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Multi-Input Multi-Output Ellipsoidal State Bounding[1]

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Abstract. Ellipsoidal state outer bounding has been considered in the literature since the late sixties. As in the Kalman filtering, two basic steps are alternated: a prediction phase, based on the approximation of the sum of ellipsoids, and a correction phase, involving the approximation of the intersection of ellipsoids. The present paper considers the general case where K ellipsoids are involved at each step. Two measures of the size of an ellipsoid are employed to characterize uncertainty, namely, its volume and the sum of the squares of its semiaxes. In the case of multi-input multi-output state bounding, the algorithms presented lead to less pessimistic ellipsoids than the usual approaches incorporating ellipsoids one by one.

Key Words. Bounded noise, ellipsoidal bounding, identification, set-membership estimation, state estimation.

1. Introduction

In the literature, most parameter or state estimation problems involving an explicit characterization of uncertainty are solved via a stochastic approach, with the perturbations assumed to be random and usually white and Gaussian. However, often the statistics of these perturbations are not known and sometimes it is more natural to assume that they belong to known compact sets, with no other hypothesis on their distributions. Then,

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one can attempt to characterize the set of all parameter or state vectors that are compatible with the data, model structure, and hypotheses on the perturbations. All elements of this feasible set are candidate solutions for the estimation problem.

Set-membership estimation was considered first in the late sixties in the context of state estimation (Refs. 1–3). Then, it was a subject of intensive research by the Russian school (Refs. 4–8) and has received a lot of attention worldwide especially in the context of parameter estimation; see, e.g., the survey papers in Refs. 9, 10, special issues of journals (Refs. 11–13), book (Ref. 14), and references therein. The solution to the problem of parameter or state bounding depends on whether the model output is linear in the parameters or initial state and on how the feasible set is going to be characterized.

The systems considered in this paper are assumed to be described by the linear discrete-time state-space model

$$x_{t+1} = A_t x_t + B_t u_t + V_t v_t, \tag{1a}$$

$$y_t = C_t x_t + D_t u_t + W_t w_t, \tag{1b}$$

where x_t is the state vector at time t, y_t the measured output vector, u_t the known input vector, v_t the process perturbation vector, and w_t the measurement noise vector. Without loss of generality, D_t will be taken as 0 in what follows. The only unknown quantities in (1) are assumed to be the state, process perturbations, and measurement noise. The information available regarding these quantities is that v_t, w_t, and x_0 belong to known compact sets. The problem is then to characterize the set of all vectors x_t that are compatible with the data given these hypotheses.

Tracking the parameters of a model whose output is linear in these parameters can be treated as a special case of (1), where A_t is the identity matrix, $B_t = 0$, and x_t stands for the parameters to be estimated; then, the particular case of time-invariant parameters can be considered by setting $V_t = 0$. All algorithms are presented in the real case, but extend trivially to complex parameter or state vectors. It will be assumed that the state dimension n is larger than one, which is not very restrictive as specific and more efficient algorithms can be derived easily for the scalar case. It will also be assumed that all components of the output vectors are corrupted by noise.

For a model described by (1), when v_t, w_t, and x_0 belong to polytopes, the feasible set is a polytope too. When the number of parameters or state variables is not too large, an exact description of this polytope can be attempted (Ref. 15). However, the set thus obtained may become extremely complicated. So, it is customary to characterize it by computing a simpler set that encloses it and is of minimum size in a sense to be specified.

Although simplexes, boxes, parallelotopes, and polytopes with limited complexity (i.e., possessing a limited number of faces and vertices) have been considered, the usual approach, also adopted in this paper, is to approximate the feasible set by an outer ellipsoid.

Classically, the size of an ellipsoid is measured by its volume, proportional to the square of the product of the lengths of its axes, which corresponds to the determinant criterion. Thus, minimal-volume outer ellipsoids are searched for. However, the determinant criterion presents some disadvantages. First, it may be minimized by a very narrow ellipsoid, i.e., uncertainty in some directions may remain extremely large even when the volume tends to zero. Second, volume optimization problems can seldom be solved explicitly, the only well-known example of an explicit solution being the intersection of an ellipsoid and a strip (Refs. 16–17). These reasons motivate the consideration of an alternative measure of size, namely, the sum of the squares of the lengths of the semiaxes, which corresponds to the trace criterion and has been the object of a renewed attention (Refs. 10 and 18–22). Although both criteria had been considered in Refs. 10 and 17, and although they are mentioned already in Ref. 23, the trace criterion has received much less emphasis in the past than the determinant criterion, which has been used in the literature overwhelmingly. For these two measures of size, it is well-known that there exists a unique smallest ellipsoid containing a compact set with nonempty interior (see Section 3.3).

Various numerical techniques are available for finding optimal or suboptimal ellipsoids containing a given set. Most focus on the simplest case where the sum or intersection of only two ellipsoids is to be computed, and solutions can be found explicitly (Refs. 5, 17, 18, 24). Ellipsoidal approximation may be converted also into convex optimization with linear matrix inequalities (LMI) as constraints (Ref. 25). Then, the powerful techniques of modern interior-point polynomial algorithms for LMI (Ref. 26) can be employed. However, this approach is less suitable for on-line state estimation or parameter tracking, where ellipsoidal approximation should be performed in real time.

In the present paper, ellipsoidal approximation with the trace or determinant criterion is studied systematically for the solution of two problems that are at the core of state bounding, i.e., optimal outer approximation of the sum and intersection of K possibly degenerate ellipsoids. Parametrized families of ellipsoids are employed that can be proved to contain this sum or intersection and lead to solving optimization problems with $K-1$ scalar parameters. Whenever possible, our approach is oriented toward an explicit solution, sometimes at the cost of suboptimality. The general case $K \geq 2$ is treated. It was mentioned already in Ref. 2, but without optimization, whereas only the case $K=2$ is considered in Ref. 18.

The paper is organized as follows. Section 2 states the problem to be solved and specifies the notation. Some basic properties of the trace and determinant functions are established in Section 3. Then, algorithms are presented for computing minimal-trace or minimal-determinant outer ellipsoids containing the sum of K ellipsoids (Section 4) or their intersection (Section 5). Thus the basic blocks of a multi-input multi-output bounded-error counterpart to the Kalman filtering are provided.

2. Problem Statement

2.1. Ideal State Bounding. The perturbation vector v_t and noise vector w_t of (1) may be partitioned respectively into I and J independent subvectors. Then, each of these subvectors is assumed to belong to a known ellipsoid. If the subvectors of v_t or w_t are not independent, then the algorithms to be presented still apply, but the ellipsoidal approximation obtained will be more pessimistic. Reciprocally, if some components of the subvectors are independent, the results will be more pessimistic than if these independent components were split into different subvectors. Let $v_t^i \in \mathbb{R}^{p_t^i}$, $i = 1, \ldots, I$, be the subvectors of v_t. Then, the state equation (1a) can be rewritten as

$$x_{t+1} = A_t x_t + B_t u_t + \sum_{i=1}^{I} V_t^i v_t^i, \tag{2}$$

where $V_t^i v_t^i$ belongs to a known ellipsoid, V_t^i being trivially deduced from V_t. Let $\mathscr{B}^n$ be the unit Euclidean ball of $\mathbb{R}^n$ centered on the origin,

$$\mathscr{B}^n = \{x \in \mathbb{R}^n: \|x\| \leq 1\},$$

where $\|x\|$ is the Euclidean norm of x. A suitable transformation of V_t^i makes it possible always to impose that v_t^i belongs to $\mathscr{B}^{p_t^i}$. In what follows, it is assumed that this transformation has been performed. After some additional transformations detailed in Section 7.1, and without loss of generality, the state and observation equations (1) can be rewritten as

$$x_{t+1} = A_t x_t + B_t u_t + \sum_{i=1}^{I} V_t^i v_t^i, \qquad v_t^i \in \mathscr{B}^{p_t^i}, \tag{3a}$$

$$y_t^j = C_t^j x_t + w_t^j, \qquad w_t^j \in \mathscr{B}^{r_t^j}, \qquad j = 1, \ldots, J, \tag{3b}$$

where y_t^j is obtained by linear combination of the components of y_t that are corrupted by the jth subvector of w_t, w_t^j is obtained by linear transformation of the jth subvector of w_t, and C_t^j is deduced from C_t. It is assumed that the initial state x_0 belongs to some known set $\Omega_{0/0}$.

