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Ellipsoidal parameter or state estimation under model uncertainty ☆

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Abstract

Ellipsoidal outer-bounding of the set of all feasible state vectors under model uncertainty is a natural extension of state estimation for deterministic models with unknown-but-bounded state perturbations and measurement noise. The technique described in this paper applies to linear discrete-time dynamic systems; it can also be applied to weakly non-linear systems if non-linearity is replaced by uncertainty. Many difficulties arise because of the non-convexity of feasible sets. Combined quadratic constraints on model uncertainty and additive disturbances are considered in order to simplify the analysis. Analytical optimal or suboptimal solutions of the basic problems involved in parameter or state estimation are presented, which are counterparts in this context of uncertain models to classical approximations of the sum and intersection of ellipsoids. The results obtained for combined quadratic constraints are extended to other types of model uncertainty.

Keywords: Bounded noise; Ellipsoidal bounding; Parameter estimation; Set-membership uncertainty; State estimation; Uncertain dynamic systems

1. Introduction

In the literature, most parameter or state estimation problems are either treated as deterministic or solved via a stochastic approach, with the state perturbations and measurement noise assumed to be random with no other uncertainty in the model. Kalman filtering is then the most widely applied technique. Often, however, the underlying probabilistic assumptions are not realistic (the main perturbation may, for instance, be deterministic). It then seems more natural to assume that the state perturbations and measurement noise are unknown but bounded and to characterize the set of all values of the parameter or state vector that are consistent with this hypothesis. This corresponds to *guaranteed estimation*, first considered at the end of 1960s and the early 1970s (Schweppe, 1968; Witsenhausen, 1968; Bertsekas & Rhodes, 1971; Schweppe, 1973). One of the main approaches, and the only one to be considered here, aims to compute ellipsoids guaranteed to contain the vector to be estimated given bounds on the perturbations and noise. The Russian school was particularly active in this domain (Kurzhanskii, 1977; Chernousko, 1981, 1994; Kurzhanskii & Valyi, 1997). Important contributions have been presented in Fogel and Huang (1982), in the context of parameter estimation. At present, the theory of guaranteed estimation is a well developed and mature area of control theory, (see, e.g. the books Milanese, Norton, Piet-Lahanier, Walter, 1996; Walter & Pronzato, 1997), special issues of journals (Walter, 1990; Norton, 1994, 1995) and the references therein. Recent results regarding ellipsoidal state estimation in a MIMO context can be found in Durieu, Walter, and Polyak (2001).

However, most of the works mentioned above deal with problems where the plant model (its structure, in the case of parameter estimation) is assumed to be precisely known and where the uncertainty only relates to state perturbations and measurement noise. This assumption seems unrealistic for many real-life problems. The lack of precise information is the fundamental paradigm of modern control theory, where the concept of robustness plays a key role. The goal of this

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paper is to develop a robust approach to guaranteed estimation, i.e. to find methods to take into account unavoidable but specified model uncertainty.

Consider a linear discrete-time dynamic system with the state equation

$$x_{k+1} = A_k x_k + w_k \tag{1}$$

and the measurement equation

$$y_k = C_k x_k + v_k. \tag{2}$$

A sequence of known inputs u_k acting through a known but possibly non-square control matrix could easily be incorporated in state Eq. (1). It is not introduced for the sake of notational simplicity. The case where w_k acts through a known but possibly non-square excitation matrix will be considered in Section 4.2. The classical unknown-but-bounded approach is based on the assumption that the matrices A_k and C_k are known while the state perturbation vector w_k and measurement noise vector v_k satisfy the constraints

$$\|w_k\| \leqslant \alpha_k, \quad \|v_k\| \leqslant \beta_k, \tag{3}$$

with α_k and β_k known. The problem is then to find a guaranteed estimate for x_k based on the measurements $y_1, \ldots, y_k$ and some prior information relating to x_0. Important particular cases are: the estimation of attainability sets (without measurements: $C_k \equiv 0$, $v_k \equiv 0$), parameter estimation (without dynamics: $A_k \equiv I$, $w_k \equiv 0$, $x_k \equiv \theta$) and parameter tracking ($A_k \equiv I$, $x_k = \theta_k$). The traditional technique is recursively to compute ellipsoids guaranteed to contain x_k.

The present research deals with a more general problem where the model matrices are also uncertain:

$$A_k \in \mathscr{A}_k, \quad C_k \in \mathscr{C}_k, \tag{4}$$

with $\mathscr{A}_k$ and $\mathscr{C}_k$ being some classes of matrices. Particular cases of this general problem have been considered in the literature (Clement & Gentil, 1990; Cerone, 1993; Kurzhanskii & Valyi, 1997; Chernousko, 1996; Rokityanskii, 1997; Norton, 1999; Chernousko & Rokityanskii, 2000). Serious difficulties have been recognized. For instance, consider a class of uncertainty described by interval matrices. An interval matrix $\mathscr{A}_{int}$ with entries a_{ij} is given by a nominal matrix A^0 with entries a^0_{ij} and associated ranges α_{ij}:

$$\mathscr{A}_{\text{int}} = \{A: \ |a_{ij} - a^0_{ij}| \leqslant \alpha_{ij}, \quad i,j = 1,\ldots,n\}. \tag{5}$$

Then the simplest attainability set X_1 for the first step with x_0 in an interval vector X_0 is given by

$$X_1 = \{x_1 = Ax_0 \ : \ A \in \mathscr{A}_{\text{int}}, \ x_0 \in X_0\}. \tag{6}$$

X_1 is already not convex and its detailed description meets combinatorial difficulties for large dimensions, see e.g. Chernousko (1996).

