Robust linear algebra and robust aperiodicity

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Abstract

We consider some simple robust linear algebra problems which provide new insight for the robust stability and aperiodicity. Uncertainties are defined via various matrix norms, which include vector-induced and component-wise norms. First, the solution set of uncertain systems of linear algebraic equations is described. Second, the radius of nonsingularity of a matrix family is calculated. Third, these results are applied for estimation of aperiodicity radius.

1 Introduction

Uncertainty plays a key role in control [1, 2, 3] as well as in numerical analysis [4, 5]. We try to present an unified framework to treat uncertainties in linear algebra. For a nominal real matrix A a family of perturbed matrices is considered in a structured form $A+B\Delta C$ where Δ is a (rectangular) real matrix, bounded in some norm. We analyze various norms, which include such widely used ones as spectral, Frobenius, interval. The main tool is given by Theorem 1, which validates that the set $\{\Delta a, ||\Delta|| \le \varepsilon\}$ is a ball in some specified norm (Section 2). Based on this result, we are in position to describe a set of all solutions of perturbed systems of linear equations $(A + B\Delta C)x = b$ (Section 3). The next issue to be addressed in Section 4 is the distance to singular matrices (nonsingularity radius). Another basic problem of robust linear algebra is pseudospectrum — set of all eigenvalues of perturbed matrices. We study a real version (i.e. the set of all real eigenvalues) of this notion in Section 5. Finally we apply the results to control. Namely, we address aperiodicity property of matrices and aperiodicity robustness (Section 6).

2 Matrix norms and preliminaries

In this Section we introduce various norms for matrix perturbations and establish their properties. All vectors and matrices in the paper are assumed to be *real*.

As usual $||x||_p$ denotes l_p norm of a vector $x \in R^n, 1 \leq p \leq \infty$, i.e. $||x||_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$. For matrices $A \in R^{m\times n}$ with entries $a_{ij}, i = 1, \dots, m, j = 1, \dots, n$ two types of norms are considered. The first is *induced norm*:

$$||A||_{p,q} = \max_{x\neq 0} \frac{||Ax||_q}{||x||_p} = \max_{||x||_p \leq 1} ||Ax||_q.$$

The second is *component-wise norm*:

$$||A||_p = \left(\sum_{i,j} |a_{ij}|^p \right)^{1/p}.$$

The most important examples are listed below; explicit expressions for norms can be found in the literature or easily validated. Vector $a_i \in R^n$ denotes i-th row of A.

$$||A||_{\infty,\infty} = \max_i \sum_j |a_{ij}|;$$

$$||A||_{1,1} = \max_j \sum_i |a_{ij}|;$$

$$||A||_{2,2} = \bar{\sigma}(A) = \max_i (\lambda_i (A^T A)^{1/2}),$$

here $\bar{\sigma}(A)$ is the largest singular value of A while $\lambda_i(B)$ are eigenvalues of B. This is widely used *spectral* or *operator* norm of a matrix;

$$||A||_{1,\infty} = ||A||_\infty = \max_{i,j} |a_{ij}|,$$

this norm is sometimes called *interval* one (the family of matrices $A+\Delta, ||\Delta||_{1,\infty} \leq \varepsilon$ is the interval matrix);

$$||A||_{2,\infty} = \max_i (\sum_j a_{ij}^2)^{1/2};$$

$$||A||_{1,2} = \max_j (\sum_i a_{ij}^2)^{1/2};$$

$$||A||_2 = ||A||_F = (\sum_{i,j} a_{ij}^2)^{1/2},$$

this is another widely used norm — *Frobenius* one;

$$||A||_1 = \sum_{i,j} |a_{ij}|;$$

$$||A||_{\infty,1} = \max_{||x||_\infty \le 1} \sum_i |(a_i, x)|;$$

$$||A||_{\infty,2} = \max_{||x||_\infty \le 1} (\sum_i (a_i, x)^2)^{1/2};$$

$$||A||_{2,1} = \max_{||x||_2 \le 1} \sum_i |(a_i, x)|;$$

Note that the first eight formulas provide explicit (or easily calculated, as for spectral norm) expressions, while three last ones reduce calculation of the norm to the optimization problems. These problems are maximization of a quadratic or piece-wise linear convex function on a convex set. They are known to be NP-hard; moreover it is proved in [6] that calculation of $||A||_{p,q}$ is NP-hard for any $p, q \ge 2, p + q > 4$. There are computationally tractable bounds for these norms with precise estimation of their tightness [6, 7, 8, 9], but we are unable to discuss this important problem here. Nevertheless, calculation of $||A||_{\infty,1}$ or $||A||_{\infty,2}$ is not a hard task for matrices of moderate size. Indeed, the solution of above optimization problems is achieved at a vertex of the unit cube, thus it suffices to check 2^n points. For $n \le 15$ this can be performed with no difficulties.

