Automation and Remote Control, Vol. 65, No. 4, 2004, pp. 1–10. Translated from Avtomatika i Telemekhanika, No. 4, 2004, pp. 70–80.

DETERMINATE SYSTEMS

Extended Superstability in Control Theory[1]

B. T. Polyak

Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia.
Received October 28, 2003

Abstract—The notion of superstability that was recently used to tackle various problems of robustness and the linear control theory was generalized to attain higher flexibility. For the continuous and discrete cases, a class of matrices E was introduced for which the superstability condition is satisfied after the diagonal transformation. Systems with these matrices have piecewise-linear Lyapunov functions $V(x) = \max_i |x_i/d_i|$. Problems such as verification of the membership $\widetilde{A} \subset E$ for the interval matrices, existence of a feedback K such that $A + BK \in E$, the best componentwise estimation, and disturbance attenuation were all of them reduced to the easily solvable linear programming problems. Efficient numerical methods were proposed to solve the arising linear inequalities.

1. INTRODUCTION

The recently introduced notion of *superstability* [1–4] was utilized in numerous applications of the automatic control theory such as robust analysis, design of static output feedback, simultaneous stabilization, robust stabilization, and disturbance attenuation. This property, however, is too rigid because we specify a fixed Lyapunov function $V(x) = \|x\|_\infty = \max_i |x_i|$. Since superstable matrices are nothing but a narrow subset of the stable matrices, superstability is very difficult to attain. A more flexible approach based on the diagonal transformation to a superstable form is proposed below. It resembles design of the diagonal quadratic Lyapunov functions (the so-called *diagonal stability*, see [5–7]) for linear systems. Here, we compile piecewise-linear Lyapunov functions as $V(x) = \max_i |x_i/d_i|$. They were examined in [8–10] in the stability check and in the proof of convergence of the numerical methods, but as far as the present author knows, they were never used for the design purposes.

We begin the next section by defining the class E of generally superstable systems (for continuous and discrete time) and present a simple description in terms of linear inequalities. We also elucidate the relations with diagonal stability. Exhaustive description of the set E is given for some examples such as the Frobenius, 2×2, and triangular matrices. In the third section, we discuss the issues of robustness and establish a necessary and sufficient condition for membership of the interval matrix in the class E, which, therefore, is sufficient for robust stability of the interval matrices. The simplest task of design—making an extended superstable system from a closed-loop system by means of a state feedback—was solved completely in Sec. 4. Sections 5 and 6 consider problems with bounded external disturbances. First, we discuss the problem of the best componentwise estimation by seeking the least invariant box for an open loop system. Then we turn to the problem of design for disturbance attenuation. Section 7 proposes special numerical methods to resolve the arising linear inequalities which are described for a special problem of simultaneous stabilization where the number of inequalities may be large.

[1] This work was supported by the Russian Foundation for Basic Research, project no. 02-01-00127, and the Program for Fundamental Research of the Presidium of the Russian Academy of Sciences.

2. EXTENDED SUPERSTABLE MATRICES

We recall that the matrix A $(n\times n)$ with real entries a_{ij} is *superstable* if

$$\sigma(A) = \min_i \left(-a_{ii} - \sum_{j\neq i} |a_{ij}|\right) > 0 \tag{1}$$

(superstability in continuous time) or

$$\|A\|_1 = \max_i \sum_j |a_{ij}| < 1 \tag{2}$$

(discrete time). To avoid confusion, we note that in many sources this norm is denoted by $\|A\|_\infty$.

For continuous systems

$$\dot{x} = Ax,$$

we perform the transformation $x = Dy$, where $D > 0$ is a diagonal matrix, and for the new variable y obtain

$$\dot{y} = \overline{A}y, \quad \overline{A} = D^{-1}AD.$$

The same transformation for discrete systems

$$x_{k+1} = Ax_k$$

can be done. It is apparent from it that

$$y_{k+1} = \overline{A}y_k, \quad \overline{A} = D^{-1}AD.$$

Definition 1. The matrix $A \in E$ (the set of extended superstable matrices) if there exists a diagonal matrix $D > 0$ such that $\overline{A} = D^{-1}AD$ is superstable. We denote E_c and E_d to discriminate between, respectively, the continuous and discrete cases.