We employ another model of uncertainty, which simplifies the analysis. It is assumed that the matrix uncertainty is combined with the uncertainty due to state perturbations and measurement noise by ellipsoidal constraints:

$$\frac{\|A_k - A_k^0\|^2}{\varepsilon_A^2} + \frac{\|w_k\|^2}{\delta_w^2} \leqslant 1, \tag{7}$$

$$\frac{\|C_k - C_k^0\|^2}{\varepsilon_C^2} + \frac{\|v_k\|^2}{\delta_v^2} \leqslant 1, \tag{8}$$

where A_k^0 and C_k^0 are nominal matrices, while ε_A, δ_w, ε_C and δ_v are prespecified weights. In (7) and (8) and below, the vector norm is understood as Euclidean: $\|x\|^2 = \sum x_i^2$, while the spectral norm is used for matrices: for $A \in \mathbb{R}^{m\times n}$, $\|A\| = \max_{\|x\|\leqslant 1} \|Ax\| = \max(\text{eig}(A^{\mathrm{T}}A))^{1/2}$. Similar models arise in other problems related to systems under uncertainty, such as total least squares (Golub & Van Loan, 1980; El Ghaoui & Lebret, 1997) and robust optimization (Ben-Tal, Ghaoui, & Nemirovskii, 2000). A more general linear-fractional representation (LFR) of uncertainty has been considered in El Ghaoui and Calafiore (1999), where the authors reduced the search of the optimal outer-bounding ellipsoid to semi-definite programming (SDP). The main contribution of our paper is the reduction of the search for an outer-bounding ellipsoid to a one-dimensional optimization. This solution is based on the fact that the set of all states x_k consistent with a given numerical value of the data vector y_k is then described by a quadratic constraint with an indefinite matrix. By using a technique to treat such constraints (Polyak, 1998), the minimal ellipsoid containing the intersection of this set with an ellipsoid can be constructed effectively. Such an approach provides an opportunity to extend ellipsoidal outer-bounding techniques to uncertain models (Section 3). Some generalizations of the above model are considered in Section 4.

2. Preliminaries

The notation $P > 0$ ($P \geqslant 0$) for a real symmetric matrix P means that this matrix is positive definite (non-negative definite). An ellipsoid is denoted by

$$E(c,P) = \{x: \ (x-c)^{\mathrm{T}} P (x-c) \leqslant 1\}, \tag{9}$$

where the vector $c \in \mathbb{R}^n$ is the center of the ellipsoid and the matrix $P \geqslant 0$ characterizes its shape and size.

A matrix $H \in \mathbb{R}^{m\times n}$ and a vector $w \in \mathbb{R}^m$ are said to be admissible (for given positive values of ε and δ) if

$$\frac{\|H\|^2}{\varepsilon^2} + \frac{\|w\|^2}{\delta^2} \leqslant 1. \tag{10}$$

Each of inequalities (7) and (8) can obviously be expressed in the form of (10). The set of all admissible pairs (H, w) will be denoted by S. In (10), $\varepsilon = 0$ is understood as $H = 0$ and $\|w\| \leqslant \delta$, while $\delta = 0$ means $w = 0$ and $\|H\| \leqslant \varepsilon$.

The following simple assertions will be used in the paper.

Lemma 1. *For any given $x \in \mathbb{R}^n$, the set of points $Hx + w$ for all admissible pairs (H, w) is a ball*:

$$\{z = Hx + w : \ (H, w) \in S\} = \{z : \ \|z\|^2 \leqslant \varepsilon^2 \|x\|^2 + \delta^2\}. \tag{11}$$

Proof. For any admissible pair (H, w), $\|Hx+w\| \leqslant \|H\|\,\|x\| + \|w\| \leqslant (\varepsilon^2\|x\|^2 + \delta^2)^{1/2}$ as can be easily seen from the definition of S. On the other hand, if $\|z\|^2 \leqslant \varepsilon^2\|x\|^2 + \delta^2$, take

$$H = \frac{\varepsilon^2}{\varepsilon^2\|x\|^2 + \delta^2} z x^{\mathrm{T}}, \qquad w = \frac{\delta^2}{\varepsilon^2\|x\|^2 + \delta^2} z; \tag{12}$$

it can then be checked that $Hx + w = z$ and (H, w) is an admissible pair. □

The next technical results relating to block matrices and quadratic functions will also be used.

Lemma 2. *Assume $B_i \in \mathbb{R}^{n\times n}$, $B_i > 0$, $i = 0, 1, 2$. Let τ_1 and τ_2 be strictly positive real numbers. Then the inequality*

$$G = \begin{pmatrix} B_0 - \tau_1 B_1 & B_0 \\ B_0 & B_0 - \tau_2 B_2 \end{pmatrix} \leqslant 0 \tag{13}$$

holds if and only if

$$B_0^{-1} \geqslant \tau_1^{-1} B_1^{-1} + \tau_2^{-1} B_2^{-1}. \tag{14}$$

Proof. Inequality (13) or (14) implies that $B_0 - \tau_1 B_1$ and $B_0 - \tau_2 B_2$ are non-singular matrices. Indeed, if (14) holds, then $\tau_2^{-1} B_2^{-1} > 0$, $B_0^{-1} > \tau_1^{-1} B_1^{-1}$, $B_0 < \tau_1 B_1$ (and similarly $B_0 < \tau_2 B_2$). If (13) holds and $(B_0 - \tau_1 B_1)x = 0$, $x \neq 0$, then for $z = (\gamma x^{\mathrm{T}}, (B_0 x)^{\mathrm{T}})^{\mathrm{T}}$ write $0 \geqslant (Gz, z) = 2\gamma\|B_0 x\|^2 + ((B_0 - \tau_2 B_2)B_0 x, B_0 x)$ for all $\gamma \in \mathbb{R}$; this function is linear in γ (with $\|B_0 x\| \neq 0$) and it cannot preserve sign for all γ, therefore $B_0 - \tau_1 B_1$ is non-singular. Similarly we prove that $B_0 - \tau_2 B_2$ is non-singular.