The objective is to find recursive ellipsoidal outer approximations of the uncertainty sets $\Omega_{t/t}$ and $\Omega_{t+1/t}$, respectively defined as the sets of all values of x_t and x_{t+1} that are compatible with all the information available at t. In principle, $\Omega_{t+1/t}$ and $\Omega_{t+1/t+1}$ could be computed recursively by alternating prediction and correction steps (as in the Kalman filtering),

$$\Omega_{t+1/t} = A_t\Omega_{t/t} + B_t u_t + \sum_{i=1}^{I} V_t^i \mathscr{B}^{p_t^i}, \tag{4a}$$

$$\Omega_{t+1/t+1} = \Omega_{t+1/t} \cap \left(\bigcap_{j=1}^{J} \mathscr{E}_{t+1}^j \right), \tag{4b}$$

where $\mathscr{E}_{t+1}^j$ is the set of all states x_{t+1} compatible with the measurement y_{t+1}^j,

$$\mathscr{E}_{t+1}^j = \{x \in \mathbb{R}^n: (y_{t+1}^j - C_{t+1}^j x) \in \mathscr{B}^{r_{t+1}^j}\}. \tag{5}$$

Of course, we shall consider only informative measurements, i.e., measurements such that $C_{t+1}^j \neq 0$. The predicted set $\Omega_{t+1/t}$ is obtained from $\Omega_{t/t}$ by a weighted Minkowski sum of sets, and the corrected set $\Omega_{t+1/t+1}$ is obtained from $\Omega_{t+1/t}$ by intersecting sets. However, in general, Eqs. (4) are too complicated to be of any practical use, especially in real time. To reduce complexity, $\Omega_{t+1/t}$ and $\Omega_{t+1/t+1}$ will be approximated recursively by outer ellipsoids. These outer ellipsoids will be assumed to be bounded and with a nonempty interior. Some of the ellipsoids involved in the algorithms to be presented will not satisfy this assumption, and a specific notation will be needed to describe ellipsoids with empty interiors (such as intervals) or unbounded ellipsoids (such as strips limited by parallel hyperplanes).

2.2. Notation. Any bounded ellipsoid $\mathscr{E}$ of $\mathbb{R}^n$ with a nonempty interior can be defined by

$$\mathscr{E} = \{x \in \mathbb{R}^n: (x-c)^{\mathrm{T}} P^{-1} (x-c) \leq 1, P > 0\}, \tag{6}$$

where c is the center of $\mathscr{E}$ and P is a positive-definite matrix (denoted by $P > 0$) that specifies its size and orientation. Of course, it is equivalent to write $\mathscr{E}$ as

$$\mathscr{E} = \{x \in \mathbb{R}^n: (x-c)^{\mathrm{T}} M (x-c) \leq 1, M > 0\}, \tag{7}$$

with $M = P^{-1}$. The two measures of the size of such an ellipsoid to be considered in this paper are $\operatorname{tr} P$ and $\det P$. As defined by (6) or (7), $\mathscr{E}$ can be written also as

$$\mathscr{E} = \{x \in \mathbb{R}^n: x = c + Vv, v \in \mathscr{B}^n\}, \tag{8}$$

where

$$M^{-1} = P = VV^{\mathrm{T}} > 0.$$

This description can be extended to accommodate ellipsoids with empty interiors by allowing v to belong to a lower-dimensional space than x,

$$\mathscr{E} = \{x \in \mathbb{R}^n: x = c + Vv, v \in \mathscr{B}^p\}. \tag{9}$$

If $p = 1$, V is a vector and $\mathscr{E}$ an interval. Equation (9) involves an affine transformation of the unit ball of $\mathbb{R}^p$ and describes an ellipsoid defined as in (8) but with $P = VV^{\mathrm{T}}$ no longer necessarily positive definite. Thus, the following description of bounded ellipsoids will be convenient at the prediction step of state estimation when summing ellipsoids, some of which may have empty interiors:

$$\mathscr{E}^+(c; P) = \{x \in \mathbb{R}^n: x = c + Vv, v \in \mathscr{B}^p, P = VV^{\mathrm{T}}\}. \tag{10}$$

In (10), the matrix P is nonnegative definite, denoted by $P \geq 0$. If P has a zero eigenvalue, then $\mathscr{E}^+(c; P)$ has an empty interior.

Sometimes, unbounded ellipsoids should also be considered. This becomes possible if the condition $M > 0$ in (7) is relaxed to $M \geq 0$. A typical example is the strip

$$\mathscr{S}(y; d) = \{x \in \mathbb{R}^n: |y - d^{\mathrm{T}}x| \leq 1\}, \tag{11}$$

which corresponds to (7) with

$$M = dd^{\mathrm{T}} \geq 0.$$

The following description of such possibly degenerate ellipsoids will be convenient when intersecting ellipsoids at the correction step of state estimation,

$$\mathscr{E}^\cap(c; M) = \{x \in \mathbb{R}^n: (x - c)^{\mathrm{T}} M (x - c) \leq 1, M \geq 0\}. \tag{12}$$

With this notation, $\mathscr{E}_t^j$ in (5) can be rewritten as $\mathscr{E}^\cap(c_t^j; M_t^j)$ with $M_t^j = C_t^{j\mathrm{T}} C_t^j$. The center c_t^j of $\mathscr{E}_t^j$ may not be defined uniquely (it must only satisfy $C_t^j c_t^j = y_t^j$), but it is not used in what follows. If y_t^j is a scalar measurement, then $\mathscr{E}_t^j$ is the strip $\mathscr{S}(y_t^j; C_t^{j\mathrm{T}})$. Note that, when $P > 0$,

$$\mathscr{E}^+(c; P) = \mathscr{E}^\cap(c; M), \qquad \text{with } P = M^{-1}.$$

2.3. Ellipsoidal State Bounding.

Initialization. The known compact set $\Omega_{0/0}$ guaranteed to contain the initial state vector will be taken as

$$\hat{\mathscr{E}}_{0/0} = \mathscr{E}^+(\hat{c}_{0/0}; \hat{P}_{0/0}).$$

As for recursive least squares, one may choose $\hat{c}_{0/0} = 0$ and $\hat{P}_{0/0} = \alpha I$, with α large enough.

Recursion on Time. Let

$$\hat{\mathscr{E}}_{t/t} = \mathscr{E}^{+}(\hat{c}_{t/t}; \hat{P}_{t/t})$$

be an ellipsoidal outer approximation of $\Omega_{t/t}$. From (4a), an ellipsoidal outer approximation of $\Omega_{t+1/t}$ is given by

$$\hat{\mathscr{E}}_{t+1/t} = \mathscr{E}^{+}(\hat{c}_{t+1/t}; \hat{P}_{t+1/t}),$$

such that either

$$\hat{\mathscr{E}}_{t+1/t} = \arg\min_{\mathscr{E} \supset \mathscr{M}_{t+1}} \operatorname{tr} P, \tag{13}$$

or

$$\hat{\mathscr{E}}_{t+1/t} = \arg\min_{\mathscr{E} \supset \mathscr{M}_{t+1}} \log\det P, \tag{14}$$

where $\mathscr{M}_{t+1}$ is the (weighted) Minkowski sum of sets,

$$\mathscr{M}_{t+1} = A_t \hat{\mathscr{E}}_{t/t} + B_t u_t + \sum_{i=1}^{I} V_t^i \mathscr{B}^{p_t^i}, \tag{15}$$

and $\mathscr{E} = \mathscr{E}^{+}(c; P)$. Equation (15) can be rewritten as

$$\mathscr{M}_{t+1} = \mathscr{E}^{+}(\hat{c}_{t/t} + B_t u_t; A_t \hat{P}_{t/t} A_t^{\mathrm{T}}) + \sum_{i=1}^{I} \mathscr{E}^{+}(0; V_t^i V_t^{i\mathrm{T}}). \tag{16}$$

Similarly, $\Omega_{t+1/t+1}$ is approximated by $\hat{\mathscr{E}}_{t+1/t+1}$, such that either

$$\hat{\mathscr{E}}_{t+1/t+1} = \arg\min_{\mathscr{E} \supset \mathscr{I}_{t+1}} \operatorname{tr} P, \tag{17}$$

or

$$\hat{\mathscr{E}}_{t+1/t+1} = \arg\min_{\mathscr{E} \supset \mathscr{I}_{t+1}} \log\det P, \tag{18}$$

where $\mathscr{I}_{t+1}$ is the intersection of sets,

$$\mathscr{I}_{t+1} = \hat{\mathscr{E}}_{t+1/t} \cap \left(\bigcap_{j=1}^{J} \mathscr{E}^{\cap}(c_{t+1}^j; C_{t+1}^{j\mathrm{T}} \; C_{t+1}^j) \right), \tag{19}$$

and

$$\mathscr{E} = \mathscr{E}^{\cap}(c; M) = \mathscr{E}^{+}(c; P).$$

Thus, characterizing the set of possible values for x_t requires summing ellipsoids at the prediction step and intersecting ellipsoids at the correction step.

Then, we look for the smallest ellipsoid, in the sense of the criterion considered, that contains the sum or intersection of ellipsoids. The corresponding optimization problem is solved usually over a parametric family of ellipsoids, at the possible cost of suboptimality. In (14) and (18), the cost function is based on the logarithm of the determinant, to ensure suitable convexity properties. Note that Schweppe proposed already parametric families of ellipsoids containing the intersection or sum of ellipsoids, but without looking for the optimal ellipsoids in these families (Ref. 23).

