THE INVARIANT ELLIPSOIDS TECHNIQUE FOR ANALYSIS AND DESIGN OF LINEAR CONTROL SYSTEMS

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ABSTRACT. In this paper an approach based on invariant ellipsoids is applied to the problem of persistent disturbance rejection by means of static state-feedback control. The dynamic system is supposed to be linear time-invariant and affected by unknown-but-bounded exogenous disturbances. Synthesis of an optimal controller that returns a minimum of the size of the corresponding invariant ellipsoid is reduced to one-dimensional convex minimization with LMI constraints.

1. INTRODUCTION. Unknown-but-bounded description of uncertain variables forms the guaranteed set-membership approach to the problems appeared in the system and control theory. This approach has received much attention as the main alternative to stochastic techniques that have been developed for estimation, control and identification. The key point of it is that there is no need to know the statistical distribution of model errors and disturbances except only its lower and upper bounds or set values. This assumption looks to be more acceptable in practice in many situations.

This approach was initially established by D. Bertsekas and F. Schweppe (see [1], [2] and [3] as basic references). Its concept is based on the analysis of reachable and feasible sets for uncertain dynamic models or on the search for their approximations by simple convex domains like boxes, polyhedra or ellipsoids, which have generated a variety of further works on this and related topics. An attractive tool within the set-membership framework is the notion of invariance and invariant (or positively invariant) sets. A set in the state space is said to be *positively invariant* for a given dynamical system if every trajectory initiated in this set remains inside it at all future time instants. Positively invariant set has the property that it always contains the reachable set of dynamic system. In spite of conservatism in approximations of reachable domains, invariant sets have a close relation to Lyapunov functions, and for this reason they are quite useful for the synthesis of feedback control. A good reference in this sense is the survey paper [4], which gives a broad outlook of applications of invariant set theory in automatic control. Among various special families of positively invariant sets, a particular class of ellipsoidal invariant sets can be emphasized. The main advantage of invariant ellipsoids lies in its simple characterization as a solution of parametrized linear matrix inequalities. Therefore the optimal control problems in this description can be reduced to semidefinite programs, i.e., to the optimization of a linear function under LMI constraints. This optimization problem is convex and is now a powerful tool in many control applications [5].

Using the technique of invariant ellipsoids in this paper, the problem of persistent disturbance rejection is considered. The question of how to compensate the effect of persistent unknown-but-bounded disturbances by means of feedback control is very important in system engineering. First type of methods appeared on this topic is founded on dynamic programming technique [1, 6, 7]. Another one is the popular l_1-optimal control theory [8, 9], which is formulated in terms of the worst-case peak-to-peak gain minimization. Alternative to the dynamic programming and l_1 approaches, which often suffer from a high complexity of an optimal solution, is the methods based on upper bounds of l_1-norm such as the so-called *-norm introduced in [10]. Minimization of the *-norm allows determination of fixed-order controllers that compensate the disturbance influence. Some properties of this norm are discussed in [11], in particular, it is shown that this norm can be very conservative upper bound for l_1-norm. Direct analogues of the *-norm are invariant sets for dynamical systems. The approach established invariant sets for the disturbance rejection has already attracted some attention [12, 13]. We believe that invariant ellipsoids technique is a challenging alternative to l_1 approach, because the last one is based on the assumption $x(0) = 0$, and nonzero initial conditions can cause serious troubles. For instance, the examples in [11] demonstrate non-robustness with respect to $x(0) \neq 0$. In contrast, invariant ellipsoids automatically cover nonzero initial conditions.

The objective of the present paper is to use the LMI technique to treat invariant ellipsoidal sets with its subsequent use for the problem of persistent disturbance rejection. A detailed comparison with the results of [10] is given below. The key point of the method is that for linear time-invariant dynamical systems the search for an optimal static state-feedback controller can be reduced to semidefinite programming and to one-dimensional convex optimization.

2. INVARIANT ELLIPSOIDS: ANALYSIS. Consider an LTI dynamical system

$$\begin{cases} \dot{x} = Ax + Bw, \\ y = Cx, \end{cases} \tag{1}$$

$x(t) \in \mathbf{R}^n$, $w(t) \in \mathbf{R}^m$, $y(t) \in \mathbf{R}^l$. It is assumed that matrix A is stable (its eigenvalues have negative real parts) and $\| w(t) \| \leq 1 \;\; \forall t \geq 0$, where $\| \cdot \|$ is the Euclidean vector norm.

Denote the non-degenerate ellipsoid in $\mathbf{R}^n$ centered at the origin as

$$E = \{x \in \mathbf{R}^n : x^T P^{-1} x \leq 1\}, \quad P > 0.$$

Definition 1. Ellipsoid E is said to be positively invariant for dynamical system (1) if $x(0) \in E$ implies $x(t) \in E \;\; \forall t \geq 0$ for every system trajectory x.

In this paper we focus only on the case of non-degenerate invariant ellipsoids (with non-empty interior). For this reason, the condition of controllability for matrix pair (A, B) is supposed to be satisfied for system (1), i.e., $[B \;\; AB \ldots A^{n-1}B] = n$.

It is obvious that for the dynamic system under consideration the invariant ellipsoid E always contains the reachable set

$$R = \{x(t) \in R^n : \;\; \dot{x} = Ax + Bw, \;\; x(0) = 0, \;\; \| w(t) \| \leq 1, \;\; t \geq 0\}.$$

This notion is used in [14] as a basic tool for construction of the ellipsoidal state estimation methods for continuous-time models. A family of invariant ellipsoids is defined by the following theorem.