By Schur formula, (13) can be rewritten as $B_0 - \tau_1 B_1 \leqslant B_0 (B_0 - \tau_2 B_2)^{-1} B_0$. According to the matrix inversion lemma, $(B_0 - \tau_2 B_2)^{-1} = B_0^{-1}[I + (\tau_2^{-1} B_2^{-1} - B_0^{-1})^{-1} B_0^{-1}]$, where I is the identity matrix. Then

$$\begin{aligned} B_0 - \tau_1 B_1 &\leqslant [I + (\tau_2^{-1} B_2^{-1} - B_0^{-1})^{-1} B_0^{-1}] B_0 \\ &= B_0 + (\tau_2^{-1} B_2^{-1} - B_0^{-1})^{-1}, \end{aligned}$$

and $\tau_1 B_1 + (\tau_2^{-1} B_2^{-1} - B_0^{-1})^{-1} \geqslant 0$. Thus, (13) is equivalent to (14). □

Lemma 3 (Schweppe, 1973). *Consider two quadratic functions $f_i(x) = (x - c_i)^{\mathrm{T}} P_i (x - c_i)$, $i = 1, 2$, and their weighted sum $f_\tau(x) = (1 - \tau) f_1(x) + \tau f_2(x)$, $0 \leqslant \tau \leqslant 1$. Then the set $E = \{x : \ f_\tau(x) \leqslant 1\}$ is an ellipsoid $E(c, P)$ with*

$$\left.\begin{aligned} &P = (1 - v)^{-1} P_\tau, \\ &P_\tau = (1 - \tau) P_1 + \tau P_2, \\ &c = P_\tau^{-1}[(1 - \tau) P_1 c_1 + \tau P_2 c_2], \\ &v = (1 - \tau) c_1^{\mathrm{T}} P_1 c_1 + \tau c_2^{\mathrm{T}} P_2 c_2 - c^{\mathrm{T}} P_\tau c, \\ &\quad v < 1, \end{aligned}\right\} \tag{15}$$

provided that $P_\tau > 0$.

Proof. Proof is by direct calculation (in a similar statement (Schweppe, 1973, p. 110) Schweppe writes "...it turns out after a lot of manipulations..."). □

Note that P_1 and P_2 are not assumed to be positive definite. Note also that if $c_1 = c_2 = 0$, then $c = 0$, $v = 0$ and $P = P_\tau$.

The so-called *S-procedure* is a well-known tool in system and control applications (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994); it has been introduced by Yakubovich at the end of the sixties. We need the following version of it. Given two quadratic forms $f_i(x) = x^{\mathrm{T}} A_i x$, $i = 1, 2$, in $\mathbb{R}^N$ and real numbers α_i, $i = 1, 2$, the problem is then to characterize all quadratic forms $f_0(x) = x^{\mathrm{T}} A_0 x$ in $\mathbb{R}^N$ and real numbers α_0 such that

$$f_1(x) \leqslant \alpha_1, \quad f_2(x) \leqslant \alpha_2 \ \Rightarrow f_0(x) \leqslant \alpha_0. \tag{16}$$

To say it another way, the problem is to describe quadratic forms such that $x \in E_1 \cap E_2$ implies $x \in E_0$, where $E_i = \{x : f_i(x) \leqslant \alpha_i\}$, $i = 0, 1, 2$. The matrices A_i are not required to be positive definite, thus the sets E_i are not necessarily ellipsoids. Taking the weighted sum of f_1 and f_2 (with weights $\tau_1 \geqslant 0$ and $\tau_2 \geqslant 0$), we obtain an obvious sufficient condition for (16) to be satisfied:

$$A_0 \leqslant \tau_1 A_1 + \tau_2 A_2, \quad \alpha_0 \geqslant \tau_1 \alpha_1 + \tau_2 \alpha_2. \tag{17}$$

More interesting is that under some mild assumptions this sufficient condition is also necessary, as indicated by the following lemma.

Lemma 4 (Polyak, 1998). *Suppose $N \geqslant 3$ and there exist μ_1 and $\mu_2 \in \mathbb{R}$, and $x^0 \in \mathbb{R}^N$ such that*

$$\mu_1 A_1 + \mu_2 A_2 > 0, \quad f_1(x^0) < \alpha_1, \quad f_2(x^0) < \alpha_2. \tag{18}$$

Then (16) holds if and only if there exist $\tau_1 \geqslant 0$ and $\tau_2 \geqslant 0$ such that the inequalities in (17) are satisfied.

The proof of this lemma was obtained in (Polyak, 1998, Theorem 4.1), where the convexity of quadratic transformations was studied.

3. Ellipsoidal state estimation

State estimation for systems under set-membership uncertainty subdivides into two standard phases that are usually alternated: a prediction step according to the dynamic state Eq. (1) and a correction step that takes measurements into account according to (2). These phases will be considered independently in this paper, but it is of course easy to construct a complete state estimation algorithm for the system described by (1) and (2) subject to the constraints (7) and (8) out of these building blocks. Note that several prediction steps can be concatenated if some observations are missing.

3.1. Prediction

Consider (1) where $x \in \mathbb{R}^n$ and A_k is split into a known matrix A and an uncertain matrix H (the time indices are dropped to simplify notation)

$$z = (A + H)x + w. \tag{19}$$

We are interested in the set F of all z, when x lies in a non-degenerate ellipsoid $E(c, P)$, $P > 0$, while (H, w) is an admissible pair. Note that, contrary to what is usual in the determination of attainability sets, we do not necessarily assume $E(c, P)$ to be centered at the origin. Then:

$$F = \left\{ (A + H)x + w : \ x \in E(c, P), \quad \frac{\|H\|^2}{\varepsilon^2} + \frac{\|w\|^2}{\delta^2} \leqslant 1 \right\}. \tag{20}$$

This set is not an ellipsoid; in most cases it is not even convex (see Examples 1 and 2 below). For the ellipsoidal technique to apply, F should be embedded in some ellipsoid $E(d, Q)$:

$$F \subset E(d, Q). \tag{21}$$

Moreover, we seek the ellipsoid with minimal size. The most natural measures of size are

$$f_1(Q) = \operatorname{tr} Q^{-1}, \tag{22}$$

$$f_2(Q) = -\ln \det Q. \tag{23}$$

Function $f_1(Q)$ is the sum of the squared lengths of the ellipsoidal semi-axes (trace criterion) and $f_2(Q)$ relates to its volume (determinant criterion). Thus the problem is to minimize (22) or (23) subject to (21). The result below reduces this problem to one-dimensional optimization.