These matrices that are well known in numerical analysis and control (see, for example, [6, 9, 11, 12] where other references can be found) seem to be first introduced by A. Ostrowski [7]. To solve the superstable system $\dot{y} = \overline{A}y$, we have the estimate $\|y(t)\|_\infty \leq e^{-\sigma(\overline{A})t}\|y(0)\|_\infty$ [1, 3]. Then, we get the estimate $\|D^{-1}x(t)\|_\infty \leq e^{-\sigma(\overline{A})t}\|D^{-1}x(0)\|_\infty$ for solving the initial system $\dot{x} = Ax$, which means that $V(x) = \max_i |x_i/d_i|$ is a Lyapunov function for $\dot{x} = Ax$ if $A \in E_c$ with the estimate $V(x(t)) \leq e^{-\sigma(\overline{A})t}V(x(0))$. Similarly, it is the Lyapunov function for the discrete system $x_{k+1} = Ax_k, A \in E_d$ with the estimate $V(x_k) \leq \|\overline{A}\|_1^k V(x_0)$.

We also note that all facts established in [1, 3] for the nonlinear and nonstationary problems are extended in a similar manner to the class of extended superstable systems, but we will not dwell on them. Now, we present the necessary and sufficient conditions for general superstability of a matrix. The entries A, $\overline{A}$, and D are denoted below, respectively, by a_{ij}, $\overline{a}_{ij}$, and d_i, $i, j = 1, \ldots, n$.

Lemma 1. *$A \in E_c$ if and only if there exists a solution $d > 0$ of the linear inequalities*

$$\sum_{j\neq i} |a_{ij}|d_j < -a_{ii}d_i, \quad i = 1, \ldots, n, \tag{3}$$

and $A \in E_d$ *if and only if there exists a solution* $d > 0$ *of the linear inequalities*

$$\sum_j |a_{ij}| d_j < d_i, \quad i = 1, \ldots, n. \tag{4}$$

Moreover, $\sigma(\overline{A}) \geq \sigma > 0$ *if there is a solution* $d > 0$ *of the linear inequalities*

$$\sum_{j \neq i} |a_{ij}| d_j \leq (-a_{ii} - \sigma) d_i, \quad i = 1, \ldots, n, \tag{5}$$

and also $\|\overline{A}\|_1 \leq q < 1$ *if there exists a solution* $d > 0$ *of the linear inequalities*

$$\sum_j |a_{ij}| d_j \leq q d_i, \quad i = 1, \ldots, n. \tag{6}$$

The proof follows immediately from conditions (1), (2) for superstability of $\overline{A}$ with regard for $\overline{a}_{ij} = a_{ij} d_j / d_i$. Therefore, it suffices to solve a system of linear inequalities to check $A \in E$. We note that (3) is satisfied only for $a_{ii} < 0, \quad i = 1, \ldots, n$, whereas (4), for $|a_{ii}| < 1$, $i = 1, \ldots, n$. These are the necessary conditions for $A \in E$. Matrices satisfying (3) were called in [9] the *quasidominant negative diagonal* matrices, whereas in [6, 12] they are called simply *quasidominant* matrices. Matrices satisfying (4) were considered in [10]. We will adhere to the terminology of [1–4] and retain the term *extended superstable matrices*. Of course, the extended superstable matrices are stable: $E \subset S$, where S is the set of stable matrices, because $A \in E$ has the same eigenvalues as the superstable matrix $D^{-1}AD$ which is stable. Then, (3) and (4) can be regarded as a sufficient stability condition. It is shown in what follows that there exists another description of the matrices E in terms of the eigenvalues of matrices and not of their entries. However, the description of Lemma 1 is more convenient for our purposes because it allows one to tackle numerous control problems.