3. Properties of Trace and Determinant

Solving minimization problems is much easier when they are convex and the first and second derivatives of their cost functions are available, which motivates what follows. Some proofs will use standard results of matrix analysis (Ref. 27), which are now recalled.

3.1. General Results. Let $\mathbb{D}$ be a convex subset of $\mathbb{R}^{n\times m}$, and let f be a function from $\mathbb{D}$ to $\mathbb{R}$. This function is convex over $\mathbb{D}$ if, for all $\lambda\in[0, 1]$ and all A and B in $\mathbb{D}$,

$$f(\lambda A+(1-\lambda)B)\leq\lambda f(A)+(1-\lambda)f(B).$$

It is strictly convex if the inequality is strict for $0<\lambda<1$ when $A\neq B$.

Let A be any interior point in $\mathbb{D}$, and let B be any point in $\mathbb{D}$. If f is such that, for any real ϵ,

$$f(A+\epsilon B)=f(A)+\epsilon\alpha_1+(1/2)\epsilon^2\alpha_2+o(\epsilon^2),$$

then f is twice directionally differentiable. Its first two derivatives at A in the direction B are

$$f'(A; B)=\alpha_1, \qquad f''(A; B)=\alpha_2.$$

For such a function, convexity is equivalent to $f''(A; B)\geq 0$, and strict convexity is guaranteed if $f''(A; B)>0$ for $B\neq 0$. If $f'(A; B)=\mathrm{tr}(CB)$, then $f(A)$ is differentiable and C is its first derivative: $f'(A)=C$.

Let $A(\lambda)\in\mathbb{D}$ be a differentiable function of a scalar parameter λ, and let $f(A)$ be a differentiable scalar function of A; then,

$$df(A(\lambda))/d\lambda=f'(A(\lambda); dA(\lambda)/d\lambda).$$

Let $A(\lambda)\in\mathbb{R}^{n\times n}$ and $B(\lambda)\in\mathbb{R}^{n\times n}$ be two differentiable functions of a scalar parameter λ; then,

$$d\,\mathrm{tr}\,A(\lambda)/d\lambda=\mathrm{tr}[dA(\lambda)/d\lambda],$$

$$d(A(\lambda)B(\lambda))/d\lambda=[dA(\lambda)/d\lambda]B(\lambda)+A(\lambda)[dB(\lambda)/d\lambda],$$

and if A is invertible,

$$dA^{-1}(\lambda)/d\lambda = -A^{-1}(\lambda)[dA(\lambda)/d\lambda]A^{-1}(\lambda).$$

3.2. Trace and Determinant Functions. Denote the linear space of all $n \times n$ real symmetric matrices by $\mathbb{S}_n$ and the subset of $\mathbb{S}_n$ consisting of the positive-definite matrices by $\mathbb{S}_n^+$.

Lemma 3.1. The function $f_t : \mathbb{S}_n^+ \to \mathbb{R}$ defined by $f_t(M) = \mathrm{tr}(M^{-1})$ is strictly convex; its first and second derivatives at $A \in \mathbb{S}_n^+$ along $B \in \mathbb{S}_n$ are

$$f'_t(A; B) = -\mathrm{tr}(A^{-1}BA^{-1}), \tag{20a}$$

$$f''_t(A; B) = 2\mathrm{tr}\,(A^{-1}BA^{-1}BA^{-1}), \tag{20b}$$

and its first derivative is

$$f'_t(A) = -A^{-2}. \tag{21}$$

Proof. A second-order Taylor expansion of f_t with respect to ϵ can be obtained as follows:

$$\begin{aligned} f_t(A + \epsilon B) &= \mathrm{tr}((I + \epsilon A^{-1}B)^{-1}A^{-1}) \\ &= \mathrm{tr}((I - \epsilon A^{-1}B + \epsilon^2 A^{-1}BA^{-1}B + o(\epsilon^2))A^{-1}) \\ &= \mathrm{tr}(A^{-1}) - \epsilon\, \mathrm{tr}(A^{-1}BA^{-1}) \\ &\quad + \epsilon^2\, \mathrm{tr}(A^{-1}BA^{-1}BA^{-1}) + o(\epsilon^2). \end{aligned} \tag{22}$$

From the results of Section 3.1, this implies (20).

Since $A \in \mathbb{S}_n^+$, it has a unique square root $A^{1/2}$. Take

$$C = A^{-1}BA^{-1}BA^{-1}, \qquad D = A^{-1/2}BA^{-1};$$

then,

$$C = D^{\mathrm{T}}D,$$

so $C \geq 0$. Thus, tr $C \geq 0$ and $f''_t(A; B) \geq 0$. Moreover, $C \neq 0$ for any $B \neq 0$; otherwise, y would be 0 for all x, which is possible only for $B = 0$. Therefore, tr $C > 0$ and $f''_t(A; B) > 0$, for all A in $\mathbb{S}_n^+$ and B in $\mathbb{S}_n$, which implies the strict convexity of f_t. Since

$$\mathrm{tr}(A^{-1}BA^{-1}) = \mathrm{tr}(A^{-2}B),$$

(21) is a direct application of Section 3.1. □

Lemma 3.2. The function $f_d : \mathbb{S}_n^+ \to \mathbb{R}$ defined by $f_d(M) = \log\det M^{-1} = -\log\det M$ is strictly convex, and its first and second derivatives at $A \in \mathbb{S}_n^+$ along $B \in \mathbb{S}_n$ are

$$f'_d(A; B) = -\mathrm{tr}(A^{-1}B), \tag{23a}$$

$$f''_d(A; B) = \mathrm{tr}(A^{-1}BA^{-1}B), \tag{23b}$$

and its first derivative is

$$f'_d(A) = -A^{-1}. \tag{24}$$

Proof. A proof of convexity can be found in Ref. 27, Theorem 7.6.7, and (23)–(24) are established in Ref. 26, Section 5.5.5. These results can also be established directly in the same way as for the trace function in Lemma 3.1. Indeed,

$$f_d(A + \epsilon B) = f_d(A) - \log\det(I + \epsilon A^{-1}B). \tag{25}$$

There exists a matrix P such that

$$T = P^{-1}A^{-1}BP$$

is triangular. Let λ_i, $i = 1, \ldots, n$, be the ith entry on the diagonal of T. Then,

$$\det(I + \epsilon A^{-1}B) = \det(I + \epsilon T) = \prod_{i=1}^{n} (1 + \epsilon\lambda_i)$$

and

$$\log\det(I + \epsilon A^{-1}B) = \epsilon \sum_{i=1}^{n} \lambda_i - (1/2)\epsilon^2 \sum_{i=1}^{n} \lambda_i^2 + o(\epsilon^2).$$

Moreover,

$$\sum_{i=1}^{n} \lambda_i = \mathrm{tr}\, T = \mathrm{tr}(A^{-1}B)$$

and

$$\sum_{i=1}^{n} \lambda_i^2 = \mathrm{tr}\, T^2 = \mathrm{tr}(A^{-1}BA^{-1}B),$$

so

$$f_d(A + \epsilon B) = f_d(A) - \epsilon\, \mathrm{tr}(A^{-1}B) + (1/2)\epsilon^2\, \mathrm{tr}(A^{-1}BA^{-1}B). \tag{26}$$

From Section 3.1, this implies (23). Let

$$C = A^{-1/2}BA^{-1/2};$$

then,

$$\mathrm{tr}(A^{-1}BA^{-1}B) = \mathrm{tr}(C^2),$$

with

$$C^2 = C^{\mathrm{T}}C \geq 0,$$

hence $\mathrm{tr}(C^2) \geq 0$ and $f''_d(A; B) \geq 0$. Moreover, $C \neq 0$ for any $B \neq 0$. Therefore, $\mathrm{tr}(C^2) > 0$ and $f''_d(A; B) > 0$ for all $B \neq 0$, which implies the strict convexity of f_d. Equation (24) is again a direct application of Section 3.1. □

3.3. Uniqueness of the Minimal Ellipsoid. Uniqueness is a well-known property of the minimal-trace or minimal-determinant ellipsoid containing a given compact set. For the determinant criterion, it corresponds to the Loewner–Behrend theorem (Ref. 28). For the trace criterion, an analogous result can be found in Ref. 29. The purpose of the present section is to give a simpler proof of this result for the trace criterion. For the determinant criterion, the proof can be established in the same way. A particular case of the following lemma will be used to prove the uniqueness of the minimal ellipsoid. The general case will be used when intersecting ellipsoids.

