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Optimization-based design of fixed-order controllers for command following ☆

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Abstract

For discrete-time scalar systems, we propose an approach for designing feedback controllers of fixed order to minimize an upper bound on the peak magnitude of the tracking error to a given command input. The work makes use of linear programming to design over a class of closed-loop systems recently proposed for the rejection of non-zero initial conditions and bounded disturbances. We incorporate performance robustness in the form of a guaranteed upper bound on the peak magnitude of the tracking error under plant coprime factor uncertainty. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

In this paper, we design fixed-order dynamic feedback controllers for the tracking of fixed inputs in scalar discrete-time systems.

Keel and Bhattacharyya (1999) have pointed out that a key issue with fixed-order controller design is that one cannot prespecify exactly the target closed-loop poles or closed-loop transfer function and have proposed a design approach based on specifying a region containing allowable closed-loop transfer function coefficients. The method involves linear programming problems. Another approach which allows fixed-order design was introduced by Blanchini and Sznaier (1997, 2000) and was named design for equalized performance. This method also uses linear programming and gives rejection of bounded disturbances, but differs from the l_1 approach of Dahleh and Pearson (1987) in that non-zero initial conditions are considered. Extensions of the approach in Blanchini and Sznaier (1997) have been carried out in Polyak and Halpern (1999b, 2001), where the term superstable system was introduced to describe the class of closed-loop systems for which equalized performance is defined.

The problem of minimizing the l_∞ norm of the tracking error to a given command input was solved by Dahleh and Pearson (1988) using minimum norm duality theory and linear programming to obtain systems with FIR closed-loop maps. Further aspects of the problem have been considered in Moore and Bhattacharyya (1990), Halpern, Evans, and Hill (1996), Casavola and Mosca (1997), Halpern (2000). These approaches all involve fixed closed-loop poles and thus are not suitable for fixed-order controller design. In addition, as a practical issue, it is clearly highly desirable to maintain tracking performance in the presence of plant uncertainty. Analysis results on robust performance when tracking fixed inputs are given by Khammash (1997) and Elia, Young, and Dahleh (1995).

In the present paper, we extend the design approach in Blanchini and Sznaier (1997, 2000), and Polyak and Halpern (1999b, 2001) to deal with the problem of tracking commands with a view to bounding the l_∞ norm of the tracking error. Firstly, a new performance measure is proposed, which is an upper bound on the l_∞ norm of a signal and is defined only for superstable systems. Like equalized performance, the new measure can be minimized with respect to controller coefficients and can be used to design controllers of fixed order. The approach can also be used to obtain

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designs giving guaranteed tracking performance under plant uncertainty.

2. Preliminary results

We begin by recalling some results from Blanchini and Sznaier (1997, 2000), and Polyak and Halpern (1999b, 2001) on a class of systems originally proposed in Blanchini and Sznaier (1997) for the problem of rejecting bounded disturbances in a nonasymptotic setting and an appropriate performance measure γ.

Consider a LTI SISO discrete-time closed-loop system described by a transfer function

$$\phi(z) = \frac{n(z)}{1 + d(z)}, \tag{1}$$

where

$$n(z) = n_0 + n_1 z + \cdots + n_N z^N,$$

$$d(z) = d_1 z + \cdots + d_D z^D.$$

We denote $\|n\|_1 = \sum_{i=0}^{N} |n_i|$; $\|d\|_1 = \sum_{i=1}^{D} |d_i|$; $\|n\|_\infty = \sup_i |n_i|$. Similarly, if $h(z) = \sum_{i=0}^{\infty} h_i z^i$, then $\|h\|_1 = \sum_{i=0}^{\infty} |h_i|$ provided $\sum_{i=0}^{\infty} |h_i| < \infty$; and $\|h\|_\infty = \sup_i |h_i|$ provided $\sup_i |h_i| < \infty$.

Definition 1. System (1) is *superstable*, if the polynomial $1 + d(z)$ is superstable, i.e. $\|d\|_1 < 1$.

A useful performance criterion for superstable systems was termed "equalized performance level" in Blanchini and Sznaier (1997). For a superstable system (1) with transfer function,

$$\phi(z) = \frac{n(z)}{1 + d(z)}$$

equalized performance, γ, is defined as

$$\gamma(\phi) = \frac{\|n\|_1}{1 - \|d\|_1}. \tag{2}$$

Some useful properties of γ are:

(1) The value of γ partially determines the response of a system to non-zero initial conditions.
(2) If $\phi(z)$ is a superstable transfer function, then

$$\gamma(\phi) \geqslant \|\phi\|_1. \tag{3}$$

(3) Fixed-order controllers minimizing γ can be designed using linear programs.