The main property of the above introduced norms is given by the following result, which will be intensively exploited.

Theorem 1 *For every $x \in R^n, \varepsilon > 0$ the set $\{y = \Delta x, ||\Delta|| \le \varepsilon\}$ is a ball, specifically*

$$\{y = \Delta x, \quad \Delta \in R^{m \times n}, ||\Delta||_{p,q} \le \varepsilon\} = \{y \in R^m : ||y||_q \le \varepsilon ||x||_p\}, \qquad (1)$$

$$\{y = \Delta x, \quad \Delta \in R^{m \times n}, ||\Delta||_p \le \varepsilon\} = \{y \in R^m : ||y||_p \le \varepsilon ||x||_{p_*}\}, \qquad (2)$$

where $1 \le p_* \le \infty$ is the index conjugate to p: $\quad 1/p + 1/p_* = 1.$

Proof. For induced norms the inequality $||\Delta x||_q \le ||\Delta||_{p,q}|| ||x||_p \le \varepsilon ||x||_p$ follows from the definition. For component-wise norm it can be validated via Holder inequality:

$$||\Delta x||_p^p = \sum_i |\sum_j \Delta_{ij} x_j|^p \le \sum_i (\sum_j |\Delta_{ij}|^p)||x||_{p*}^p = ||\Delta||_p^p ||x||_{p*}^p$$

hence $||y||_p \le \varepsilon ||x||_{p*}$. On the other hand, if $y \in R^m, ||y||_q \le \varepsilon ||x||_p$, then take $\Delta = yv^T$, where $v \in R^n$ is the vector such that $v^T x = 1, ||v||_{p*}||x||_p = 1$. Then $\Delta x = yv^T x = y$ and for induced norms

$$||\Delta||_{p,q} = \max_{||z||_p \le 1} ||y||_q |v^T z| \le ||y||_q ||v||_{p*} \le \varepsilon ||x||_p ||v||_{p*} = \varepsilon.$$

Similarly for component-wise norms if $y \in R^m, ||y||_p \le \varepsilon ||x||_{p*}$, take $\Delta = yv^T, v \in R^n, v^T x = 1, ||v||_p ||x||_{p*} = 1$ and for this Δ

$$||\Delta||_p = (\sum_{i,j} |y_i|^p |v_j|^p)^{1/p} = ||y||_p ||v||_p \le \varepsilon ||x||_{p*} ||v||_p = \varepsilon.$$

Thus for both kinds of norms the equivalence of two sets in (1),(2) is validated.

3 Solution set of perturbed linear algebraic equations

Let $A \in R^{n \times n}, B \in R^{n \times m}, C \in R^{r \times n}$ be given matrices, A is nonsingular, $b \in R^n$ is a given vector, $\varepsilon > 0$. The set

$$S_\varepsilon = \{x \in R^n : \exists \Delta \in R^{m \times r}, ||\Delta|| \le \varepsilon, (A + B\Delta C)x = b\} \quad (3)$$

is called *the solution set* for the nominal equation $Ax = b$ under structured perturbations. The level of perturbations is given by ε and the norm $||\cdot||$ should be specified. The result below provides the closed-form expression for S_ε.

Theorem 2 *If* $||\Delta|| = ||\Delta||_{p,q}$ *then*

$$S_\varepsilon = \{x = A^{-1}(b - By) : ||y||_q \le \varepsilon ||CA^{-1}(b - By)||_p\} \quad (4)$$

and if $||\Delta|| = ||\Delta||_p$ *then*

$$S_\varepsilon = \{x = A^{-1}(b - By) : ||y||_p \le \varepsilon ||CA^{-1}(b - By)||_{p*}\}. \quad (5)$$

Proof. If $(A + B\Delta C)x = b$ then denoting $y = \Delta Cx$ we get $x = A^{-1}(b - By)$ and $y = \Delta CA^{-1}(b - By)$. Due to Theorem 1 all y satisfying the last equation with $||\Delta|| \le \varepsilon$ coincide with the set $||y|| \le \varepsilon||CA^{-1}(b - By)||$ with corresponding norms in the left- and right-hand sides. Substituting these norms and returning to x variables we arrive to the Theorem assertions.