Lemma 2. a. [12, 13] $A \in E_c$ *if and only if* $\overline{A}$ (*with the entries* $\overline{a}_{ij} = |a_{ij}|$, $j \neq i, \overline{a}_{ii} = a_{ii}$) *is a Hurwitz matrix.*

b. [14] $A \in E_d$ *if and only if* $\overline{A}$ (*with the entries* $\overline{a}_{ij} = |a_{ij}|$) *is a Schur matrix.*

Now, we describe the relations between the matrices of E and diagonal stability.

Definition 2 [6]. The matrix $A \in \mathcal{D}$ (set of the *diagonally stable* matrices) if there exists a positive diagonal solution of the Lyapunov inequalities. Namely, $A \in \mathcal{D}_c$ (continuous time) if there exists a diagonal matrix $P > 0$ such that $PA + A^T P < 0$, and $A \in \mathcal{D}_d$ (discrete time) if there exists a diagonal matrix $P > 0$ such that $A^T PA - P < 0$.

It is common knowledge [6, 12] that $E \subset \mathcal{D}$, that is, $A \in E_c$ ($A \in E_d$) implies that $A \in \mathcal{D}_c$ ($A \in \mathcal{D}_d$). However, these matrix sets do not coincide. For example, $A = \begin{pmatrix} -0.5 & 0.75 \\ -0.75 & -0.5 \end{pmatrix}$ belongs to $\mathcal{D}_d$ and not to E_d [6]. If $a_{ij} \geq 0$ for all i, j, then $E_d = \mathcal{D}_d = S_d$ (set of Schur stable matrices) [6, Lemma 2.7.25]. It deserves noting that the check for diagonal stability comes to checking for existence of a solution of a linear matrix inequality, whereas the check of $A \in E$ comes to checking the (vector) linear inequalities (3), (4), which are much simpler.

Example 1. Let us consider a matrix reduced to the Frobenius form

$$A = \begin{pmatrix} 0 & 1 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \\ a_1 & a_2 & \ldots & a_n \end{pmatrix}.$$

It is not superstable (in the discrete sense) for any a_i because condition (2) is violated. However, if $\sum\limits_i |a_i| < 1$ (Cohn condition), then $A \in E_d$ (by taking a sufficiently small $d_1 > d_2 > \ldots > d_n = 1, d_1 - 1$). Moreover, it is readily verified that the Cohn condition is also the necessary condition for $A \in E_d$.

Example 2. $n = 2$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Then, $a_{11} < 0$, $a_{22} < 0$, $a_{11}a_{22} > |a_{12}a_{21}|$ and $|a_{11}| < 1$, $|a_{22}| < 1$, $(1-|a_{11}|)(1-|a_{22}|) > |a_{12}a_{21}|$ will be the necessary and sufficient conditions, respectively, for $A \in E_c$ and $A \in E_d$. The superstability conditions [1, 3] are more rigid: $a_{11} < 0$, $a_{22} < 0$, $-a_{11} > |a_{12}|$, $-a_{22} > |a_{21}|$ (continuous time) and $|a_{11}| < 1$, $|a_{22}| < 1$, $1 - |a_{11}| > |a_{12}|$, $1 - |a_{22}| > |a_{21}|$ (discrete time).

Example 3. Triangular matrix $a_{ij} = 0$, $j < i$. $A \in E_c$ if and only if $a_{ii} < 0$ for all i. Indeed, let us take $\tau_i = \min\left\{1, -a_{ii}/\sum\limits_{j>i} |a_{ij}|\right\}$ and choose $d_n = 1$, $0 < d_{i+1} < \tau_i d_i$, $i = n-1, \ldots, 1$. For these d_i, we get $\sum\limits_{j\neq i} |a_{ij}| d_j = \sum\limits_{j>i} |a_{ij}| d_j < d_{i+1} \sum\limits_{j>i} |a_{ij}| < \tau_i d_i \sum\limits_{j>i} |a_{ij}| \leq -a_{ii} d_i$, $i = 1, \ldots, n-1$. Consequently, for these i condition (3) is satisfied, and for $i = n$ it is obvious. Then, for these matrices the extended superstability is tantamount to stability.

Now, we can turn to various problems of control that are related with the properties of generall superstability.