Theorem 1. *Each ellipsoid in the family $E(d(\tau), Q(\tau))$ with*

$$d(\tau) = (1 - \delta^2\tau) A Q_\tau^{-1} P c, \tag{24}$$

$$Q(\tau) = \frac{\{A Q_\tau^{-1} A^{\mathrm{T}} + \tau^{-1} I\}^{-1}}{1 - \xi(\tau)}, \tag{25}$$

where

$$Q_\tau = (1 - \delta^2\tau)P - \tau\varepsilon^2 I, \tag{26}$$

$$\xi(\tau) = (1 - \delta^2\tau) c^{\mathrm{T}} P c - (1 - \delta^2\tau)^2 c^{\mathrm{T}} P Q_\tau^{-1} P c, \tag{27}$$

contains F for all τ such that

$$0 < \tau < \tau^* = \frac{\lambda_{\min}}{\delta^2 \lambda_{\min} + \varepsilon^2}, \tag{28}$$

where $\lambda_{\min}$ is the minimal eigenvalue of P.

Proof. Any point $x^+ \in F$ as defined by (20) satisfies $x^+ = Ax + z$, where $x \in E(c, P)$ and $z = Hx + w$, $(H, w) \in S$. Then $Ax \in E(Ac, (A^{-1})^{\mathrm{T}} P A^{-1})$ and z satisfies the quadratic inequality $\|z\|^2 \leqslant \varepsilon^2 \|x\|^2 + \delta^2$ due to Lemma 1. (For simplicity, it is assumed in this proof that A is invertible; otherwise we may consider (19) in a subset of $\mathbb{R}^n$ where A is not degenerate.)

Thus the problem is to check when two quadratic inequalities $(x - c)^{\mathrm{T}} P (x - c) \leqslant 1$ and $\|z\|^2 \leqslant \varepsilon^2 \|x\|^2 + \delta^2$ imply the third $(Ax + z - d)^{\mathrm{T}} Q (Ax + z - d) \leqslant 1$. To write this in standard form, introduce the vector $s = ((Ax)^{\mathrm{T}}, z^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{2n}$, vectors $s_i \in \mathbb{R}^{2n}$ and matrices $M_i \in \mathbb{R}^{2n \times 2n}$, $i = 0, 1, 2$, with

$$M_0 = \begin{pmatrix} Q & Q \\ Q & Q \end{pmatrix}, \quad s_0 = \begin{pmatrix} d \\ 0 \end{pmatrix}, \tag{29}$$

$$M_1 = \begin{pmatrix} (A^{-1})^{\mathrm{T}} P A^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad s_1 = \begin{pmatrix} Ac \\ 0 \end{pmatrix}, \tag{30}$$

$$M_2 = \begin{pmatrix} -\varepsilon^2 (A^{-1})^{\mathrm{T}} A^{-1} & 0 \\ 0 & I \end{pmatrix}, \quad s_2 = 0, \tag{31}$$

and the functions $f_i(s) = (s - s_i)^{\mathrm{T}} M_i (s - s_i)$, $i = 0, 1, 2$. Now, the condition $x^+ \in E(d, Q)$ for all admissible data can be rewritten as

$$f_1(s) \leqslant 1, \quad f_2(s) \leqslant \delta^2 \ \Rightarrow f_0(s) \leqslant 1. \tag{32}$$

For any $\tau_1 \geqslant 0$ and $\tau_2 \geqslant 0$, consider the linear combination of the quadratic functions

$$f_{\tau_{1,2}}(s) \doteq \tau_1 f_1(s) + \tau_2 f_2(s). \tag{33}$$

Define

$$\mathscr{T} = \{(\tau_1, \tau_2)^{\mathrm{T}} : \ \tau_1 \geqslant 0, \ \tau_2 \geqslant 0, \ \tau_1 + \delta^2 \tau_2 \leqslant 1\}. \tag{34}$$

Obviously, for all $(\tau_1, \tau_2)^{\mathrm{T}} \in \mathscr{T}$:

$$f_1(s) \leqslant 1, \quad f_2(s) \leqslant \delta^2 \ \Rightarrow f_{\tau_{1,2}}(s) \leqslant 1. \tag{35}$$

Let $\mathscr{E}_{\tau_{1,2}} = \{s : \ f_{\tau_{1,2}}(s) \leqslant 1, \ (\tau_1, \tau_2)^{\mathrm{T}} \in \mathscr{T}\}$. Consider the inequality $f_{\tau_{1,2}}(s) \leqslant 1$. After simple transformations, it can equivalently be written as

$$(s - s_{\tau_{1,2}})^{\mathrm{T}} M_{\tau_{1,2}} (s - s_{\tau_{1,2}}) \leqslant 1 - \xi_{\tau_{1,2}}, \tag{36}$$

where

$$M_{\tau_{1,2}} = \tau_1 M_1 + \tau_2 M_2,$$

$$s_{\tau_{1,2}} = \tau_1 M_{\tau_{1,2}}^{-1} M_1 s_1, \tag{37}$$

$$\xi_{\tau_{1,2}} = \tau_1 s_1^{\mathrm{T}} M_1 s_1 - s_{\tau_{1,2}}^{\mathrm{T}} M_{\tau_{1,2}} s_{\tau_{1,2}}. \tag{38}$$

Since $P > 0$, there exists $(\tau_1, \tau_2)^T \in \mathscr{T}$ such that $M_{\tau_{1,2}} > 0$ and therefore $1 - \xi_{\tau_{1,2}} > 0$, so

$$(s - s_{\tau_{1,2}})^T \frac{M_{\tau_{1,2}}}{1 - \xi_{\tau_{1,2}}} (s - s_{\tau_{1,2}}) \leqslant 1. \tag{39}$$