Now we introduce a quantity analogous to γ for the minimum amplitude tracking of a given input. Let $h(z)$ be any causal superstable rational transfer function. It can be factored as $h(z) = n(z)c(z)$ where

$$n(z) = \sum_{i=0}^{N} n_i z^i$$

and

$$c(z) = \frac{1}{1 + \sum_{i=1}^{D} d_i z^i},$$

is superstable, that is to say, $\sum_{i=1}^{D} |d_i| < 1$. We then have the following lemma.

Lemma 1. *An upper bound for $\|h\|_\infty$ is given by*

$$\beta(h) := \frac{\|n\|_\infty}{1 - \|d\|_1}.$$

Proof. Consider the impulse response h of the transfer function $h(z)$. Now

$$\|h\|_\infty \leqslant \|n\|_\infty \|c\|_1 \tag{4}$$

$$\leqslant \|n\|_\infty \gamma(c) \tag{5}$$

$$= \frac{\|n\|_\infty}{1 - \|d\|_1} = \beta(h). \tag{6}$$

Thus, for superstable $h(z)$, we see that $\beta(h)$ is an easily computed upper bound on the l_∞ norm of h. We show later that it can be minimized using linear programs.

It is of interest to examine the sharpness of the inequalities (4), (5). Equality is obtained in (4) when $c(z) = 1$, that is when $h(z)$ is a polynomial.

A condition for equality between the r.h.s. of (4) and (5) is obtained next. It is noted in Polyak and Halpern (1999b) that $\gamma(x) = \|x\|_1$ for polynomial $x(z)$. Here we indicate that such equality is also obtained for certain all pole rational functions.

Lemma 2. *Assume $c(z) = 1/(1 + d(z))$ is superstable.*

(a) *If all $d_i \leqslant 0$, then $\gamma(c) = c(1) = \|c\|_1$.*
(b) *If all $-1^i d_i \leqslant 0$, then $\gamma(c) = c(-1) = \|c\|_1$.*

For example if $c(z) = 1/(1 - 0.2z - 0.3z^2 - 0.1z^3 - 0.2z^8)$ *or* $c(z) = 1/(1 + 0.2z - 0.3z^2 + 0.1z^3 - 0.2z^8)$, *then* $\gamma(c) = \|c\|_1 = 5.0$.

3. Main results

Here, we are given a plant $P(z)$ and a command $w(z)$ and our goal is to design a controller to force the output of the plant to track the command. The plant $P(z)$ has input u and output y and is described by its transfer function:

$$P(z) = \frac{b(z)}{a(z)} = \frac{b_1 z + b_2 z^2 + \cdots + b_B z^B}{1 + a_1 z + \cdots + a_A z^A}, \tag{7}$$

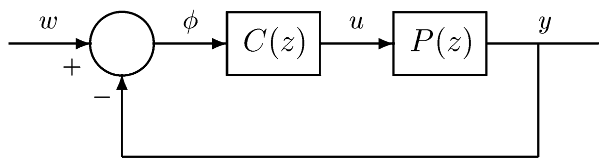

Fig. 1. 1-DOF feedback configuration for tracking.

with $y=Pu$, while the command has transfer function $w(z)=w_1(z)/w_2(z)$, where $w_1(z)$ and $w_2(z)$ are polynomials with $w_2(0)=1$. Here, $w(z)$ is allowed to be unstable. Our treatment of the problem will require the *tracking error* $\phi(k)$, which is given by

$$\phi = w - y$$

to be superstable. Since $\phi(z) = T(z)w(z)$ where $T(z)$ is a stable closed-loop transfer function, it is clear that if $w(z)$ is not superstable, then at least some of its poles will need to be cancelled by zeros of $T(z)$ in order to have $\phi(z)$ superstable. Although these cancellations can involve unstable poles; they do not affect stability; they only affect the response to initial conditions at the input and output of the command generator $w(z)$. We assume these initial conditions are zero-valued.