Let us consider some particular cases of the above result.

1. *Unstructured perturbations.* $B = C = I$. Then we can express y via $x : y = b - Ax$ and the solution set becomes

$$S_\varepsilon = \{x : ||Ax - b||_q \le \varepsilon||x||_p\} \tag{6}$$

for induced norms and

$$S_\varepsilon = \{x : ||Ax - b||_p \le \varepsilon||x||_{p*}\} \tag{7}$$

for component-wise norms.

2. *Spectral or Frobenius norm.* For $p = q = 2$ (induced norms) or $p = 2$ (component-wise norm) we get the same expression

$$S_\varepsilon = \{x = A^{-1}(b - By) : ||y||_2 \le \varepsilon||CA^{-1}(b - By)||_2\}. \tag{8}$$

This leads to the following conclusion.

Proposition 1 *If $\varepsilon < 1/\bar{\sigma}(CA^{-1}B)$, then the solution set for perturbations bounded in spectral or Frobenius norm is an ellipsoid.*

Proof. Indeed under above assumption on ε the quadratic form $||y||_2^2 - \varepsilon^2||CA^{-1}By||_2^2$ is positive definite, thus the set of y defined by (8) is an ellipsoid. Hence the set S_ε is an ellipsoid as well, being a linear image of the ellipsoid.

3.*Solution set for interval equations.* Suppose $B = C = I, p = 1, q = \infty$, that is S_ε is solution set for interval equations:

$$S_\varepsilon = \{x : \exists|\Delta_{ij}| \le \varepsilon, i, j = 1, \dots, n, \quad (A + \Delta)x = b.\} \tag{9}$$

Then according to Theorem 2

$$S_\varepsilon = \{x : ||Ax - b||_\infty \le \varepsilon||x||_1\}. \tag{10}$$

This is a polytopic set; however it can be non convex for arbitrary small $\varepsilon > 0$. For instance for $n = 2, A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ the set $S_\varepsilon = \{x \in R^2 : \max\{|x_1 - 1|, |x_2|\} \le \varepsilon(|x_1| + |x_2|)\}$ is nonconvex for any $\varepsilon > 0$.

The structure of all solutions for interval equations has been described first in [10]. More details can be found in the monograph [11].

In the above analysis we supposed that the right hand side of the equation — vector b — remains unperturbed. It is not hard to incorporate more general case, but we do not address the issue here.

4 Nonsingularity radius

We consider the same framework as above: let $A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{r\times n}$ be given matrices, A is nonsingular. The problem is to find the margin of perturbations Δ which preserve nonsingularity of the matrix $A + B\Delta C$. More rigorously, we define the *nonsingularity radius* as

$$\rho(A) = \min\{||\Delta|| : A + B\Delta C \text{ is singular. }\}$$

The norm in the above definition should be specified; we denote the radius as $\rho(A)_{p,q}$ or $\rho(A)_p$ for induced and component-wise norms respectively. Notice that standard definition deals with complex unstructured ($B = C = I$) perturbations and spectral norm while we address real structured perturbations and arbitrary norms.

Theorem 3 *The nonsingularity radius is given by*

$$\rho(A)_{p,q} = 1/||CA^{-1}B||_{q,p} \tag{11}$$

$$\rho(A)_p = 1/||CA^{-1}B||_{p,p_*} \tag{12}$$

Proof. Matrix $A+B\Delta C$ is singular if and only if the equation $(A+B\Delta C)x = 0$ has nonvanishing solution, that is if the solution set for this equation contains a nonzero point. Consider induced norms case first. From Theorem 2 with $b = 0$ it means that the inequality $||y||_q \le \varepsilon||CA^{-1}By||_p$ holds for $y \ne 0, \varepsilon = ||\Delta||_{p,q}$. This is equivalent to $||\Delta||_{p,q} \ge 1/||CA^{-1}B||_{q,p}$. Thus nonsingularity arises if the last inequality holds; this leads to (11). The case of component-wise norms is treated similarly.

Some particular cases are of interest.