Lemma 3.3. If $c_k \in \mathbb{R}^n$, $M_k \geq 0$, and $\alpha_k \geq 0, k = 1, \ldots, K$, are such that $\sum_{k=1}^K \alpha_k M_k > 0$, then

$$\delta \stackrel{\Delta}{=} \sum_{k=1}^K \alpha_k c_k^{\mathrm{T}} M_k c_k - \left(\sum_{k=1}^K \alpha_k M_k c_k\right)^{\mathrm{T}} \left(\sum_{k=1}^K \alpha_k M_k\right)^{-1} \left(\sum_{k=1}^K \alpha_k M_k c_k\right) \geq 0. \tag{27}$$

Proof. For any $M > 0, c \in \mathbb{R}^n$, and $\gamma \in \mathbb{R}$, from the extension of the Schur inequality (Ref. 30), the following conditions are equivalent:

$$P = \begin{bmatrix} M & c \\ c^{\mathrm{T}} & \gamma \end{bmatrix} \geq 0 \Leftrightarrow \gamma \geq c^{\mathrm{T}} M^{-1} c. \tag{28}$$

Take

$$M = \sum_{k=1}^K \alpha_k M_k, \qquad c = \sum_{k=1}^K \alpha_k M_k c_k, \qquad \gamma = \sum_{k=1}^K \alpha_k c_k^{\mathrm{T}} M_k c_k;$$

then,

$$\delta = \gamma - c^{\mathrm{T}} M^{-1} c.$$

From (28), $\delta \geq 0$ can be rewritten as $P \geq 0$. Take $x^T = (x_1^T, x_2^T)$ to get

$$x^T P x = \sum_{k=1}^{K} \alpha_k (x_1 - x_2 c_k)^T M_k (x_1 - x_2 c_k). \tag{29}$$

So,

$$x^T P x \geq 0, \qquad \text{for all } x,$$

which proves that $P \geq 0$ and thus that $\delta \geq 0$. □

Theorem 3.1. Let $\mathscr{C} \subset \mathbb{R}^n$ be a compact set with a nonempty interior, and let $\mathscr{E} = \mathscr{E}(c; M) \subset \mathbb{R}^n$ be an ellipsoid defined as in (7). Then, each of the minimization problems $\min_{\mathscr{E} \supset \mathscr{C}} \operatorname{tr} M^{-1}$ (Problem T) and $\min_{\mathscr{E} \supset \mathscr{C}} \log\det M^{-1}$ (Problem D) has a unique solution.

Proof. Only the proof for the trace criterion will be detailed. First, let us prove that a solution exists. Since $\mathscr{C}$ has a nonempty interior, it contains a ball $\{x \in \mathbb{R}^n : \|x - c_0\| \leq r_{\min}\}$ with $r_{\min} > 0$ and $\|M\| \leq r_{\min}^{-2}$, where $\|M\|$ is the (operator) norm of M, equal to its maximum eigenvalue. The boundedness of $\mathscr{C}$ implies that there exist $d_{\max}$ such that

$$\|c_0\| \leq d_{\max}$$

and $r_{\max}$ such that

$$\|M^{-1}\| \leq r_{\max}^2.$$

Since $c_0 \in \mathscr{E}$, we have

$$\|c_0 - c\| \leq r_{\max}, \qquad \|c\| \leq r_{\max} + d_{\max} = d_0.$$

Hence, Problems T and D can be considered subject to the extra constraints

$$\|M\| \leq r_{\min}^{-2}, \; \|M^{-1}\| \leq r_{\max}^2, \qquad \|c\| \leq d_0,$$

which define a compact set. Now, the cost functions tr M^{-1} and log det M^{-1} are continuous on this set since they are differentiable. Therefore, by standard continuity considerations, Problems T and D possess solutions.

Assume now that there are two distinct solutions of Problem T or D, namely

$$\mathscr{E}_1 = \mathscr{E}(c_1; M_1), \qquad \mathscr{E}_2 = \mathscr{E}(c_2; M_2).$$

Then, for any x in $\mathscr{C}$, one has

$$(1/2) \sum_{k=1}^{2} \|M_k^{1/2}(x - c_k)\|^2 \leq 1.$$

This inequality is equivalent to

$$(x-c)^{\mathrm{T}}M(x-c)\leq 1-\delta,$$

with

$$M=(1/2)(M_1+M_2),$$

$$c=(1/2)M^{-1}(M_1c_1+M_2c_2),$$

$$\delta=(1/2)(c_1^{\mathrm{T}}M_1c_1+c_2^{\mathrm{T}}M_2c_2)-c^{\mathrm{T}}Mc.$$

Since

$$(x-c)^{\mathrm{T}}M(x-c)\geq 0,$$

and since $\mathscr{C}$ is a nonempty set, $\delta<1$. Lemma 3.3 with $K=2$ and $\alpha_1=\alpha_2=1/2$ implies that $\delta\geq 0$. Therefore, $\mathscr{C}$ is contained in $\mathscr{E}(c;M)$, but also in $\mathscr{E}(c;(1-\delta)^{-1}M)$.

For the trace criterion, the two solutions satisfy

$$\operatorname{tr} M_1^{-1}=\operatorname{tr} M_2^{-1}=t^*.$$

The strict convexity of the trace function implies that

$$\operatorname{tr}((1-\delta)M^{-1})\leq(1-\delta)t^*,$$

the equality being satisfied only if

$$M_1=M_2.$$

So the optimality of M_1 and M_2 implies that $M_1=M_2$ and $\delta=0$. Now, if $M_1=M_2=M$, then

$$\delta=(1/4)(c_1-c_2)^{\mathrm{T}}M(c_1-c_2),$$

so $\delta=0$ implies that $c_1=c_2$; hence, $\mathscr{E}_1=\mathscr{E}_2$. Thus, the minimum-trace ellipsoid is unique. □

4. Sum

The ellipsoids considered in this section are bounded but may have empty interiors; this is why they will be described in the form (10). Let

$$\mathscr{E}=\mathscr{E}^+(c;P).$$

Given K ellipsoids of $\mathbb{R}^n$

$$\mathscr{E}_k=\mathscr{E}^+(c_k;P_k),\qquad k=1,\ldots,K,$$

and their sum

$$\mathcal{M}_K = \sum_{k=1}^{K} \mathcal{E}_k,$$

which is a convex set, the problem is then to find either

$$\mathcal{E}^* = \arg\min_{\mathcal{E} \supset \mathcal{M}_K} \operatorname{tr} P \quad \text{(Problem T}^+\text{)}$$

or

$$\mathcal{E}^* = \arg\min_{\mathcal{E} \supset \mathcal{M}_K} \log\det P \quad \text{(Problem D}^+\text{)}.$$

From Theorem 3.1, this ellipsoid exists and is unique.

Theorem 4.1. The center of the optimal ellipsoid $\mathcal{E}^*$ for both Problems T^+ and D^+ is given by

$$c^* = \sum_{k=1}^{K} c_k. \tag{30}$$

Proof. The support function $s_{\mathcal{E}}: \mathbb{R}^n \to \mathbb{R}$ of $\mathcal{E} = \mathcal{E}^+(c; P)$ is given by (Ref. 23)

$$s_{\mathcal{E}}(\eta) = \max_{x \in \mathcal{E}} \eta^{\mathrm{T}} x = \eta^{\mathrm{T}} c + \sqrt{\eta^{\mathrm{T}} P \eta}. \tag{31}$$

Since the support function of a sum of convex sets is the sum of the support functions of each of them, the support function of $\mathcal{M}_K$ is

$$s_{\mathcal{M}_K}(\eta) = \eta^{\mathrm{T}} \sum_{k=1}^{K} c_k + \sum_{k=1}^{K} \sqrt{\eta^{\mathrm{T}} P_k \eta}. \tag{32}$$

Moreover, $\mathcal{E} = \mathcal{E}^+(c; P)$ contains $\mathcal{M}_K$ if and only if

$$s_{\mathcal{E}}(\eta) \geq s_{\mathcal{M}_K}(\eta), \qquad \text{for all } \eta.$$

So, a necessary and sufficient condition to have $\mathcal{M}_K \subset \mathcal{E}$ is that

$$\eta^{\mathrm{T}} c + \sqrt{\eta^{\mathrm{T}} P \eta} \geq \eta^{\mathrm{T}} \sum_{k=1}^{K} c_k + \sum_{k=1}^{K} \sqrt{\eta^{\mathrm{T}} P_k \eta}, \qquad \forall \eta \in \mathbb{R}^n. \tag{33}$$

Assume that the ellipsoid $\mathcal{E}^+(c_+; P)$, with

$$c_+ = \sum_{k=1}^{K} c_k + \Delta,$$

contains $\mathscr{M}_K$. Equation (33) can then be rewritten as

$$\eta^{\mathrm{T}}\Delta + \sqrt{\eta^{\mathrm{T}} P \eta} \geq \sum_{k=1}^{K} \sqrt{\eta^{\mathrm{T}} P_k \eta}, \qquad \forall \eta \in \mathbb{R}^n. \tag{34}$$

Replace η by $-\eta$ in (34) to get

$$-\eta^{\mathrm{T}}\Delta + \sqrt{\eta^{\mathrm{T}} P \eta} \geq \sum_{k=1}^{K} \sqrt{\eta^{\mathrm{T}} P_k \eta}, \qquad \forall \eta \in \mathbb{R}^n, \tag{35}$$

which implies that $\mathscr{E}^+(c_-; P)$, with

$$c_- = \sum_{k=1}^{K} c_k - \Delta,$$

also contains $\mathscr{M}_K$. Since P takes the same value in $\mathscr{E}^+(c_+; P)$ and $\mathscr{E}^+(c_-; P)$, they are both equally optimal, which is in contradiction with the uniqueness of the minimal determinant or trace ellipsoid containing $\mathscr{M}_K$, unless $\Delta = 0$. So, the center of $\mathscr{E}^*$ is given by

$$c^* = \sum_{k=1}^{K} c_k. \qquad \square$$

The following theorem provides a parametrized family of ellipsoids over which optimization can be carried out. It is a slight modification of a result given in Ref. 23, Exercise 4.14.