3. ROBUSTNESS

Let us consider the family of interval matrices

$$\widetilde{A} = A + \Delta, \quad |\Delta_{ij}| \leq m_{ij} \quad i, j = 1, \ldots, n.$$

A question emerges as to whether it is possible to make all matrices $D^{-1}\widetilde{A}D$ superstable by using a single matrix D?

Lemma 3. *$\widetilde{A} \in E$ with the common $D > 0$ if and only if the following linear inequalities have solution $d > 0$:*

$$\sum_{j\neq i} (|a_{ij}| + m_{ij}) d_j < (-a_{ii} - m_{ii}) d_i, \quad i = 1, \ldots, n \tag{7}$$

(*continuous time*) *and*

$$\sum_j (|a_{ij}| + m_{ij}) d_j < d_i, \quad i = 1, \ldots, n \tag{8}$$

(*discrete time*).

Indeed, let us apply conditions (3), (4) to the entries $\widetilde{a}_{ij}$ of the matrix $\widetilde{A}$ and choose the worst-case values Δ_{ij}. We note that conditions (7), (8) can be rearranged in $\overline{A} + M \in E$, where $\overline{A}$ is the matrix from Lemma 2. These conditions are less rigid than those for robust stability of the interval matrices obtained in [1, 3] on the basis of superstability. We bear in mind that $A \in E$ implies $A \in \mathcal{D}$ and obtain the sufficient conditions for diagonal stability (compare with [6, Theorem 3.4.17]).

4. STABILIZATION

Given are the continuous system

$$\dot{x} = Ax + Bu$$

or the discrete system

$$x_{k+1} = Ax_k + Bu_k.$$

How to choose the state feedback $u = Kx$ so as to make the closed-loop system extended superstable? We denote by b_{is}, k_{sj} the entries of the matrices B, K.

Theorem 1. *Matrix K such that $A+BK \in E$ exists if and only if the following linear inequalities have a solution $d_i > 0, y_{sj}$:*

$$\sum_{j\neq i}\left|a_{ij}d_j + \sum_s b_{is}y_{sj}\right| < -a_{ii}d_i - \sum_s b_{is}y_{si}, \quad i = 1,\dots,n \tag{9}$$

(*continuous time*) *and*

$$\sum_{j}\left|a_{ij}d_j + \sum_s b_{is}y_{sj}\right| < d_i, \quad i = 1,\dots,n \tag{10}$$

(*discrete time*). *Then, $D = diag(d_i), k_{sj} = y_{sj}/d_j$.*

The proofs of this and the following theorems are given in the Appendix.

We note that if conditions (9), (10) are replaced by the conditions

$$\sum_{j\neq i}\left|a_{ij}d_j + \sum_s b_{is}y_{sj}\right| \leq (-a_{ii} - \sigma)d_i - \sum_s b_{is}y_{si}, \quad i = 1,\dots,n \tag{11}$$

$$\sum_{j}\left|a_{ij}d_j + \sum_s b_{is}y_{sj}\right| \leq qd_i, \quad i = 1,\dots,n \tag{12}$$

with some $\sigma > 0, q < 1$, then $\sigma(D^{-1}\widetilde{A}D), \|D^{-1}\widetilde{A}D\|_1$ can be estimated as $\sigma(D^{-1}\widetilde{A}D) \geq \sigma$, $\|D^{-1}\widetilde{A}D\|_1 \leq q$ (compare with a similar result of Lemma 1).

Example 4. Let us consider a discrete system in the controllable canonical form

$$x_{k+1} = Ax_k + Bu_k, \quad A = \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{pmatrix}.$$

Then, there always exists $K = (k_1, \dots, k_n)$ such that $A + BK \in E_d$. Indeed, if we take k_i so that $\sum_i |a_i - k_i| < 1$, then according to Example 1 the matrix $A + BK \in E_d$.

