcal{D}$. We have

$$A(x + \lambda y, \Delta) = \widehat{A}(\lambda) + \Delta(\lambda),$$

where it is denoted

$$\widehat{A}(\lambda) \doteq A_0 + \sum_{i=1}^{\ell}(x_i + \lambda y_i)A_i, \quad \Delta(\lambda) \doteq \Delta_0 + \sum_{i=1}^{\ell}(x_i + \lambda y_i)\Delta_i,$$

and by Theorem 9, the matrix $\widehat{A}(\lambda) + \Delta(\lambda)$ remains nonsingular (hence, negative definite) for all $\Delta \in \mathcal{D}$ satisfying $\left\|\left(\widehat{A}(\lambda)\right)^{-1}\right\| < 1/\|\Delta(\lambda)\|$. Since the perturbations Δ_i are independent, the estimate $\|\Delta(\lambda)\| \leqslant \|\Delta_0\| + \sum_{i=1}^{\ell} |x_i + \lambda y_i| \, \|\Delta_i\| = \varepsilon_0 + \sum_{i=1}^{\ell} |x_i + \lambda y_i| \varepsilon_i$ is sharp. hence, by considering the two scalar functions

$$\varphi(\lambda) \doteq \left\| \left(A_0 + \sum_{i=1}^{\ell}(x_i + \lambda y_i)A_i \right)^{-1} \right\|, \quad \varepsilon(\lambda) \doteq \frac{1}{\varepsilon_0 + \sum_{i=1}^{\ell} |x_i + \lambda y_i| \, \varepsilon_i}, \tag{58}$$

the interval $(\underline{\lambda}^{rob}, \overline{\lambda}^{rob})$ of robust definiteness of the family $A(x+\lambda y, \Delta)$ can be found numerically as $\{\lambda\colon \varphi(\lambda) \leqslant \varepsilon(\lambda)\}$.

Theorem 12 [25] . *let $A(x,0) < 0$. For any $y \in \mathbb{R}^\ell$, the maximal and minimum values of λ retaining the negative definiteness of the matrix $A(x+\lambda y, \Delta)$ for all admissible perturbations Δ are given by the two solutions of the equation $\varphi(\lambda) = \varepsilon(\lambda)$ (58) over the segment $[\underline{\lambda}, \overline{\lambda}]$, where $[\underline{\lambda}, \overline{\lambda}]$ are the bounds of negative definiteness of the matrix $A(x+\lambda y, 0)$ (49).*

Similarly, to solve a simpler problem of checking if $x \in D_n^{rob}$ for some $x \in \mathbb{R}^\ell$, it suffices to consider the unperturbed matrix at the point x: $A(x,0) = A_0 + \sum\limits_{i=1}^{\ell} x_i A_i$ and check if the inequality $\varepsilon_0 + \sum\limits_{i=1}^{\ell} |x_i|\varepsilon_i < 1/\|(A(x,0))^{-1}\|$ holds.

Another randomized method mentioned above is Shake-and-Bake that allows generating points not inside but in the boundary of a set $\mathcal{K}$. We assume the boundary oracle that provided some extra information compare to one used for Hit-and-Run. Namely, for the boundary point $k \in \partial\mathcal{K}$ the boundary oracle provides the internal normal $N = N(k)$ for $\partial\mathcal{K}$ at this point and for the internal direction $d \in \mathbb{R}^n$, $(d, N) > 0$ computes $\overline{\lambda} = \overline{\lambda}(k, d) = \sup\{\lambda > 0 : k + td \in \mathcal{K}, \forall t \in (0, \lambda)\}$. In the other words, $\overline{\lambda}$ is a point of the first intersection of the ray $k + \lambda d$, $\lambda > 0$ and the boundary $\mathcal{K}$.

Shake-and-Bake algorithm is the following:

Step 1. Given an initial point $k^0 \in \partial\mathcal{K}$.

Step 2. At the point $k^i \in \partial\mathcal{K}$ compute the unit internal normal N^i for $\partial\mathcal{K}$ and generate random direction d^i as follows: $d^i = \alpha_i N^i + c^i$, where c^i is uniformly distributed in $\|c^i\| = 1$, $(c^i, N^i) = 0$; $\alpha_i > 0$ is a scalar random variable, $\alpha = \left(1 - \xi^{\frac{2}{n-1}}\right)^{1/2}$, where $\xi = \texttt{rand}$ is uniformly random in $[0, 1]$.

Step 3. Using the boundary oracle find $\overline{\lambda}_i = \overline{\lambda}(k^i, d^i)$ and generate a new point

$$k^{i+1} = k^i + \overline{\lambda}_i d^i.$$

Step 4. Go to Step 2 and increase i.