Theorem 4.2. Let $\mathscr{D}^{+*}$ be the convex set of all vectors $\alpha \in \mathbb{R}^K$ with all $\alpha_k > 0$ and $\sum_{k=1}^{K} \alpha_k = 1$. For any $\alpha \in \mathscr{D}^{+*}$, the ellipsoid $\mathscr{E}_\alpha = \mathscr{E}^+(c^*; P_\alpha)$, with c^* defined by (30) and

$$P_\alpha = \sum_{k=1}^{K} \alpha_k^{-1} P_k, \tag{36}$$

contains $\mathscr{M}_K$.

Proof. A necessary and sufficient condition to have

$$\mathscr{M}_K \subset \mathscr{E}_\alpha = \mathscr{E}^+(c^*; P_\alpha)$$

is given by

$$s_{\mathscr{E}_\alpha}(\eta) \geq s_{\mathscr{M}_K}(\eta), \qquad \text{for all } \eta.$$

From (33) with $c = c^*$ defined as in (30) and $P = P_\alpha$ in (36), this condition is equivalent to

$$\sqrt{\eta^T P_\alpha \eta} \geq \sum_{k=1}^{K} \sqrt{\eta^T P_k \eta}, \qquad \text{for all } \eta.$$

This condition is also equivalent to

$$\sqrt{\left(\sum_{k=1}^{K} a_k^2\right)\left(\sum_{k=1}^{K} b_k^2\right)} \geq \sum_{k=1}^{K} a_k b_k, \tag{37}$$

with

$$a_k = \alpha_k^{1/2}, \qquad b_k = \alpha_k^{-1/2} \sqrt{\eta^T P_k \eta},$$

which is satisfied for any $\alpha \in \mathscr{D}^{+*}$ as a trivial consequence of the Schwarz inequality; so, $\mathscr{M}_K \subset \mathscr{E}_\alpha$. □

Note that the center of $\mathscr{E}_\alpha$ is optimal according to Theorem 4.1 and does not depend on α or the measure of size considered. In what follows, the optimal ellipsoid $\mathscr{E}^*$ of Problem T^+ or D^+ will be approximated by the optimal ellipsoid $\mathscr{E}_{\alpha*}$ in the parametrized family $\mathscr{E}_\alpha$, which leads to computing either

$$\alpha^* = \arg\min_{\alpha \in \mathscr{D}^{+*}} \operatorname{tr} P_\alpha \quad (\text{Problem } T_\alpha^+)$$

or

$$\alpha^* = \arg\min_{\alpha \in \mathscr{D}^{+*}} \log\det P_\alpha \quad (\text{Problem } D_\alpha^+).$$

Theorem 4.3. The optimization problems T_α^+ and D_α^+ are convex and their cost functions are twice differentiable. Let $\mathscr{D}^*$ be the set of all $\alpha \in \mathbb{R}^K$, with $\alpha_k > 0, k = 1, \ldots, K$, which contains $\mathscr{D}^{+*}$. Define P_α as in (36), but with $\alpha \in \mathscr{D}^*$. Let $\varphi_t(\alpha) = \operatorname{tr} P_\alpha$. The ith entry of its gradient is given by

$$\partial\varphi_t(\alpha)/\partial\alpha_i = -\alpha_i^{-2} \operatorname{tr} P_i, \tag{38}$$

and its Hessian H_t is diagonal, with

$$H_t(\alpha) = 2\,\operatorname{diag}(\operatorname{tr} P_1/\alpha_1^3, \ldots, \operatorname{tr} P_K/\alpha_K^3). \tag{39}$$

Let $\varphi_d(\alpha) = \log\det P_\alpha$. The ith entry of its gradient is given by

$$\partial\varphi_d(\alpha)/\partial\alpha_i = -\alpha_i^{-2} \operatorname{tr}(P_\alpha^{-1} P_i), \tag{40}$$

and the entries of its Hessian H_d are given by

$$\partial^2\varphi_d(\alpha)/\partial\alpha_i^2 = 2\alpha_i^{-3} \operatorname{tr}(P_\alpha^{-1} P_i) - \alpha_i^{-4} \operatorname{tr}(P_\alpha^{-1} P_i P_\alpha^{-1} P_i), \tag{41a}$$

$$\partial^2\varphi_d(\alpha)/\partial\alpha_i\partial\alpha_j = -\alpha_i^{-2}\alpha_j^{-2} \operatorname{tr}(P_\alpha^{-1} P_i P_\alpha^{-1} P_j), \qquad \forall i \neq j. \tag{41b}$$

Proof. Equations (38) and (39) are obtained by direct calculation. Since

$$\varphi_d(\alpha) = -\log \det P_\alpha^{-1},$$

the results of Section 3.1 with $f' = \varphi'_d$ can be used to establish (40)–(41). Since $P_k \geq 0$ and $P_k \neq 0$,

$$\operatorname{tr} P_k > 0, \qquad k = 1, \ldots, K.$$

Therefore, $H_t(\alpha)$ as given by (39) is a strictly positive-definite matrix and $\varphi_t(\alpha)$ is strictly convex over $\mathscr{D}^*$ and thus over $\mathscr{D}^{+*}$. It follows that Problem T_α^+ is convex. The demonstration for Problem D_α^+ is more complicated and will be detailed in Section 7.2. □

The following result will be used to prove that replacing Problem T^+ or D^+ by Problem T_α^+ or D_α^+ yields often a suboptimal solution of the initial problem.

Lemma 4.1. A necessary condition for an ellipsoid $\mathscr{E}_{\alpha^*}$ to be an optimal solution of Problem T^+ or D^+ is that there exists η such that $\alpha_k^{*-2}\eta^{\mathrm{T}} P_k \eta$ does not depend on k.

Proof. A necessary optimality condition is that there exist contact points between $\mathscr{E}_{\alpha^*}$ and $\mathscr{M}_K$. These contact points correspond to vectors η such that

$$s_{\mathscr{E}_{\alpha^*}}(\eta) = s_{\mathscr{M}_K}(\eta).$$

The proof of Theorem 4.2 shows that this is satisfied if and only if $a_k = \alpha_k^{*1/2}$ is proportional to $b_k = \alpha_k^{*-1/2}\sqrt{\eta^{\mathrm{T}} P_k \eta}$ for all k, i.e., if and only if $\alpha_k^{*-2}\eta^{\mathrm{T}} P_k \eta$ does not depend on k for the values of η that correspond to contact points. □

The next two subsections address the optimization Problems T_α^+ and D_α^+ in turn.

4.1. Trace Criterion. A considerable advantage of the trace criterion is that an explicit solution for α^* can be given, and the following theorem is the main result of this section.

Theorem 4.4. In the family $\mathscr{E}_\alpha = \mathscr{E}^+(c^*; P_\alpha)$, the minimal-trace ellipsoid containing the sum of the ellipsoids $\mathscr{E}_k = \mathscr{E}^+(c_k; P_k)$, $k = 1, \ldots, K$, is obtained for

$$P_{\alpha^*} = \left(\sum_{k=1}^{K} \sqrt{\operatorname{tr} P_k} \right) \left(\sum_{k=1}^{K} P_k / \sqrt{\operatorname{tr} P_k} \right). \tag{42}$$

Proof. Let

$$t_k = \operatorname{tr} P_k, \qquad t_\alpha = \operatorname{tr} P_\alpha,$$

with

$$P_\alpha = \sum_{k=1}^{K} \alpha_k^{-1} P_k, \qquad \alpha \in \mathscr{D}^{+*};$$

thus,

$$t_\alpha = \sum_{k=1}^{K} \alpha_k^{-1} t_k.$$

Problem T_α^+ can be solved easily by introducing a Lagrange multiplier λ and minimizing

$$L(\alpha) = \sum_{k=1}^{K} \alpha_k^{-1} t_k + \lambda \left(\sum_{k=1}^{K} \alpha_k - 1 \right).$$

A necessary condition for $L(\alpha^*)$ to be a minimum is that

$$\partial L / \partial \alpha_k = 0, \qquad k = 1, \ldots, K,$$

at $\alpha = \alpha^*$. This implies that

$$\alpha_k^* = \lambda^{-1/2} \sqrt{t_k}.$$

Then, the constraint

$$\sum_{k=1}^{K} \alpha_k^* = 1$$

leads to

$$\sqrt{\lambda} = \sum_{k=1}^{K} \sqrt{t_k},$$

so

$$\alpha_k^* = \left(\sum_{k=1}^{K} \sqrt{t_k} \right)^{-1} \sqrt{t_k}, \qquad k = 1, \ldots, K.$$

Equation (42) is finally obtained by replacing α in (36) by its optimal value α^*. □

Similar results could be derived with the Chernousko parametrization (Refs. 10, 18, 29, 31). The next result shows that the solution is not necessarily optimal.