Example 5. $n = 2$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, the case of a continuous-time system. This example was considered in [2, 3], and the conditions were established for existence of a matrix $K = (k_1, k_2)$ such that $A + BK$ is superstable. For this example, conditions (9) are as follows: $|a_{12}d_2 + y_2| < -a_{11}d_1 - y_1$, $|a_{21}d_1 + y_1| < -a_{22}d_2 - y_2$. By acting as in [2, 3], we get that these conditions are equivalent to $(a_{11} - a_{21})d_1 + (a_{22} - a_{12})d_2 < 0$; the last inequality has a solution $d > 0$ if and only if $\min\{a_{11} - a_{21}, a_{22} - a_{12}\} < 0$. In [2, 3], $a_{11} - a_{21} + a_{22} - a_{12} < 0$ was the condition for superstabilization by state feedback, which is much more restrictive. Similar calculations for the discrete case provide $A \in E_d$ if and only if $\min\{|a_{11} - a_{12}|, |a_{21} - a_{22}|\} < 1$.

Theorem 1 offers simple calculation tools for verifying whether a matrix can be made extended superstable by a feedback. However, no general analytical conditions for this reduction are known. The superstability-based approach can be used to design not only a state feedback, but also an output feedback [2, 3]. Unfortunately, it is not possible to do so for general superstability in the general case. Example 4 with output feedback is one of the exceptions for the discrete systems.

Example 6. Let us consider a discrete system in the canonical form and with the output $y_k \in R^m$

$$x_{k+1} = Ax_k + Bu_k,$$
$$y_k = Cx_k,$$

where A and B are the same as in Example 4. The problem lies in determining a matrix $K = (k_1, \ldots, k_m)$, if any, such that $A + BKC \in E_d$. Solvability of the linear inequalities $\sum_{i=1}^{n} |a_i + \sum_s k_s c_{si}| < 1$ is the necessary and sufficient condition for existence of such matrix. Indeed, it is the Cohn condition (see Example 1) for the matrix $A + BKC$ which also has the Frobenius form. The corresponding results on diagonal stability can be found in [6, p. 218].

Theorem 3.2 of [2] suggests the following conclusion about the impossibility of static output feedback which makes the discrete system extended superstable.

Lemma 4. *Let there be a matrix Z and a diagonal matrix $D > 0$ such that $TrA^TZ \geq 1$, $\|D^{-1}ZD\|_\infty \leq 1$, $CZB = 0$. Then, $A + BKC \notin E_d$ for all K.*

We recall that $\|A\|_\infty = \sum_j \max_i |a_{ij}|$.

5. THE BEST STATE ESTIMATE

Given are the continuous

$$\dot{x} = Ax + Cw$$

or discrete

$$x_{k+1} = Ax_k + Cw_k$$

systems with the bounded external disturbances

$$\|w(t)\|_\infty \leq 1, \quad \|w_k\|_\infty \leq 1.$$

Then, it is possible to estimate an *invariant box* $\mathbf{B}$ for which $x(t) \in \mathbf{B}$ follows from $x(0) \in \mathbf{B}$ for all $t \geq 0$ and arbitrary $\|w(t)\|_\infty \leq 1$ or $x_k \in \mathbf{B}$ follows from $x_0 \in \mathbf{B}$ for all $k \geq 0$ and any $\|w_k\|_\infty \leq 1$. Needed is to choose the "least," in a sense, box. In what follows, $d_{\max}$ and $d_{\min}$ stand for the greatest and the least values of d_i.

Theorem 2. *If there exists a diagonal matrix $D > 0$ such that $\widetilde{A} = D^{-1}AD$ is superstable, then it follows from the condition*

$$\|x(0)\|_\infty \le \gamma = \frac{d_{\max}\|C\|_1}{d_{\min}\sigma(\widetilde{A})}, \quad \left(\|x_0\|_\infty \le \gamma = \frac{d_{\max}\|C\|_1}{d_{\min}(1-\|\widetilde{A}\|_1)}\right) \tag{13}$$

that $\|x(t)\|_\infty \le \gamma$, $t \ge 0$ ($\|x_k\|_\infty \le \gamma$, $k \ge 0$) for the continuous (discrete) systems. The least γ is obtained by solving the following parametric linear programming problem:

$$\begin{gathered} \min \beta/\sigma \\ \sum_{j\ne i} |a_{ij}|d_j \le (-a_{ii}-\sigma)d_i, \quad i = 1,\ldots,n, \\ 1 \le d_i \le \beta, \quad i = 1,\ldots,n \end{gathered} \tag{14}$$