Thus, the set $\mathscr{E}_{\tau_{1,2}}$ is an ellipsoid $E(s_{\tau_{1,2}}, M_{\tau_{1,2}}/(1 - \xi_{\tau_{1,2}}))$ in R^{2n} for all $(\tau_1, \tau_2)^T \in \mathscr{T}$ such that $M_{\tau_{1,2}} > 0$. The condition $M_{\tau_{1,2}} > 0$ is equivalent to $\tau_1 P - \tau_2 \varepsilon^2 I > 0$. The implication (32) is satisfied, i.e. $f_0(s) \leqslant 1$, if $s_0 = s_{\tau_{1,2}}$ and $M_0 \leqslant M_{\tau_{1,2}}/(1 - \xi_{\tau_{1,2}})$. For this reason, we write

$$s_0 = \tau_1 M_{\tau_{1,2}}^{-1} M_1 s_1,$$

$$M_0 \leqslant \frac{M_{\tau_{1,2}}}{1 - \xi_{\tau_{1,2}}}. \tag{40}$$

The matrix inequality in (40) is equivalent to

$$\begin{pmatrix} Q - \dfrac{(A^{-1})^T[\tau_1 P - \tau_2 \varepsilon^2 I]A^{-1}}{1 - \xi_{\tau_{1,2}}} & Q \\ Q & Q - \dfrac{\tau_2 I}{1 - \xi_{\tau_{1,2}}} \end{pmatrix} \leqslant 0. \tag{41}$$

The application of Lemma 2 to (41) implies that

$$Q^{-1} \geqslant (1 - \xi_{\tau_{1,2}}) \{A[\tau_1 P - \tau_2 \varepsilon^2 I]^{-1} A^T + \tau_2^{-1} I\} \tag{42}$$

is equivalent to (41), provided that $\tau_1 \geqslant 0$, $\tau_2 > 0$, $\tau_1 + \delta^2 \tau_2 \leqslant 1$ and $\tau_1 P - \tau_2 \varepsilon^2 I > 0$. We seek the minimal ellipsoid in the family $E(d, Q)$ according to (22) or (23). Therefore, the matrix inequality (42) is converted into an equality while parameters $\tau_1 \geqslant 0$ and $\tau_2 > 0$ satisfy the equation $\tau_1 + \delta^2 \tau_2 = 1$, which leads to a one-parametric family of ellipsoids. Indeed, let $\tau_2 = \tau$, $\tau_1 = 1 - \delta^2 \tau$ with $\tau > 0$. The condition $\tau_1 P - \tau_2 \varepsilon^2 I = (1 - \delta^2 \tau) P - \tau \varepsilon^2 I > 0$ is then equivalent to $\tau < \tau^* = \lambda_{\min}/(\delta^2 \lambda_{\min} + \varepsilon^2)$, where $\lambda_{\min}$ is the minimal eigenvalue of P. For all $\tau : 0 < \tau < \tau^*$, the equation

$$Q^{-1} = (1 - \xi(\tau)) \{A Q_\tau^{-1} A^T + \tau^{-1} I\} \tag{43}$$

gives (25), where Q_τ is given by (26) and $\xi(\tau)$ by (27). The scalar function $\xi(\tau)$ is such that $\xi(\tau) < 1$ for all τ in the interval defined by (28); hence, the matrix $Q(\tau)$ is positive definite over this interval. The equation giving s_0 in (40) leads to (24). This completes the proof. □

Now, minimizing a function $\varphi_1(\tau) = \operatorname{tr} Q(\tau)^{-1}$ or $\varphi_2(\tau) = -\ln \det Q(\tau)$ over $0 < \tau < \tau^*$, we obtain a suboptimal ellipsoid that contains the set F. This is a complete analog of the situation with no model uncertainty, where one-parametric optimization is required to construct the best ellipsoid for the prediction step, compare Fogel and Huang (1982), Chernousko (1994), Kurzhanskii and Valyi (1997), Polyak (1998), Durieu et al. (2001). Note that proving the convexity of $\varphi_1(\tau)$ and $\varphi_2(\tau)$ over $0 < \tau < \tau^*$ is an open problem.

Theorem 1 provides a useful suboptimal solution for the prediction step of ellipsoidal state estimation, and this algorithm can be incorporated directly into a state estimator for system with measurements.

A simpler method of outer approximation of attainability sets was proposed by Chernousko and Rokityanskii (2000). Their algorithm computes the ellipsoid $E \supset F$ as an approximation of the sum of $E(Ac, (A^{-1})^T P A^{-1})$ and a ball of radius r given by $E(0, r^{-2} I)$, with

$$r^2 = \varepsilon^2 \max_{x \in E(c,P)} \|x\|^2 + \delta^2$$

$$= \varepsilon^2 \left(\frac{1}{\lambda_{\min}(P)} + 2\sqrt{c^T P^{-1} c} + \|c\|^2 \right) + \delta^2. \tag{44}$$

Indeed, the sum of these two ellipsoids is already convex and contains F, and it is easy to construct an ellipsoid $E(Ac, \tilde{Q})$ that contains this sum. The simplicity of such an approach is obviously attractive. But, as illustrated by the next examples, such an estimation may be much more conservative than the one obtained by our method.

Example 1. For $P = \operatorname{diag}\{1/9, 1\}$, $c = (1, 2)^T$, $A = \operatorname{diag}\{1, 1\}$, $\varepsilon = 1$ and $\delta = 0.5$, the attainability set F is non-convex (see Fig. 1) and the ellipsoid $E(d(\tau), Q(\tau))$ with $d(\tau)$ from (24) and $Q(\tau)$ from (25) contains F for all τ such that $0 < \tau < \tau^* = \frac{4}{37}$. For simplicity, we omit the exact expressions for $d(\tau)$ and $Q(\tau)$. The minimal trace ellipsoid $E(d, Q)$ in the family of Theorem 1 (solid line on Fig. 1) gives an outer approximation of the attainability set F.

Compare with the significantly more conservative outer approximation obtained as in Chernousko and Rokityanskii (2000), and represented on Fig. 1 by a dotted line. The difference in conservatism between the two algorithms is even more pronounced if they are used recursively for long-range prediction.