Proposition 4.1. In general, the ellipsoid obtained by solving Problem T^+_α is only suboptimal for Problem T^+.

Proof. Take

$$P_1 = dd^T,$$

with

$$d = (1, 1)^T, \qquad P_2 = \mathrm{diag}(1, 0), \qquad P_3 = \mathrm{diag}(0, 1),$$

so

$$\alpha_1^* = (\sqrt{2} + 2)^{-1}\sqrt{2}, \qquad \alpha_2^* = \alpha_3^* = (\sqrt{2} + 2)^{-1}.$$

Then, the condition of Lemma 4.1 becomes

$$2^{-1/2}|\eta_1 + \eta_2| = |\eta_1| = |\eta_2|,$$

and it is impossible to find η such that this condition is satisfied. □

The main advantage of Theorem 4.4 is the explicit form of the solution. Another benefit of this solution is its transitive nature, which makes it easy to consider a recursive version of Problem T^+_α when the ellipsoids $\mathscr{E}_k$ are made available one after the other. Suppose that the approximating ellipsoid $\mathscr{E}^r_k = \mathscr{E}^+(c^r_k; P^r_k)$ has been obtained after processing the first k ellipsoids $\mathscr{E}_1, \ldots, \mathscr{E}_k$. The next approximation is to find $\mathscr{E}^r_{k+1} = \mathscr{E}^+(c^r_{k+1}; P^r_{k+1})$ containing $\mathscr{E}^r_k + \mathscr{E}_{k+1}$. From (42) for $K = 2$, one gets the recursive algorithm

$$c^r_{k+1} = c^r_k + c_{k+1}, \tag{43a}$$

$$P^r_{k+1} = (\sqrt{\mathrm{tr}\, P^r_k} + \sqrt{\mathrm{tr}\, P_{k+1}}) \times (P^r_k / \sqrt{\mathrm{tr}\, P^r_k} + P_{k+1} / \sqrt{\mathrm{tr}\, P_{k+1}}), \tag{43b}$$

initialized at $c^r_1 = c_1$ and $P^r_1 = P_1$.

Theorem 4.5. Once all the ellipsoids of the sum have been taken into consideration, the recursive and nonrecursive algorithms generate the same approximating ellipsoid.

Proof. This is shown by direct calculation. □

Note that the Kalman filter shares this property of equivalence of the results obtained recursively and nonrecursively.

4.2. Determinant Criterion. Contrary to the minimum-trace case, no general explicit solution is available for the minimal-determinant approximation. The optimal value of α is obtained by solving a convex optimization problem of dimension $K-1$ (Theorem 4.3). Then, standard iterative methods for solving convex constrained optimization problems can be applied, such as gradient projection, conditional gradient, and constrained Newton methods (Ref. 32). A damped Newton method for self-concordant functions (Ref. 26) can also be applied.

Proposition 4.2. In general, the ellipsoid obtained by solving Problem D_α^+ is only suboptimal for Problem D^+.

Proof. Consider the same example as in the proof of Proposition 4.1. The symmetry of the problem implies

$$\alpha_2^* = \alpha_3^* = \beta$$

and then the constraint $\alpha \in \mathscr{D}^{+*}$ implies

$$\alpha_1^* = 1 - 2\beta.$$

Solving Problem D_α^+ with respect to β yields

$$\beta = 1/3.$$

Then, the condition of optimality in Lemma 4.1 becomes

$$|\eta_1 + \eta_2| = |\eta_1| = |\eta_2|,$$

and it is impossible to find η such that this condition is satisfied. □

The recursive algorithm reads

$$c_{k+1}^r = c_k^r + c_{k+1}, \tag{44a}$$

$$P_{k+1}^r = \alpha_k^{*\,-1} P_k^r + (1 - \alpha_k^*)^{-1} P_{k+1}, \tag{44b}$$

with

$$\alpha_k^* = \arg\min_{0<\alpha<1} \log\det(\alpha^{-1} P_k^r + (1 - \alpha)^{-1} P_{k+1}). \tag{45}$$

It is initialized at $c_1^r = c_1$ and $P_1^r = P_1$.

The next proposition will be useful for the prediction step of the state estimation in the special case where there is a single scalar bounded input, as it gives an explicit expression for α_k^* in (45).

Proposition 4.3. When $P_{k+1} = d_{k+1} d_{k+1}^{\mathrm{T}}$, with d_{k+1} a vector, $\mathscr{E}_{k+1}$ is an interval. Let $\gamma = d_{k+1}^{\mathrm{T}} (P_k^r)^{-1} d_{k+1}$. If $\gamma \neq 1$, then

$$\alpha_k^* = [\gamma(1-n) + 2n - \sqrt{\gamma^2(1-n)^2 + 4\gamma n}]/[2n(1-\gamma)]; \tag{46}$$

else, if $\gamma = 1$, then

$$\alpha_k^* = n/(n+1).$$

Proof. Let

$$l(\alpha) = \log\det(\alpha^{-1} P_k^r + (1-\alpha)^{-1} d_{k+1} d_{k+1}^{\mathrm{T}}).$$

Since

$$\det(I + uv^{\mathrm{T}}) = 1 + u^{\mathrm{T}} v,$$

it is easy to establish that

$$l(\alpha) = \log\det P_k^r - n\log\alpha + \log(1 + \alpha(1-\alpha)^{-1}\gamma).$$

A necessary condition for $l(\alpha)$ to be minimum at α_k^* is that $l'(\alpha_k^*) = 0$. The numerator of $l'(\alpha)$ is

$$m(\alpha) = n(\gamma - 1)\alpha^2 + (\gamma(1-n) + 2n)\alpha - n.$$

Note that $m(0) = -n < 0$ and $m(1) = \gamma > 0$. Thus, the second-order polynomial equation $m(\alpha) = 0$ has only one solution for α in $]0, 1[$, which is trivial to obtain and corresponds to α_k^* as given by the proposition. □

5. Intersection

Some of the ellipsoids considered in this section may be unbounded but have nonempty interiors; this is why they will be described in the form (12).

Let

$$\mathscr{E} = \mathscr{E}^{\cap}(c; M).$$

Given K ellipsoids

$$\mathscr{E}_k = \mathscr{E}^{\cap}(c_k; M_k) \in \mathbb{R}^n, \qquad k = 1, \ldots, K,$$

and given

$$\mathscr{I}_K = \bigcap_{k=1}^{K} \mathscr{E}_k,$$

which is a convex set, the problem is then to find either

$$\mathscr{E}^* = \arg\min_{\mathscr{E} \subset \mathscr{I}_K} \operatorname{tr} M^{-1} \text{ (Problem T}^{\cap}\text{)}$$

or

$$\mathscr{E}^* = \arg\min_{\mathscr{E} \supset \mathscr{I}_K} \log\det M^{-1} \text{ (Problem D}^{\cap}\text{).}$$

From Lemma 3.1, this ellipsoid exists and is unique.

Theorem 5.1. Let $\mathscr{D}^+$ be the convex set of all vectors $\alpha \in \mathbb{R}^K$, with $\alpha_k \geq 0, k = 1, \ldots, K$, and $\sum_{k=1}^K \alpha_k = 1$. Take any $\alpha \in \mathscr{D}^+$ such that $\sum_{k=1}^K \alpha_k M_k > 0$. Define

$$M_\alpha = \sum_{k=1}^{K} \alpha_k M_k, \qquad c_\alpha = M_\alpha^{-1} \sum_{k=1}^{K} \alpha_k M_k c_k,$$

$$\delta_\alpha = \sum_{k=1}^{K} \alpha_k c_k^{\mathrm{T}} M_k c_k - c_\alpha^{\mathrm{T}} M_\alpha c_\alpha.$$

Then, the ellipsoid

$$\mathscr{E}_\alpha = \mathscr{E}^{\cap}(c_\alpha; (1-\delta_\alpha)^{-1} M_\alpha)$$

contains $\mathscr{I}_K$.