(*with the variables d, β and parameter $\sigma > 0$; $\gamma_{opt} = \beta\|C\|_1/\sigma$*) *for the continuous case and*

$$\begin{gathered} \min \beta/(1-q) \\ \sum_{j} |a_{ij}|d_j \le q d_i, \quad i = 1,\ldots,n, \\ 1 \le d_i \le \beta, \quad i = 1,\ldots,n \end{gathered} \tag{15}$$

(*with the variables d, β and parameter $q < 1$; $\gamma_{opt} = \beta\|C\|_1/(1-q)$*) *for the discrete case.*

In the above analysis, by the "*best*" box was meant that having the minimum greatest side among the feasible ones. Other optimality criteria (for example, the volume of box) lead to other optimization problems. We note that the proposed estimates γ_{opt} are nothing but the upper bounds for $\max_w \max_t \|x(t)\|_\infty$.

6. DISTURBANCE ATTENUATION

Let us consider linear systems with control and bounded external disturbances:

$$\dot{x} = Ax + Bu + Cw$$

(continuous time) or

$$x_{k+1} = Ax_k + Bu_k + Cw_k$$

(discrete time), where

$$\|w(t)\|_\infty \le 1, \quad \|w_k\|_\infty \le 1.$$

It is required to determine a feedback $u = Kx$ such that for the closed-loop system it would be possible to guarantee the estimate $\|x(t)\|_\infty \le \gamma \quad (\|x_k\|_\infty \le \gamma)$ with the least γ by means of the estimation technique from the last section. Combination of (11), (12) and Theorem 2 provides the following result.

Theorem 3. *Let us solve the parametric linear programming problem*

$$\begin{gathered} \min \beta/\sigma, \\ \sum_{j\ne i} \left| a_{ij}d_j + \sum_s b_{is}y_{sj} \right| \le (-a_{ii}-\sigma)d_i - \sum_s b_{is}y_{si}, \quad i = 1,\ldots,n, \\ 1 \le d_i \le \beta, \quad i = 1,\ldots,n \end{gathered} \tag{16}$$

with the variables β, d, Y and parameter $\sigma > 0$. If it is solvable, then the solution is optimized in σ. The corresponding optimal values are denoted by β^*, d^*, Y^*, and σ^*. Then, for the feedback $u = Kx$, where $K = Y^* D^{-1}$, $D = diag(d_1^*, \ldots, d_n^*)$, the estimate

$$\|x(t)\|_\infty \le \frac{\beta^* \|C\|_1}{\sigma^*}, \quad 0 \le t < \infty$$

of the solution of the closed-loop continuous system can be guaranteed, provided that $x(0)$ satisfies this inequality. Similarly, for the discrete case we solve the problem

$$\min \beta/(1-q)$$

$$\sum_j \left| a_{ij} d_j + \sum_s b_{is} y_{sj} \right| \le q d_i, \quad i = 1, \ldots, n, \tag{17}$$

$$1 \le d_i \le \beta, \quad i = 1, \ldots, n$$

and optimize in the parameter $q < 1$. For the optimal values β^*, d^*, Y^*, and q^* we assume that K is the same as in the last case. Then, the inequality

$$\|x_k\|_\infty \le \frac{\beta^* \|C\|_1}{1 - q^*}, \quad 0 \le k < \infty$$

is satisfied for the closed-loop system with $u = Kx$, provided that it is satisfied for x_0.

7. NUMERICAL METHODS

All the above problems were reduced to linear inequalities which in turn are solvable by the usual methods of linear programming. However, the number of inequalities can be sufficiently large, and the iterative methods similar to those proposed in [15] can prove to be more efficient. We demonstrate their application to the problem of *simultaneous stabilization* which was not considered so far.