The behavior of this method in the general situation is given by the following

Theorem 13 [44]. *Let $\mathcal{K}$ be a connected open bounded set with a normal mostly at every point. Then the distribution of k^i tends to the uniform distribution at $\partial\mathcal{K}$ as $i \to \infty$.*

We note that the set $\mathcal{K}$ is not assumed to be convex. Figure 11 depicts the result of the generating points at the boundary of the nonconvex set. Indeed, the points look approximately uniformly distributed.

For the problems with stability domain $\mathcal{K}$, Shake-and-Bake technique works, for instance, for the matrix case considered in Section 6. Let

$$\widetilde{\mathcal{K}} = \{K \in \mathbb{R}^{r \times m} : A + BKC \quad \text{be stable}\},$$

$\mathcal{K}$ be a simply connected component of $\widetilde{\mathcal{K}}$ that contains the certain K^0 (for instance, $K^0 = 0$ for stable matrix A). The boundary oracle for Shake-and-Bake algorithm can be produced numerically using Eq. (39) of the D-decomposition for matrices. Finally, we obtain matrices K approximately

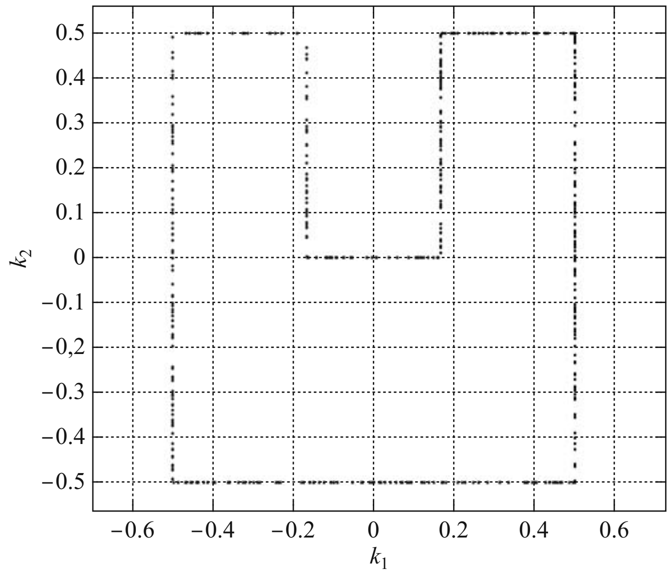

Fig. 11. Points generated in the boundary of nonconvex domain.

uniformly distributed on the boundary of $\mathcal{K}$. Shake-and-Bake method also works for LMI; the required boundary oracle is defined explicitly.

Finally, proposed randomized algorithms allows one to fill in the stability domain (or its boundary) with randomly generated points. This procedure may be regarded as an multidimensional alternative for graphical algorithms.

9. CONCLUSION

We provided the simple and effective techniques to describe the stability domain in the parameter space. This is an extension of the D-decomposition method for polynomials. Simultaneously with the stability domain we construct all the root (eigenvalue) invariant regions, i.e., simply connected regions of the domains with the invariant number of like-sign roots (eigenvalues) of the system matrix. This technique can be helpful for the low-order controllers design and for a detailed robustness analysis.

After more than 50 years the D-decomposition idea remains actual and there appear new applications for various control problems. The conducted survey shows that the domain of applicability for D-decomposition is much wider than the original problem statement for polynomials.

Besides new results on the structure of D-decomposition for polynomials, we discuss the method for description of the root invariant regions for uncertain polynomials. The modifications of the technique in order to describe the regions in the parameter space corresponding to the fixed-order controllers that satisfy some H_∞-specifications are considered. The main limitation is the small number of parameters but in some spacial cases (see Section 3) is can be overcome. In these cases, D-decomposition allows characterizing the multidimensional stability domain as a system of linear inequalities and it gives a straightforward solution of the stability radius problem.

The extension of the D-decomposition for the systems with matrix transfer functions is of particular importance. The condition of alteration of the number of stable eigenvalues of the parameter depending matrices is formulated in terms of singularity of the new matrix. The result has much in common with the basic D-decomposition theorem that describes when the stable root number changes but it is applicable for more general system structure. We pick out several classes of parameter depending matrices and get the analytical solution of D-decomposition equations for them. When the number of parameters do not exceed two the graphical representation of the result is possible.

The D-decomposition technique is applied in the theory of linear matrix inequalities. It is a tool to construct the regions such that the affine family of symmetric matrices has a fixed number of like-sign eigenvalues inside a region. Finally, we propose randomized algorithms exploiting the D-decomposition idea in the multidimensional case.

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