Proof. Let

$$\varphi_k(x) = (x - c_k)^{\mathrm{T}} M_k (x - c_k), \qquad k = 1, \ldots, K;$$

then,

$$\mathscr{I}_K = \{x \in \mathbb{R}^n : \varphi_k(x) \leq 1, k = 1, \ldots, K\}.$$

Of course, if $x \in \mathscr{I}_K$, then

$$\varphi(x) = \sum_{k=1}^{K} \alpha_k \varphi_k(x) \leq 1, \qquad \text{for any } \alpha \in \mathscr{D}^+.$$

After simple transformations, $\varphi(x)$ can be written as

$$\varphi(x) = (x - c_\alpha)^{\mathrm{T}} M_\alpha (x - c_\alpha) + \delta_\alpha;$$

then, $\mathscr{I}_K$ can be rewritten as

$$\mathscr{I}_K = \{x \in \mathbb{R}^n: (x - c_\alpha)^T M_\alpha (x - c_\alpha) \leq 1 - \delta_\alpha\}.$$

Since $\mathscr{I}_K$ is a nonempty set, $\delta_\alpha < 1$ and thus,

$$\mathscr{I}_K \subset \mathscr{E}^\cap(c_\alpha; (1 - \delta_\alpha)^{-1} M_\alpha). \qquad \square$$

Proposition 5.1. For any $\alpha \in \mathscr{D}^+$, the ellipsoid $\mathscr{E}^\cap(c_\alpha; M_\alpha)$ contains also $\mathscr{I}_K$.

Proof. One has $\delta_\alpha < 1$ and, from Lemma 3.3, $\delta_\alpha \geq 0$; thus, $\delta_\alpha \in [0, 1[$ and $\mathscr{E}^\cap(c_\alpha; M_\alpha)$ contains $\mathscr{E}^\cap(c_\alpha; (1 - \delta_\alpha)^{-1} M_\alpha)$. $\square$

Thus, two parametrized families of outer ellipsoids may be considered to find the optimal value of α, namely

$$\mathscr{E}_\alpha = \mathscr{E}^\cap(c_\alpha; (1 - \delta_\alpha)^{-1} M_\alpha)$$

and

$$\mathscr{E}'_\alpha = \mathscr{E}^\cap(c_\alpha; M_\alpha).$$

This leads to four problems depending on the criterion and the family considered:

$$\alpha^* = \arg\min_{\alpha \in \mathscr{D}^+} \operatorname{tr}((1 - \delta_\alpha)^{-1} M_\alpha)^{-1} \text{ (Problem } \mathrm{T}_\alpha^\cap), \tag{47a}$$

$$\alpha^* = \arg\min_{\alpha \in \mathscr{D}^+} \operatorname{tr} M_\alpha^{-1} \text{ (Problem } \mathrm{T'}_\alpha^\cap), \tag{47b}$$

$$\alpha^* = \arg\min_{\alpha \in \mathscr{D}^+} \log\det((1 - \delta_\alpha)^{-1} M_\alpha)^{-1} \text{ (Problem } \mathrm{D}_\alpha^\cap), \tag{47c}$$

$$\alpha^* = \arg\min_{\alpha \in \mathscr{D}^+} \log\det M_\alpha^{-1} \text{ (Problem } \mathrm{D'}_\alpha^\cap). \tag{47d}$$

Optimization within the family $\mathscr{E}_\alpha$ leads to a better value of the criterion considered and so to a better ellipsoid than the one obtained with the family $\mathscr{E}'_\alpha$, at the cost of more computation. Even if the optimization of α takes place within the family $\mathscr{E}'_\alpha$, the final ellipsoidal approximation is improved by taking

$$\mathscr{E}_{\alpha^*} = \mathscr{E}^\cap(c_{\alpha^*}; (1 - \delta_{\alpha^*})^{-1} M_{\alpha^*}).$$

Then, in most examples treated so far, the improvement obtained by optimizing within the family $\mathscr{E}_\alpha$ becomes marginal.

Theorem 5.2. The optimization problems $T'^{\cap}_{\alpha}$ and $D'^{\cap}_{\alpha}$ are convex and their cost functions are twice differentiable. Let $\mathscr{D}$ be the set of all $\alpha \in \mathbb{R}^K$, with $\alpha_k \geq 0, k = 1, \ldots, K$, which contains $\mathscr{D}^+$. Define M_α as in Theorem 5.1, but with $\alpha \in \mathscr{D}$. Let $\varphi_t(\alpha) = \operatorname{tr} M_\alpha^{-1}$. The ith entry of its gradient is given by

$$\partial \varphi_t(\alpha)/\partial \alpha_i = -\operatorname{tr} M_\alpha^{-1} M_i M_\alpha^{-1}, \tag{48}$$

and the entries of its Hessian are given by

$$\partial^2 \varphi_t(\alpha)/\partial \alpha_i \partial \alpha_j = 2 \operatorname{tr} M_\alpha^{-1} M_i M_\alpha^{-1} M_j M_\alpha^{-1}. \tag{49}$$

Let $\varphi_d(\alpha) = \log \det M_\alpha^{-1}$. The ith entry of its gradient is given by

$$\partial \varphi_d(\alpha)/\partial \alpha_i = -\operatorname{tr} M_\alpha^{-1} M_i, \tag{50}$$

and the entries of its Hessian are given by

$$\partial^2 \varphi_d(\alpha)/\partial \alpha_i \partial \alpha_j = \operatorname{tr} M_\alpha^{-1} M_i M_\alpha^{-1} M_j. \tag{51}$$

Proof. These relations are obtained by applying the results of Section 3.1 with $f' = \varphi'_t$ or $f' = \varphi'_d$ given by (20) or (23).

From Lemma 3.1, tr M^{-1} is a strictly convex function, $\varphi_t(\alpha)$ is strictly convex over $\mathscr{D}$ and thus over $\mathscr{D}^+$, so Problem $T'^{\cap}_{\alpha}$ is convex. The function $\log \det M^{-1}$ is also strictly convex and a similar reasoning proves that Problem $D'^{\cap}_{\alpha}$ is also convex. □

The same type of optimization method as mentioned in Section 4.2 can be employed. Problems $T'^{\cap}_{\alpha}$ and $D'^{\cap}_{\alpha}$ are similar from a computational point of view; none of them yields an explicit solution in general.

Consider now a recursive version of the problems in (47). Let

$$\mathscr{E}^r_k = \mathscr{E}^{\cap}(c^r_k; M^r_k)$$

be the approximate ellipsoid obtained after processing the first k ellipsoids $\mathscr{E}_1, \ldots, \mathscr{E}_k$. The next approximation is to find

$$\mathscr{E}^r_{k+1} = \mathscr{E}^{\cap}(c^r_{k+1}; (1-\delta_\alpha)^{-1} M^r_{k+1})$$

containing $\mathscr{E}^r_k \cap \mathscr{E}_{k+1}$. From Theorem 5.1, the following recursive algorithm can be obtained:

$$M^r_{k+1} = M^r_{k+1}(\alpha^*_{k+1}), \tag{52a}$$

$$c^r_{k+1} = c^r_{k+1}(\alpha^*_{k+1}), \tag{52b}$$

where

$$\alpha^*_{k+1} = \arg \min_{0 \leq \alpha \leq 1} \varphi(\alpha),$$

with

$$\varphi(\alpha) = \mathrm{tr}((1-\delta_\alpha)^{-1} M^r_{k+1})^{-1},$$

$$\varphi(\alpha) = \mathrm{tr}(M^r_{k+1})^{-1},$$

$$\varphi(\alpha) = \log\det((1-\delta_\alpha)^{-1} M^r_{k+1})^{-1},$$

or

$$\varphi(\alpha) = \log\det(M^r_{k+1})^{-1},$$

depending on the criterion and family considered, and where

$$M^r_{k+1}(\alpha) = \alpha M^r_k + (1-\alpha) M_{k+1}, \tag{53a}$$

$$c^r_{k+1}(\alpha) = (M^r_{k+1}(\alpha))^{-1}(\alpha M^r_k c^r_k + (1-\alpha) M_{k+1} c_{k+1}), \tag{53b}$$

$$\delta_\alpha = \alpha c^{r\mathrm{T}}_k M^r_k c^r_k + (1-\alpha) c^{\mathrm{T}}_{k+1} M_{k+1} c_{k+1} - c^{r\mathrm{T}}_{k+1}(\alpha) M^r_{k+1}(\alpha) c^r_{k+1}(\alpha). \tag{53c}$$

The algorithm is initialized at $c^r_1 = c_1$ and $M^r_1 = M_1$. It generates a more pessimistic final ellipsoid than the nonrecursive algorithm.

An important special case is when $\mathscr{E}_{k+1}$ is a strip,

$$\mathscr{E}_{k+1} = \mathscr{S}(y_{k+1}; d_{k+1}),$$

because an expression of α^* over the better parametrized family $\mathscr{E}_\alpha$ can then be established. This corresponds to the well-known results of Fogel and Huang (Ref. 17). The trace criterion requires finding the unique feasible solution of a cubic equation. With the determinant criterion, this equation is only quadratic. When one limiting hyperplane of the strip does not cut into $\mathscr{E}^r_k$, it is well-known that translating the nonintersecting hyperplane to make it tangent to the intersected ellipsoid makes it possible to obtain a better result (Ref. 16), which turns out to be optimal (Ref. 33). This confirms that the solution obtained even with the best parametrized family is sometimes suboptimal. This strip tightening has the additional advantage (Ref. 34) of making useless a test of the algorithm for the trace criterion proposed by Fogel and Huang. The complexity of both criteria then becomes equivalent for the intersection of an ellipsoid and a strip.