1. *Unstructured perturbations, induced norms with $p = q$.* For $B = C = I, p = q$ we obtain from (11):

$$\rho(A)_{p,p} = 1/||A^{-1}||_{p,p}.$$

This is the classical result due to Kahan [12].

2. *Interval norm.* Taking $p = 1, q = \infty, B = C = I$ we get

$$\rho(A)_{1,\infty} = \rho(A)_{\infty} = 1/||A^{-1}||_{\infty,1}.$$

Say it another way, nonsingularity radius for interval perturbations is reciprocal to the $(\infty, 1)$-norm of the inverse matrix. As we have mentioned, calculation of such norm is NP-hard problem, however it requires to compute 2^n numbers. Thus the problem is tractable for moderate n, say $n \leq 15$.

3. *Scalar perturbation.* If $m = r = 1, B = e_i, C = e_j^T, e_i$ is i-th ort, then $B\Delta C$ is the matrix with the only ij-th entry nonvanishing (equal to $\Delta \in R^1$) and all other entries equal to 0. Thus

$$\min\{|\Delta| : A + \Delta E \text{ is singular }\} = 1/|m_{ji}|,$$

where $E = ((e_{kl})), k, l = 1, \dots, n, e_{ij} = 1, e_{kl} = 0, (k, l) \neq (i, j), A^{-1} = ((m_{kl})), k, l = 1, \dots, n.$

5 Real pseudospectrum

We proceed to investigation of spectrum of perturbed matrices, which is often called *pseudospectrum.* In contrast with numerous works on this subject [13, 14, 15] we deal with *real* perturbations and eigenvalues. We call *(real) pseudospectrum* of a matrix $A \in R^{n \times n}$ the set

$$\Lambda_{\varepsilon}(A) = \{\lambda \in R^1 : \exists \Delta \in R^{m \times r}, ||\Delta|| \leq \varepsilon, \lambda \text{ is an eigenvalue of } A + B\Delta C.\} \tag{13}$$

The level of perturbation $\varepsilon > 0$ and the norm in the above definition should be specified; we denote the pseudospectra as $\Lambda_{\varepsilon}(A)_{p,q}$ or $\Lambda_{\varepsilon}(A)_p$ for induced and component-wise norms respectively. Below we use notation

$$G(\lambda) = C(A - \lambda I)^{-1}B$$

Theorem 4 *The real pseudospectra is given by*

$$\Lambda_{\varepsilon}(A)_{p,q} = \{\lambda \in R^1 : 1/||G(\lambda)||_{q,p} \leq \varepsilon\} \tag{14}$$

$$\Lambda_{\varepsilon}(A)_p = \{\lambda \in R^1 : 1/||G(\lambda)||_{p,p_*} \leq \varepsilon\}. \tag{15}$$

Proof. λ is an eigenvalue of a matrix $A + B\Delta C$ if and only if $A - \lambda I + B\Delta C$ is singular. Thus we can apply Theorem 3 to the matrix $A - \lambda I; \lambda$ belongs to the pseudospectrum if and only if $\varepsilon \leq \rho(A - \lambda I)$. This coincides with the assertion of Theorem 4.

Corollary 1 *Suppose matrix $A \in R^{n\times n}$ has no real eigenvalues. Then*

$$\min\{||\Delta||_{p,q} : A + B\Delta C \quad \text{has a real eigenvalue}\} = \min_{\lambda\in R^1}\{1/||G(\lambda)||_{q,p}\} \quad (16)$$

$$\min\{||\Delta||_{p} : A + B\Delta C \quad \text{has a real eigenvalue}\} = \min_{\lambda\in R^1}\{1/||G(\lambda)||_{p,p_*}\}. \quad (17)$$

For Frobenius norm we can provide another characterization of the distance to matrices with real eigenvalues, which avoids λ gridding. Denote

$$D_\varepsilon = \begin{pmatrix} A & -\varepsilon BB^T \\ -\varepsilon CC^T & A^T \end{pmatrix}.$$

Proposition 2

$$\varepsilon^* = \min\{||\Delta||_2 : A + B\Delta C \quad \text{has a real eigenvalue}\} \quad (18)$$
$$= \min\{\varepsilon \in R^1 : D_\varepsilon \quad \text{has a real eigenvalue}\}. \quad (19)$$

Moreover, matrix D_ε has a real eigenvalues for all $\varepsilon \geq \varepsilon^*$.