Given are m systems obeying the equations

$$x_{k+1} = A^l x_k + B^l u_k, \quad l = 1, \ldots, m. \tag{18}$$

Needed is to verify whether there exists a feedback K such that all matrices $A^l + B^l K \in E_d$ with a common matrix $D > 0$. We consider only the discrete case; the continuous case is considered along similar lines. Theorem 1 asserts that a solution exists if and only if all m systems of linear inequalities (9) corresponding to each system (18) have a common solution, that is,

$$\sum_j \left| a_{ij}^l d_j + \sum_s b_{is}^l y_{sj} \right| < d_i, \quad i = 1, \ldots, n, \quad l = 1, \ldots, m. \tag{19}$$

Then, the following iterative algorithm can be used.

Algorithm. 1. Take an initial approximation D^0, Y^0 (for example, $D^0 = I$, $Y^0 = 0$).

2. At step t, we get D^t, Y^t and choose randomly (for example, with equal probabilities $1/m$) one of the systems $l(t)$.

3. If the inequalities $\sum\limits_j |a_{ij}^{l(t)} d_j^t + \sum\limits_s b_{is}^{l(t)} y_{sj}^t| < d_i^t$, $i = 1, \ldots n$ are satisfied, then $D^{t+1} = D^t$, $Y^{t+1} = Y^t$, and we pass to Step 2. Otherwise, we determine the index $i(t)$ of the most violated inequality (19).

4. We assume that

$$d_j^{t+1} = \left(d_j^t - \gamma_t a_{i(t)j}^{l(t)} sign(\varepsilon_j)\right)_+, \quad y_{sj}^{t+1} = y_{sj}^t - \gamma_t b_{i(t)s}^{l(t)} sign(\varepsilon_j),$$

$$\gamma_t = \frac{\sum\limits_j \varepsilon_j - d_{i(t)}^t + \delta}{\sum\limits_j \left(a_{i(t)j}^{l(t)}\right)^2 + n \sum\limits_s \left(b_{i(t)s}^{l(t)}\right)^2},$$

where $\varepsilon_j = \left| a_{i(t)j}^{l(t)} d_j^t + \sum\limits_s b_{i(t)s}^{l(t)} y_{sj} \right|$, $\alpha_+ = \max\{0, \alpha\}$, $\delta > 0$, is sufficiently small.

This algorithm realizes the general scheme for inequalities (19) substantiated in [15]. If they are solvable, the algorithm is completed in a finite number of steps with the probability 1. Similar algorithms are applicable to *robust stabilization*, for example, if A is an interval matrix. In this case, the number of inequalities is extremely large, and the proposed method which at each step deals with one inequality becomes computationally attractive.

8. CONCLUSIONS

Proposed was a new approach to analysis and design in the theory of automatic control which extends the use of superstability [1–4]. The problems are reduced to solving linear inequalities either by a method of linear programming or special iterative methods (in the case of large-dimensional problems).

ACKNOWLEDGMENTS

The author is grateful to V.L. Kharitonov for his constructive criticism of the notion of superstability which stimulated the proposed generalization and also to A.A. Tremba and P.S. Scsherbakov for their useful comments.

APPENDIX

Proof of Theorem 1. We denote $\widetilde{A} = A + BK$ with the entries $\widetilde{a}_{ij} = a_{ij} + \sum\limits_s b_{is}k_{sj}$; then, condition (3) for $\widetilde{A} \in E_c$ is as follows:

$$\sum_{j \neq i} \left| a_{ij}d_j + \sum_s b_{is}k_{sj}d_j \right| < -a_{ii}d_i - \sum_s b_{is}y_{si}, \quad i = 1, \ldots, n.$$

If we denote $y_{sj} = k_{sj}d_j$, then we arrive to (9). The discrete case is studied similarly. ■

Proof of Theorem 2. After the change of variables $x = Dy$, the system $\dot{x} = Ax + Cw$ takes the form $\dot{y} = \widetilde{A}y + D^{-1}Cw$ with the superstable matrix $\widetilde{A} = D^{-1}AD$. For these systems [1, 3], we obtain the estimate $\|y(t)\|_\infty \le \|D^{-1}B\|_1/\sigma(\widetilde{A})$, $0 \le t < \infty$, if $\|y(0)\|_\infty \le \|D^{-1}C\|_1/\sigma(\widetilde{A})$

$$\|x(t)\|_\infty = \|Dy(t)\|_\infty \le \|D\|_1\|D^{-1}C\|_1/\sigma(\widetilde{A}) \le \frac{d_{\max}\|C\|_1}{d_{\min}\sigma(\widetilde{A})}$$

and for the continuous case get inequality (13). For the discrete case, the equation is considered similarly using the estimate $\|y_k\|_\infty \le \|D^{-1}C\|_1/(1 - \|\widetilde{A}\|_1)$.