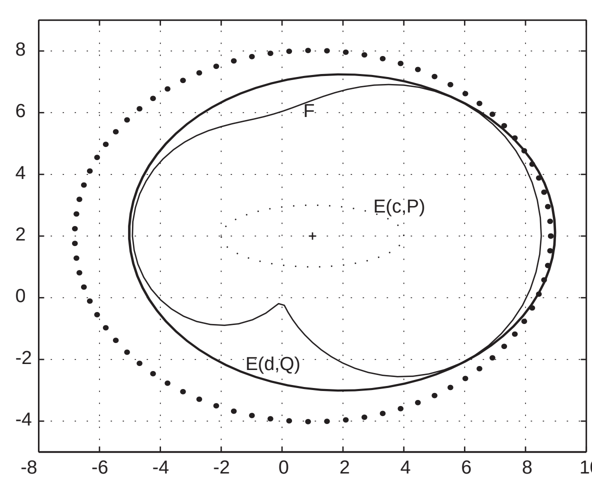

Fig. 1. Outer approximations of attainability set.

The initial ellipsoid $E(c,P)$ can sometimes be assumed to be centered at the origin, i.e. $c=0$. This assumption is natural, for instance when dealing with a reachability problem with no measurements. Indeed, if $x_0 \in E(0,P_0)$, then $x_k \in E(0,P_k)$ due to the symmetry of the reachability set at each step. Under this assumption the optimal ellipsoid in the one-parameter family happens to be optimal among all ellipsoids containing the set F, and Theorem 1 is transformed into a stronger result.

Theorem 2. *If the ellipsoid $E(c,P)$ is centered at the origin ($c=0$) and $\delta > 0$, then the one-parameter family $E(0,Q(\tau))$ with*

$$Q(\tau) = \{A[(1-\delta^2\tau)P - \tau\varepsilon^2 I]^{-1}A^{\mathrm{T}} + \tau^{-1}I\}^{-1} \tag{45}$$

contains F for all τ satisfying (28). The minimization of the one-dimensional smooth and convex function $\varphi_1(\tau) = \operatorname{tr} Q(\tau)^{-1}$ or $\varphi_2(\tau) = -\ln\det Q(\tau)$ subject to (28) provides the minimal-trace or minimal-volume ellipsoid (21) containing F.

Proof. The proof follows the same lines as in Theorem 1. The condition $c=0$ implies $s_1=0$. Then f_0, f_1 and f_2 in (32) are homogeneous quadratic forms. For $s^0=0$, $f_1(s^0)=0 < \alpha_1$ with $\alpha_1=1$ and $f_2(s^0)=0<\alpha_2$ with $\alpha_2=\delta^2>0$. So, the last two inequalities in (18) are satisfied. Moreover, it is always possible to choose $\mu_2>0$ small enough so that the first inequality in (18) is satisfied for $\mu_1=1$. Therefore, Lemma 4 applies provided that $N=2n>3$, i.e. $n \geqslant 2$. According to Lemma 4 condition (32) is equivalent to

$$M_0 \leqslant \tau_1 M_1 + \tau_2 M_2, \tag{46}$$

where $\tau_1 \geqslant 0$, $\tau_2 \geqslant 0$ and $\tau_1 + \delta^2\tau_2 \leqslant 1$. After simple transformations, this linear matrix inequality can be written as

$$Q^{-1} \geqslant A(\tau_1 P - \tau_2\varepsilon^2 I)^{-1}A^{\mathrm{T}} + I/\tau_2 \tag{47}$$

with $0 \leqslant \tau_1 \leqslant 1-\delta^2\tau_2$, $0<\tau_2$, provided that $\tau_1 P - \tau_2\varepsilon^2 I > 0$. We seek the minimal ellipsoid in terms of (22) or (23). Therefore, the solution will be achieved for $\tau_1 = 1-\delta^2\tau_2$ and equality instead of inequality. Thus (47) is converted into a one-parameter family of matrices (45) with τ satisfying (28).

The smoothness and convexity of the objective functions $\varphi_1(\tau) = \operatorname{tr} Q(\tau)^{-1}$ and $\varphi_2(\tau) = -\ln\det Q(\tau)$ with $Q(\tau)$ defined by (45) can be proved as was done in (Durieu et al., 2001) for similar functions. Its minimization gives the best ellipsoidal approximation in terms of the trace or determinant criterion. □

The following example illustrates the application of Theorem 2.

Example 2. For $P = \operatorname{diag}\left\{\frac{1}{9}, 1\right\}$, $A = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$ and $\varepsilon=\delta=0.5$, the ellipsoids $E(0,Q(\tau))$ with $Q(\tau)$ given by (25) contain the attainability set F for all τ such that $0<\tau<\tau^* = \frac{2}{5}$.

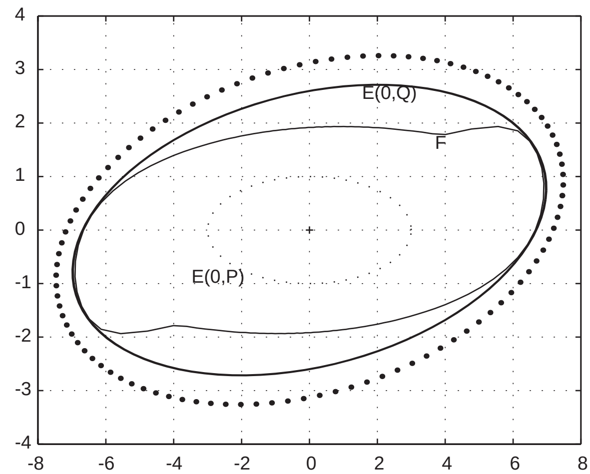

Fig. 2. Outer approximations of attainability set ($c=0$).

In order to find the minimal ellipsoid according to (22) or (23), we need to solve a simple polynomial equation which has a unique root $\tau_{\min}$ in the interval $(0, \frac{2}{5})$. The resulting ellipsoid is the optimal outer-bounding approximation of the set F indicated by a solid line on Fig. 2.

Compare again with the significantly more conservative outer approximation obtained as in Chernousko and Rokityanskii (2000) and represented on Fig. 2 by a dotted line.