6. Conclusions

Algorithms for the ellipsoidal approximation of the sum and intersection of K ellipsoids are building blocks for a multi-input multi-output bounding counterpart to the Kalman filtering. Those presented in this paper can accommodate situations where the perturbations and measurement noise are multidimensional and consist of independent subvectors. Since sets of ellipsoidal constraints can be taken into account, the method is not limited to linear inequalities. At each step, the ellipsoid is chosen optimally in a family, each member of which is guaranteed to contain the set to be characterized. However, the optimal ellipsoid that would be obtained without constraints is not guaranteed to belong to this family, so the overall result may be suboptimal.

To measure the size of the ellipsoids obtained, and thus the quality of the approximation, it turns out that the trace criterion has several advantages over the determinant criterion more classically used. The trace criterion is less prone to yielding narrow ellipsoids with small volumes but large parameter uncertainty intervals. In the case of the summation of ellipsoids during the prediction phase, and contrary to the determinant criterion, it leads to an explicit and transitive solution, which means that ellipsoids can be added successively, with the same final result as if they were all taken into account simultaneously. This is especially interesting if several prediction steps must take place before the occurrence of a measurement allowing a correction step. No explicit solution is available in general for the intersection of ellipsoids during the correction phase with either criterion. One of the two parametrized families that have been proposed here has the advantage of leading to convex problems, with a seemingly marginal decrease in performance. The extension of the approach to deal with uncertainty in the matrices A, B, C, D is under investigation.

7. Appendix

7.1. Reformulation of the Observation Equation. The purpose of this section is to show how the observation equation (1b) can be rewritten as (3b). First, let

$$w_t^j \in \mathbb{R}^{q_t^j}, \qquad j = 1, \ldots, J,$$

be the independent subvectors of w_t assumed to belong to known ellipsoids, and let $y_t^j \in \mathbb{R}^{r_t^j}$ be the r_t^j components of y_t corrupted by w_t^j. Then, the observation equation (1b) can be rewritten as

$$y_t^j = C_t^j x_t + W_t^j w_t^j, \qquad j = 1, \ldots, J, \tag{54}$$

where $W_t^j w_t^j$ belongs to a known ellipsoid, C_t^j and W_t^j being trivially deduced from C_t and W_t. A suitable transformation of W_t^j makes it possible always to impose that w_t^j belongs to the unit ball $\mathscr{B}^{q_t^j}$ and (54) becomes

$$y_t^j = C_t^j x_t + W_t^j w_t^j, \qquad w_t^j \in \mathscr{B}^{q_t^j}, \qquad j = 1, \ldots, J. \tag{55}$$

In what follows, it is assumed that this transformation has been performed.

Assume also that

$$\dim w_t^j \geq r_t^j, \qquad \operatorname{rank} W_t^j = r_t^j.$$

This amounts to assuming that no linear combination of the components of y_t^j is noise free. As a result, the set of all x_t such that (55) is satisfied has a nonempty interior. As in the Kalman filtering, noise-free data would require a special treatment.

Then, let us show that it is possible to eliminate the matrix W_t^j, without loss of generality. First, perform a singular-value decomposition of W_t^j as RST^{T}, where R and T are unitary matrices and $S=(D,0)$, with $D=\operatorname{diag}(\lambda_1,\ldots,\lambda_{r_t^j})$, λ_i, $i=1,\ldots,r_t^j$, being the (nonzero) singular values of W_t^j. Since T is unitary, the vector

$$w = T^{\mathrm{T}} w_t^j$$

belongs to $\mathscr{B}^{q_t^j}$. Partition w as

$$w^{\mathrm{T}} = (w_t'^{j\mathrm{T}}, w_t''^{j\mathrm{T}}),$$

in a way compatible with the dimensions of the blocks of S. The projection $w_t'^j$ of w belongs to $\mathscr{B}^{r_t^j}$, and (55) can be rewritten as

$$y_t^j = C_t^j x_t + RDw_t'^j, \qquad w_t'^j \in \mathscr{B}^{r_t^j}.$$

Since R is unitary and D invertible, left multiply this equation by $D^{-1}R^{\mathrm{T}}$ to get

$$y_t'^j = C_t'^j x_t + w_t'^j, \qquad w_t'^j \in \mathscr{B}^{r_t^j},$$

with

$$y_t'^j = D^{-1}R^{\mathrm{T}} y_t^j, \qquad C_t'^j = D^{-1}R^{\mathrm{T}} C^j{}_t.$$

To simplify notation, it is assumed in the paper that these transformations have been performed already and that $y_t'^j, C_t'^j, w_t'^j$ are written as y_t^j, $C^j{}_t, w_t^j$.

7.2. Proof of Theorem 4.3. The demonstration of the convexity of $\varphi_d(\alpha)$ uses the function Δ_d: $[0,1] \to \mathbb{R}$ defined by

$$\Delta_d(\lambda) = \varphi_d(\lambda\alpha + (1-\lambda)\beta) - \lambda\varphi_d(\alpha) - (1-\lambda)\varphi_d(\beta), \tag{56}$$

with $\alpha \in \mathscr{D}^*$ and $\beta \in \mathscr{D}^*$. This function satisfies

$$\Delta_d(0) = \Delta_d(1) = 0.$$

From Lemma 3.2 and Section 3.1,

$$\Delta'_d(\lambda) = \mathrm{tr}\,(P_\gamma^{-1} Q_\gamma) - \varphi_d(\alpha) + \varphi_d(\beta),$$

with

$$\gamma = \lambda\alpha + (1-\lambda)\beta \quad \text{and} \quad Q_\gamma = dP_\gamma/d\lambda = -\sum_{k=1}^{K} \delta_k \gamma_k^{-2} P_k,$$

where

$$\delta = \alpha - \beta.$$

It is then shown easily that

$$\Delta''_d(\lambda) = \mathrm{tr}\, S_\gamma,$$

with

$$S_\gamma = P_\gamma^{-1/2} T_\gamma P_\gamma^{-1/2}, \qquad T_\gamma = R_\gamma - Q_\gamma P_\gamma^{-1} Q_\gamma,$$

$$R_\gamma = dQ_\gamma/d\lambda = 2\sum_{k=1}^{K} \delta_k^2 \gamma_k^{-3} P_k.$$

By construction, T_γ is symmetric and thus S_γ is symmetric too. Since $P_\gamma > 0$, proving that $S_\gamma > 0$ is equivalent to proving that $T_\gamma > 0$. The proof is based on the generalization of the Schur inequality (Ref. 27, Theorem 7.7.6). For any $P_\gamma > 0$, the following conditions are equivalent:

$$U_\gamma = \begin{bmatrix} P_\gamma & Q_\gamma \\ Q_\gamma^{\mathrm{T}} & R_\gamma \end{bmatrix} > 0 \Leftrightarrow R_\gamma > Q_\gamma^{\mathrm{T}} P_\gamma^{-1} Q_\gamma. \tag{57}$$

Since $Q_\gamma = Q_\gamma^{\mathrm{T}}$, $T_\gamma > 0$ is equivalent to $U_\gamma > 0$. U_γ can be rewritten as

$$U_\gamma = \sum_{k=1}^{K} U_k,$$

with

$$U_k = \begin{bmatrix} \gamma_k^{-1} P_k & -\delta_k \gamma_k^{-2} P_k \\ -\delta_k \gamma_k^{-2} P_k & 2\delta_k^2 \gamma_k^{-3} P_k \end{bmatrix}. \tag{58}$$

Since P_k is symmetric, U_k is symmetric too. Take $x^{\mathrm{T}} = (x_1^{\mathrm{T}}, x_2^{\mathrm{T}})$, with x_1 and x_2 in $\mathbb{R}^n$, to get

$$\begin{aligned} x^{\mathrm{T}} U_k x = {} & (\gamma_k^{-1/2} x_1 - \delta_k \gamma_k^{-3/2} x_2)^{\mathrm{T}} P_k (\gamma_k^{-1/2} x_1 - \delta_k \gamma_k^{-3/2} x_2) \\ & + \delta_k^2 \gamma_k^{-3} x_2^{\mathrm{T}} P_k x_2. \end{aligned} \tag{59}$$

Since $P_k \geq 0$, $k = 1, \ldots, K$, and since $P_\gamma > 0$,

$$x^{\mathrm{T}} U_\gamma x > 0, \qquad \text{for all } x,$$

which proves that $U_\gamma > 0$ and $S_\gamma > 0$. Thus, $\operatorname{tr} S_\gamma > 0$ and therefore, $\Delta_d''(\lambda) > 0$ and $\Delta_d(\lambda) < 0$ for all $\lambda \in]0, 1[$ and $\alpha \neq \beta$, which is equivalent to

$$\varphi_d(\gamma) < \lambda \varphi_d(\alpha) + (1 - \lambda)\varphi_d(\beta);$$

so, φ_d is strictly convex and Problem D^+ is strictly convex over $\mathscr{D}^*$ and thus over $\mathscr{D}^{+*}$. □

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