Proof. Optimal Δ in (18) is the solution of the optimization problem

$$\min ||\Delta||_2^2, \quad (A + B\Delta C)x = \lambda x, \quad x \neq 0, \lambda \in R^1, x \in R^n. \quad (20)$$

The Lagrange function for the problem is

$$L(x, \Delta, y) = ||\Delta||_2^2 + (y, (A + B\Delta C)x - \lambda x).$$

Writing the derivatives with respect to the vector variable x and the matrix variable Δ we have

$$L_\Delta = \Delta + B^T y x^T C^T = 0,$$
$$L_x = (A^T + C^T\Delta^T B^T)y - \lambda y = 0.$$

Excluding Δ we obtain equations

$$Ax - ||Cx||_2^2 BB^T y = \lambda x \quad (21)$$
$$A^T y - ||B^T y||_2^2 C^T Cx = \lambda y. \quad (22)$$

These equations remain invariant if one replaces x with $\gamma x, y$ with y/γ for arbitrary $\gamma \neq 0$. We can choose γ so that $||Cx||_2 = ||B^T y||_2$. Recall that $\Delta = -B^T y x^T C^T$, thus for optimal solution $||\Delta||_2 = ||Cx||_2||B^T y||_2 = \varepsilon^*$ and (21), (22) becomes

$$Ax - \varepsilon^* BB^T y = \lambda x \quad (23)$$
$$A^T y - \varepsilon^* C^T Cx = \lambda y. \quad (24)$$

Now consider the real eigenvalue problem (19): $D_\varepsilon w = \lambda w$; for $w = (u, v)$ it can be written as

$$Au - \varepsilon BB^T v = \lambda u \tag{25}$$
$$A^T v - \varepsilon C^T Cu = \lambda v \tag{26}$$

and completely coincides with (21), (22) for $\varepsilon = \varepsilon^*$. Thus if ε^* is the solution of (18) then D_{ε^*} has a real eigenvalue. Multiplying (25) by v and (26) by u we conclude that $||B^T v||_2 = ||Cu||_2$. If we take

$$x = \sqrt{\varepsilon} u/||Cu||_2, \quad y = \sqrt{\varepsilon} v/||B^T v||_2,$$

then such x, y satisfy (21), (22) and $\Delta = -B^T y x^T C^T$ has the norm equal to ε. Thus existence of a real eigenvalue of the matrix D_ε implies that optimality conditions (21), (22) hold.

To prove the last assertion of Proposition 2 we can rewrite (25), (26) as

$$(A - \lambda I)u = \varepsilon BB^T v \tag{27}$$
$$(A - \lambda I)v = \varepsilon C^T Cu \tag{28}$$

or denoting $t = Cu$

$$\varepsilon^{-2} t = G(\lambda)G(\lambda)^T t, \quad G(\lambda) = C(A - \lambda I)^{-1} B.$$

Thus ε^{-2} is an eigenvalue of the symmetric nonnegative definite matrix $G(\lambda)G(\lambda)^T$ if and only if (25), (26) hold. We conclude that if $\varepsilon \geq 1/\max_\lambda \overline{\sigma}(G(\lambda)) = \varepsilon^*$ then D_ε has a real eigenvalue.

Let us apply the above expressions of matrix pseudospectrum for finding zero sets of perturbed polynomials. If a family of polynomials has the form

$$\mathcal{P}_p = \{P(s, a) = a_1 + a_2 s + \cdots + a_n s^{n-1} + s^n, \quad ||a - a^*||_p \leq \varepsilon\} \tag{29}$$

where $a^* \in R^n$ are the coefficients of the nominal polynomial $P(s, a^*)$, then its *(real) zero set* is

$$Z_\varepsilon = \{\lambda \in R^1 : \exists P(s, a) \in \mathcal{P}, \quad P(\lambda, a) = 0\}. \tag{30}$$

Such sets are sometimes called *spectral sets*, see [3]; usually their complex counterparts for $p = 2$ are studied. To apply Theorem 4 for calculation of Z_ε let us take $m = 1, r = n, C = I$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad \Delta = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_{n-1} \\ \delta_n \end{pmatrix}^T.$$