Now, we substitute the upper bound $\sigma(\widetilde{A}) \ge \sigma$ for $D > 0$ satisfying (5) and obtain $\gamma \le d_{\max}\|C\|_1/d_{\min}\sigma$. It is possible to scale D (the matrix $\widetilde{A} = D^{-1}AD$ remains unchanged under the transformation $D \Rightarrow \alpha D$) so that $d_{\min} = 1$. Additionally, it is possible to replace $d_{\max}$ by the upper bound $\beta : d_i \le \beta, i = 1, \ldots, n$. Then, optimization of γ is equivalent to (14). In the same way we obtain (15) for the discrete case. ■

REFERENCES

1. Polyak, B.T. and Shcherbakov, P.S., Superstable Linear Control Systems. I. Analysis, *Avtom. Telemekh.*, 2000, no. 8, pp. 37–53.
2. Polyak, B.T. and Shcherbakov, P.S., Superstable Linear Control Systems. II. Design, *Avtom. Telemekh.*, 2002, no. 11, pp. 56–75.
3. Polyak, B.T. and Shcherbakov, P.S., *Robastnaya ustojchivost i upravlenie* (Robust Stability and Control), Moscow: Nauka, 2002.
4. Polyak, B., Sznaier, M., Halpern, M., and Scherbakov, P., Superstable Control Systems, *Proc. 15th IFAC World Congress. Barcelona, Spain*, 2002, pp. 799–804.
5. Boyd, S., El Ghaoui, L., Feron, E., and Balakrishnan, V., *Linear Matrix Inequalities in System and Control Theory*, Philadelphia: SIAM, 1994.
6. Kazkurewicz, E. and Bhaya, A., *Matrix Diagonal Stability in Systems and Computation*, New York: Springer, 2000.
7. Ostrowski, A., Uber die Determinanten mit uberwiegender Hauptdiagonale, *Comment. Math. Helv.*, 1937–1938, vol. 10, pp. 69–96.
8. Rosenbrock, H., A Lyapunov Function for some Naturally Occurring Linear Homogeneous Time-dependent Equations, *Automatica*, 1963, vol. 1, pp. 97–109.
9. Siljak, D., *Large-scale Dynamic Systems: Stability and Structure*, New York: North Holland, 1978.
10. Kaszkurewicz, E., Bhaya, A., and Siljak, D., On the Convergence of Parallel Asynchronous Block-iterative Computations, *Linear Algebra Appl.*, 1990, vol. 131, pp. 139–160.
11. Fiedler, M. and Ptak, V., Diagonally Dominant Matrices, *Czechosl. Math. J.*, 1967, vol. 92, no, 17, pp. 429–433.
12. Moylan, P., Matrices with Positive Principal Minors, *Linear Algebra Appl.*, 1977, vol. 17, pp. 53–58.
13. Willems, J., Lyapunov Functions for Diagonally Dominant Systems, *Automatica*, 1976, vol. 12, no. 5, pp. 519–523.
14. Bertsekas, D. and Tsitsiklis, J., *Parallel and Distributed Computation: Numerical Methods*, Englewood Cliffs, New Jersey: Prentice-Hall, 1989.
15. Polyak, B., Random Algorithms for Solving Convex Inequalities, in Butnariu, D., Censor, Y., and Reich, S., Eds., *Inherently Parallel Algorithms in Feasibility and Optimiz. and Their Appl.*, Elsevier, 2001, pp. 409–422.

This paper was recommended for publication by L.B. Rapoport, a member of the Editorial Board