To illustrate some of the ideas involved, we set $w(z) = 1/(1-z)$, which is not superstable. We consider a 1-Degree-of-freedom (1-DOF) system, as in Fig. 1. For a step command, this configuration gives a controller with integral action. Our goal is to design a controller of fixed structure:

$$C(z) = \frac{g(z)}{(1-z)f(z)} = \frac{g_0 + g_1 z + \cdots + g_G z^G}{(1-z)(1 + f_1 z + \cdots + f_F z^F)} \tag{8}$$

(with prescribed orders F, G and $f(0) = 1$) which ensures super stability of the closed-loop system and minimizes the performance index $\beta(\phi)$ where

$$\phi(z) = \frac{1}{1-z}\left(\frac{a(1-z)f}{(1-z)af + bg}\right).$$

As it stands, this transfer function cannot be made superstable through selection of f, g since the denominator of w is not superstable. In order to obtain a superstable transfer function, which is required for the development in this paper, it is necessary to remove the factor $(1-z)$ from the denominator, by cancelling with the numerator as discussed earlier. After this cancellation, the tracking error becomes

$$\phi(z) = \frac{af}{(1-z)af + bg}$$

for which

$$\beta(\phi) = \frac{\|af\|_\infty}{1 - \|(1-z)af + bg - 1\|_1} \tag{9}$$

provided $(1-z)af + bg$ is superstable.

Next we show how to minimize $\beta(\phi)$ with respect to the controller coefficients. Recall that controller orders F, G are fixed.

Theorem 1. *Minimization of* (9) *is equivalent to a parametric linear programming problem*

$$\beta^* = \min_{0 \leqslant \mu < 1} \min_{f,g} \frac{\|af\|_\infty}{1-\mu} : \tag{10}$$

$$\|(1-z)af + bg - 1\|_1 \leqslant \mu. \tag{11}$$

If the admissible set in (11) *is non-empty, then the closed-loop system with the controller, found as the solution of this optimization problem, is superstable with* $\beta(\phi) = \beta^*$.

For fixed μ, the above problem can be cast, using standard techniques, as a linear program. Hence, the solution over $\mu \in [0, 1)$ involves a one-parameter family of linear programs.

4. Some extensions

4.1. 2-Degree-of-freedom systems

The preceding results are readily extended to the case of 2-DOF controller structures. In this case, the control input u is given by

$$(1-z)f(z)u(k) = s(z)w(k) - g(z)y(k), \tag{12}$$

where $s(z)$ is a polynomial to be determined along with $g(z)$ and $f(z)$.

4.2. Robust performance design

Here, we use the same problem setup as in Polyak and Halpern (1999a, b) with plant coprime factor uncertainty bounded in l_1 norm. Consider the family of plants $\mathscr{P}$ with coprime factor uncertainty

$$P(z) = \frac{b(z)}{a(z)} = \frac{b^0(z) + \delta b(z)}{a^0(z) + \delta a(z)},$$

$$\|\delta b\|_1 \leqslant \varepsilon_B, \quad \|\delta a\|_1 \leqslant \varepsilon_A. \tag{13}$$

Here, $b(z), b^0(z), \delta b(z)$ are polynomials, with $b^0(0) = \delta b(0) = 0$; $a(z), a^0(z), \delta a(z)$ are polynomials with $a^0(0) = 1, \delta a(0) = 0$.

Omitting details, we obtain the following robust performance result for 1-DOF systems.

Theorem 2. *Suppose the parametric linear programming problem*

$$\beta^* = \min_{0 \leqslant \mu < 1} \min_{f,g} \frac{\|a^0 f\|_\infty + \varepsilon_A \|f\|_\infty}{1-\mu} :$$

$$\|a^0(1-z)f + b^0 g - 1\|_1 + \varepsilon_B \|g\|_1 + \varepsilon_A \|(1-z)f\|_1 \leqslant \mu,$$

has a solution f^*, g^*. Then the controller $C^* = g^*/((1-z)f^*)$ makes all closed-loop systems with plants $P \in \mathscr{P}$ robustly superstable with $\|\phi\|_\infty \leqslant \beta^*$.

Note in the presence of plant uncertainty, β^*, as defined here, is an upper bound on $\beta(\phi)$.

5. Examples

Example 1. Consider the example from Casavola and Mosca (1997) of designing a 1-DOF controller for plant,

$$P(z) = \frac{-10z(z-0.5)}{(1-10z)(1-0.5z)}$$

tracking a step.