3.2. Correction

Consider now a linear dynamic system with measurements (2). An appropriate operation for the correction step of classical ellipsoidal state estimation is the intersection of ellipsoids. In general, ellipsoids are considered possibly degenerate, i.e. their matrices are only positive semi-definite. A scalar observation y, for example, defines a set of possible locations of the state vector x that is a strip in $\mathbb{R}^n$ if the observation matrix C is certain. The basic tool for the classical approximation of the intersection of ellipsoids is Lemma 3. However, we assume here that the observation matrix C_k in (2) is split into a known matrix C and an uncertain matrix H:

$$y = (C+H)x + w, \tag{48}$$

with $y \in \mathbb{R}^m$, $C \in \mathbb{R}^{m\times n}$ and (H,w) an admissible pair. For a given vector of measurements y and a given nominal observation matrix C, we need to estimate the set of all state vectors x that are consistent with the above data. According to Lemma 1

$$\|y - Cx\|^2 \leqslant \varepsilon^2\|x\|^2 + \delta^2, \tag{49}$$

or equivalently

$$x^{\mathrm{T}}(C^{\mathrm{T}}C - \varepsilon^2 I)x - 2x^{\mathrm{T}}C^{\mathrm{T}}y + y^{\mathrm{T}}y - \delta^2 \leqslant 0. \tag{50}$$

Assume that ε^2 is not an eigenvalue of C^TC, i.e. $C^TC - \varepsilon^2 I$ is invertible, which is generically true. Then (50) can be rewritten in terms of a quadratic form:

$$(x-d)^T M(x-d) \leqslant 1, \tag{51}$$

where

$$\left.\begin{aligned} M &= (y^T C R^{-1} C^T y - y^T y + \delta^2)^{-1} R,\\ d &= R^{-1} C^T y,\\ R &= C^T C - \varepsilon^2 I. \end{aligned}\right\} \tag{52}$$

The matrices R and M may not be positive or even non-negative definite. Therefore, the set of all x that satisfy (51) is not necessarily an ellipsoid or a strip. It depends on the values of C and ε. Nevertheless, an ellipsoidal technique can be used to deal with the intersection of this set with some non-degenerate ellipsoid. The main result of this section can now be stated.

Theorem 3. *If x belongs to $E(c,P)$, $P>0$ and satisfies $y=(C+H)x+w$, where (H,w) is an admissible pair, then x also belongs to the ellipsoid $E(g(\tau),Q(\tau))$ with*

$$\left.\begin{aligned} Q(\tau) &= (1-\nu_\tau)^{-1} Q_\tau,\\ Q_\tau &= (1-\tau)P + \tau M,\\ g(\tau) &= Q_\tau^{-1}[(1-\tau)Pc + \tau M d],\\ \nu_\tau &= (1-\tau)c^T P c + \tau d^T M d - g(\tau)^T Q_\tau g(\tau) \end{aligned}\right\} \tag{53}$$

for all τ such that $0 \leqslant \tau < \tau^* = \min\{1,\ 1/(1-\lambda_{\min})\}$, where M and d are defined by (52) *and $\lambda_{\min}$ is the minimal generalized eigenvalue of the matrix pair (M,P). (The generalized eigenvalues λ_i and eigenvectors v_i of the matrix pair (M,P) are defined as $Mv_i = \lambda_i P v_i$.)*

Proof. Since $x \in E(c,P) = \{x\ :\ (x-c)^T P(x-c) \leqslant 1\}$, $P>0$ and inequality (51) holds, then

$$f_\tau(x) = (1-\tau)(x-c)^T P(x-c) + \tau(x-d)^T M(x-d) \leqslant 1, \quad \forall \tau : 0 \leqslant \tau \leqslant 1. \tag{54}$$

Let $Q_\tau = (1-\tau)P + \tau M$ and $\lambda_{\min} = \min \operatorname{eig}(M,P)$. Since $P>0$, it is trivial to conclude that

$$\forall \tau:\ 0 \leqslant \tau < \tau^*, \quad Q_\tau > 0, \tag{55}$$

where $\tau^* = \min\{1,\ 1/(1-\lambda_{\min})\}$.

Further, consider the set $E = \{x\ :\ f_\tau(x) \leqslant 1\}$. Then Lemma 3 can be applied under condition (55) for parameter τ to obtain the final formulas in (53). □

Optimizing $E(g(\tau),Q(\tau))$ with respect to τ with (22) or (23) gives the minimal ellipsoid in this parametrized family containing the intersection. However, it may be a suboptimal estimate in the class of all ellipsoids in the same way as the solution for the prediction step (Theorem 1). It will be optimal only if the centers c and d of the sets to be intersected become equal; this can be proved relying on results in Polyak (1998). But this assumption is not as natural as for the computation of attainability sets. Note that when c and d differ the center g of the resulting ellipsoid is no longer equal to c. Notice also that the problem of proving the convexity of the scalar functions $\varphi_1(\tau) = \operatorname{tr} Q(\tau)^{-1}$ and $\varphi_2(\tau) = -\ln\det Q(\tau)$ over $0 \leqslant \tau < \tau^*$ for $E(g(\tau),Q(\tau))$ remains open.

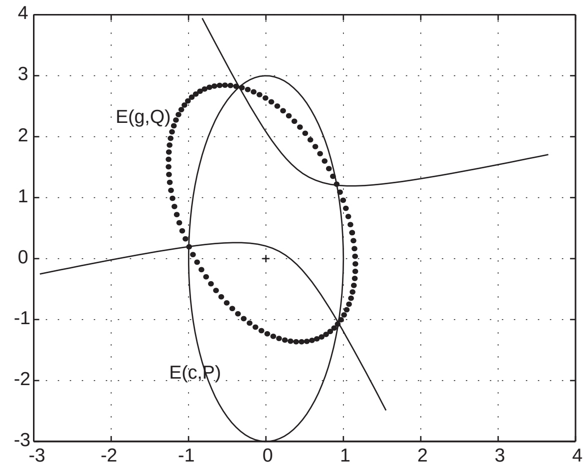

Fig. 3. Outer approximation of intersection.