Then $||\Delta||_p$ is the same as $||\delta||_p, \delta \in R^n$ and the matrix $A + B\Delta C$ has the same form as A with the last row a replaced with $a+\delta$. Hence the eigenvalues of $A + B\Delta C$ are equal to the zeros of $P(s, a+\delta)$ and the zero set coincides with pseudospectrum. It is easy to show that

$$G(\lambda) = (1, \lambda, \lambda^2, \dots, \lambda^n)^T / P(\lambda, a)$$

Then exploiting Theorem 4 we obtain:

Proposition 3 *The zero set of a polynomial family $\mathcal{P}_p$ is equal to*

$$Z_\varepsilon = \{\lambda : |P(\lambda, a)| \le \varepsilon ||(1, \lambda, \lambda^2, \dots, \lambda^n)^T||_{p_*}\} \tag{31}$$

For $p = 2$ this result (and its complex extension) was known [3], Theorem 16.3.4.

6 Aperiodicity radius

A matrix $A \in R^{n\times n}$ is called *aperiodic*, if its eigenvalues are all real, negative and distinct. Such matrices play role in control, because solutions of a system $\dot{x} = Ax$ with aperiodic A are stable and do not oscillate (each component of $x(t)$ change sign not more than n times). *Robust aperiodicity* problem is to check aperiodicity of a family of perturbed matrices $A + B\Delta C, ||\Delta|| \le \varepsilon$. The similar problem for polynomials is well studied, see e.g. [16, 17, 18]. However the matrix version of the problem remained open; just particular results have been obtained [19].

Define *aperiodicity radius* for an aperiodic matrix A as

$$\nu(A) = \min\{||\Delta|| : A + B\Delta C \quad \text{is not aperiodic}\}. \tag{32}$$

For specific norm $||\cdot||_{p,q}$ or $||\cdot||_p$ we obtain $\nu(A)_{p,q}$ and $\nu(A)_p$ respectively. If $\lambda_i \in R^1, i = 1, \dots, n$ are eigenvalues of A, then the functions

$$\phi(\lambda)_{p,q} = 1/||G(\lambda)||_{p,q}, \quad \phi(\lambda)_p = 1/||G(\lambda)||_{p,p_*}$$

are vanishing at points λ_i and positive for all other λ. Computationally it is not hard to find

$$\phi^i_{p,q} = \max_{\lambda_i \le \lambda \le \lambda_{i+1}} \phi(\lambda)_{p,q}, \phi^i_p = \max_{\lambda_i \le \lambda \le \lambda_{i+1}} \phi(\lambda)_p$$

for $i = 1, \dots, n-1$.

Theorem 5 *Aperiodicity radius is estimated by formulas*

$$\nu(A)_{p,q} \geq \min\{\phi^1_{p,q}, \dots, \phi^{n-1}_{p,q}, \phi(0)_{p,q}\} \tag{33}$$

$$\nu(A)_p \geq \min\{\phi^1_p, \dots, \phi^{n-1}_p, \phi(0)_p\} \tag{34}$$

Proof. It follows from Theorem 4 that for ε less than right hand side of (33), (34) the corresponding pseudospectrum consists of n distinct intervals, all located in the negative half-axis. Thus matrices $A + B\Delta C, ||\Delta|| \leq \varepsilon$ remain aperiodic.

We conjecture that the lower bound in the Theorem coincides with the upper bound, i.e. equality holds in (33), (34).

Of course the above result can be easily extended to *discrete-time aperiodicity* when eigenvalues λ_i of A are assumed to be real, distinct and $-1 < \lambda_i < 0$. Then we should include $\phi(-1)$ in the right hand side of (33), (34).

Also we can obtain results on robust aperiodicity of polynomials by using the same technique as at the end of the previous Section.

7 Conclusions

In the paper we presented an unified approach to analysis of perturbations for typical problems of linear algebra (solving systems of equations, checking nonsingularity, finding of eigenvalues). However this approach was restricted with real perturbations and real eigenvalues only. While the transition to complex perturbations often simplifies the research, the case of complex eigenvalues and real perturbations is much harder. For instance, the challenging problem of finding real stability radius has been solved just recently [20] for spectral norms only. From technical point of view the difference is that critical perturbations are rank-one matrices in the real case (as in this paper) and rank-two matrices in complex case (as in [20]). Of course the presented approach allows to obtain various bounds for the stability radius and related problems with different norms (including the famous problem of robust stability of interval matrices). This is the direction for future work.

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