Following the procedures in Theorems 1 and 2, we design 1-DOF controllers to minimize $\beta(\phi)$ or an upper bound, in the case of an uncertain plant, for some controller orders and plant uncertainty levels as shown below:

	F	2	3	4	5	6
	G	2	3	4	5	6
$\varepsilon_B = \varepsilon_A = 0$;	β^*	40.0	21.6	16.9	15.0	14.2
	v_0	40.0	21.6	16.9	15.0	14.2
$\varepsilon_B = \varepsilon_A = 0.01$;	β^*	48.9	25.9	20.0	17.9	16.9
	v_0	40.0	21.6	16.9	15.0	14.2
$\varepsilon_B = \varepsilon_A = 0.05$;	β^*	431	93.0	67.6	50.1	44.4
	v_0	40.0	26.1	24.6	16.8	15.8

The table shows the values of β^*, which is our minimized upper bound on $\|\phi\|_\infty$; and v_0 which we introduce here to be the value of $\|\phi\|_\infty$ achieved with the nominal plant. The values of both of these quantities can be compared with the lowest possible value of $\|\phi\|_\infty$ achievable with the nominal plant and given command and 1-DOF controller structure, namely 13.5 from Casavola and Mosca (1997) and Halpern (2000).

We show the results for controller orders $G=3, F=3$ in detail. With no plant uncertainty, the optimal controller minimizing β using Theorem 1 is

$$\frac{g}{(1-z)f} = \frac{2.672 - 1.448z - 2.896z^2 + 1.472z^3}{(1-z)(1-1.86z-2.94z^2)}$$

(notice $f_3 = 0$) giving a tracking error

$$\phi(z) = 1 - 12.362z + 21.602z^2 + 21.602z^3 - 14.719z^4.$$

Now $v_0 = \|\phi\|_\infty \approx 21.6$, and, since $\phi(z)$ is FIR, $\beta^* = v_0$.

We now incorporate plant uncertainty by setting $\varepsilon_B = \varepsilon_A = 0.01$. Using Theorem 2 to compute β^*, we find β^* is achieved at $\mu = 0.16376$, even though the controller and nominal tracking error are the same as the previous result obtained with no uncertainty. Increasing the uncertainty level to $\varepsilon_B = \varepsilon_A = 0.05$ gives an optimum at $\mu = 0.718$, and a different controller

$$\frac{g}{(1-z)f} = \frac{2.744 - 2.276z - 1.780z^2 + 1.112z^3}{(1-z)(1-2.221z-2.224z^2)},$$

giving a nominal tracking error

$$\phi(z) = 1 - 12.721z + 26.101z^2 + 12.243z^3 - 11.119z^4,$$

with $\beta^* = 93.0$, $v_0 = 26.1$.

Example 2. In the preceding examples, the nominal tracking errors, obtained by minimizing β or its upper bound, are FIR. In this example, β is minimized with different closed-loop poles. With plant

$$P(z) = \frac{-10.5z + 26z^2 - 10z^3}{1 - 10.75z + 7.625z^2 - 1.25z^3},$$

which has zeros at $z \in \{0, 0.5, 2.1\}$ and poles at $z \in \{0.1, 2, 4\}$, using $F = G = 3$, a feasible solution to the problem in Theorem 1 can be found for $\mu = 0$, but β is minimized at $\mu \approx 0.05$, giving a nominal tracking error

$$\phi(z) = \frac{1 - 12.58z + 23.95z^2 + 21.25z^3 - 23.95z^4 + 4.51z^5}{1 - 0.00481147z + 0.0451885z^7}.$$

In this example, the plant has a stable pole and stable zero near each other. In such situations, controller design by closed-loop pole placement can be problematic since the pole placement Diophantine equation can be poorly conditioned, giving rise to large controller coefficient magnitudes unless a closed-loop pole is chosen carefully near the offending plant pole and zero (Halpern, 1988). The automatic closed-loop pole placement arising from the minimization of β, which to some extent penalizes large controller coefficients, can alleviate this problem.

6. Conclusions

We have extended results on the design of superstable systems to the problem of tracking a fixed command. Our approach involves minimizing an upper bound on the l_∞ norm of the tracking error and readily allows the incorporation of plant coprime factor uncertainty to enable the design of fixed-order controllers for guaranteed tracking performance. In this paper, we have focussed on the case of a step command ($w(z) = 1/(1-z)$) but other commands can be treated similarly.

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