To illustrate specific features of measurements with an uncertain observation matrix, a scalar output, i.e. $m=1$, is considered. In this case an admissible vector x for fixed y lies inside a hyperboloid in $\mathbb{R}^n$; while in the situation with no model uncertainty ($\varepsilon=0$) it lies inside a strip in $\mathbb{R}^n$.

Example 3. Take $m=1$, $n=2$, $y=1$, $C=(1,2)$, $\varepsilon=1.5$ and $\delta=0.5$. Consider the non-degenerate prior ellipsoid $E(0,P)$ with $P=\operatorname{diag}\{1,\frac{1}{9}\}$. The two eigenvalues of R as defined by (52) have different signs, so neither R nor M is positive or non-negative definite. Therefore the set of all $x \in \mathbb{R}^n$ such that $(x-d)^T M(x-d) \leqslant 1$ is the interior of a hyperbola (see Fig. 3). We calculate $\lambda_{\min} = \min\operatorname{eig}(M,P) = -2.95$ and $\tau^* = 0.253 < 1$. The optimal ellipsoid in the family provided by Theorem 3 can be computed in terms of the trace or determinant criterion. Fig. 3 shows the resulting approximation for the trace criterion.

4. Some extensions

4.1. Frobenius norm for perturbation

Lemma 1 still holds true if the spectral matrix norm is replaced with the Frobenius norm in the model uncertainty constraints. The proof remains the same because $\|Hx\| \leqslant \|H\|_F\|x\|$ and $\|zx^T\|_F = \|z\|\,\|x\|$, where x and z are vectors. For this reason, no difference should be found in the formulation of the state estimation problems and resulting algorithms. Thus, all estimation algorithms derived from Theorems 1–3 can obviously be extended to models with uncertainties in (7) and (8) with the Frobenius matrix norm.

4.2. Approximation of sum for separate uncertainties

The combined quadratic constraint (7) may sometimes seem too restrictive and inflexible. Often, only separate inequalities on model uncertainty and additive disturbances are available. The methods proposed in this paper could also help to construct reliable ellipsoidal estimates in this case.

Consider the following situation with a linear state-space model

$$x_{k+1} = (A_k + \Delta_k)x_k + W_k w_k, \tag{56}$$

with $x_0 \in E(c_0, P_0)$, $\|\Delta_k\| \leqslant \varepsilon_k$ and $\|w_k\| \leqslant \delta_k$. The spectral norm is used for matrices. Note that the model (56) covers the case with non-square excitation matrices W_k. Without loss of generality, a one-step prediction for such a process

$$z = (A + \Delta)x + Ww;$$
$$x \in E(c, P), \quad \|\Delta\| \leqslant \varepsilon, \quad \|w\| \leqslant \delta \tag{57}$$

can be studied. Then $Ax \in E(Ac, (A^{-1})^{\mathrm{T}}PA^{-1})$, $\|\Delta x\| \leqslant \varepsilon\|x\|$ and $Ww \in E_1$, where $E_1 = \{Ww: \ \|w\| \leqslant \delta\}$ is a possibly degenerate ellipsoid. In order to find an ellipsoid $E(d, Q)$ that contains all possible vectors z under the given data (see (57)), it suffices to perform the next two operations:

(a) find an ellipsoid $E(d_a, Q_a)$ containing the vector sum $z_a = Ax + \Delta x$ directly from Theorem 1 with $\delta = 0$;
(b) compute an optimal outer approximation of the sum of the ellipsoids $E(d_a, Q_a)$ and E_1 to get the final approximation $E(d, Q)$ of vector $z = z_a + Ww$ (see, for instance (Durieu et al., 2001)).

Each step produces a suboptimal approximation, therefore the final ellipsoid gives a suboptimal estimate for the reachable set of dynamic system. Note that the order of operations in the procedure can be inverted; this would lead to a different suboptimal solution.

Hence, the dynamic system (56) with general separate constraints on model uncertainty and additive disturbance vector is also tractable. The proposed ellipsoidal technique involves two steps, one is based on Theorem 1 and the other is the classical ellipsoidal approximation of a sum.

4.3. Approximation of intersection for separate uncertainties

When we deal with dynamic systems with measurements, the combined quadratic constraint (8) leads to the quadratic inequality (49). Unfortunately, this is no longer the case when the constraints are separate. Indeed, the set of all state vectors x consistent with the measurement equation

$$y = (C + \Delta)x + w, \quad \text{with} \quad \|\Delta\| \leqslant \varepsilon, \|w\| \leqslant \delta, \tag{58}$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, is equivalently defined by the inequality

$$\|y - Cx\| \leqslant \varepsilon\|x\| + \delta, \tag{59}$$

which is not a quadratic constraint. For this reason, Theorem 3 does not apply directly. Nevertheless, (59) implies that

$$\|y - Cx\|^2 \leqslant \varepsilon^2\|x\|^2 + \left(\delta^2 + 2\varepsilon\delta \max_{x \in E(c,P)} \|x\|\right). \tag{60}$$

Theorem 3 can then be applied with δ^2 replaced by $\tilde{\delta}^2 = \delta^2 + 2\varepsilon\delta \max_{x \in E(c,P)}\|x\|$, which can be easily computed.

An ellipsoidal approximation of intersection at the correction step for dynamic system with separate constraints (58) on model uncertainty and additive disturbances can thus be obtained. Of course, the resulting approximation is suboptimal.

5. Conclusions

In the present paper an outer-bounding ellipsoidal technique for the estimation of the state for a linear discrete-time dynamic system under model uncertainty characterized by (7) and(8) has been proposed. As usual, two operations have been considered, which are at the core of the state estimator. The first one is some generalized sum of ellipsoids and the second is an intersection. A suboptimal solution for each operation is obtained, which reduces the search for an outer-bounding ellipsoid to a one-dimensional optimization. For the particular case of reachability problems under zero initial conditions where the dynamic system has no measurements and no deterministic inputs the resulting ellipsoid is a one-step optimal estimate of reachable sets. The above techniques form the basic blocks of a recursive state estimator. Moreover, as shown in Section 4, the proposed algorithms readily extend to more general models of uncertainty for dynamic systems with measurements.

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