Theorem 1 [5, 10]. The non-degenerate ellipsoid E is positively invariant for the stable dynamical system (1) with $\| w(t) \| \leq 1 \; \forall t \geq 0$, if and only if $P > 0$ satisfies

$$AP + PA^T + \alpha P + \frac{1}{\alpha} BB^T \leq 0 \tag{3}$$

for some $\alpha > 0$, where (A, B) is controllable.

We are interested in the minimal invariant ellipsoids. If the dynamical system is stable, then there exists a unique invariant ellipsoid, which minimizes some certain criterion $\varphi(P)$. The most natural measures of size for ellipsoids are: $f_1(P) = \mathrm{Tr}\,P$, $f_2(P) = \det P$, or $f_3(P) = \| P \|$ – spectral matrix norm. As a criterion in this paper we take $\varphi(P) = \mathrm{Tr}CPC^T$. Minimization of this objective function corresponds to the search for a minimal trace invariant ellipsoid for the system output $y(t) = Cx(t)$.

Theorem 2 [10]. All minimal invariant ellipsoids of the stable dynamical system (1) with controllable pair (A, B) belong to the one-parameter family of ellipsoids with matrices $P(\alpha)$ for $0 < \alpha < \alpha^*$, where $P(\alpha) > 0$ is the solution of the Lyapunov equation

$$AP + PA^T + \alpha P + \frac{1}{\alpha} BB^T = 0,$$

and $\alpha^* = -2 \max \mathrm{Re}\, \lambda_i(A)$, $\lambda_i(A)$ are eigenvalues of matrix A. Moreover, the function $\varphi(\alpha) = \mathrm{Tr}CP(\alpha)C^T$ is strictly convex on the interval $0 < \alpha < \alpha^*$.

Therefore the one-dimensional minimization

$$\begin{gathered} \min_{0<\alpha<\alpha^*} \mathrm{Tr}\, CPC^T \\ \text{subject to } AP + PA^T + \alpha P + \alpha^{-1} BB^T = 0 \end{gathered} \tag{2}$$

is strictly convex over $0 < \alpha < \alpha^*$ and has a unique solution on this interval.

Most results of the analysis of the invariant sets and its application in control deal with continuous-time dynamical systems. Similar results can also be obtained for the dicrete-time case, see [16].

Example 1. Take the discrete-time system $\begin{cases} x_{k+1} = A_{x_k} + Bw_k, \\ y_k = Cx_k, \end{cases}$ with $A = 0.5 \cdot I$, $B = C = I$, where I is the identity 2×2-matrix. Since matrix A is stable (its spectral radius $\rho(A) = 0.5 < 1$) and matrix pair (A, B) is controllable, the Lyapunov equation (8) has a unique positive definite solution for every fixed α such that $\alpha \in (\rho^2(A), 1)$. These solutions form a one-parameter family of invariant ellipsoids of the system. The minimal invariant ellipsoid $E_{\min}$ of the system belongs to this family and is obtained via a scalar convex optimization. Here $E_{\min}$ represents a ball of radius $r = 2$ (thick line in Fig. 1). As an example, the system trajectory initiated at $x_0 = (1, 1)^T$ and distorted by the disturbance vector $w = Ax/\| Ax \|$ is shown in the figure. The disturbance $w(t)$ is the unit vector that is antiparallel to Ax for all $t \geq 0$. Therefore, this trajectory approaches the boundary of $E_{\min}$ as $k \to \infty$, but never intersects the boundary of this minimal invariant ellipsoid.

3. INVARIANT ELLIPSOIDS: SYNTHESIS. In order to compensate the influence of persistent unknown-but-bounded disturbances on the model output in

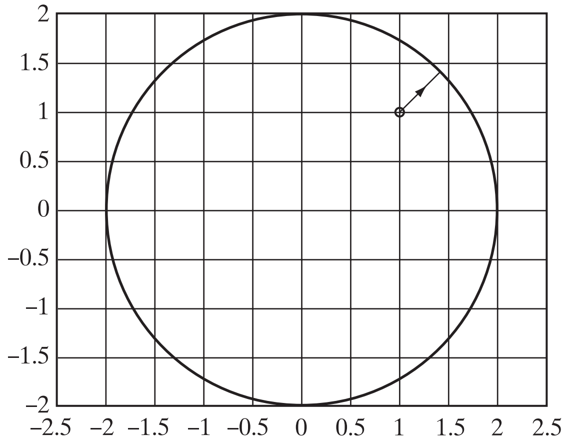

Fig. 1. The minimal invariant ellipsoid

the LTI dynamical system, a static state-feedback control is introduced in this section. Minimization of this influence leads to the search for the minimal invariant ellipsoid for the closed-loop system as described previously. Let us consider the dynamical system governed by

$$\begin{aligned} \dot{x} &= Ax + B_1 u + Dw, \\ y &= Cx + B_2 u, \\ u &= Kx. \end{aligned} \tag{3}$$

Matrix A is not assumed to be stable, but the pair (A, B_1) is supposed to be controllable, and $\| w(t) \| \leq 1$. In addition, assume $B_2^T C = 0$. The goal is to find a static controller K, which stabilizes the closed-loop system and optimally rejects the influence of unknown-but-bounded disturbances $w(t) \in \mathbf{R}^m$ in the sense of minimizing the size of the invariant ellipsoid $E_y = \{y \in \mathbf{R}^m : y^T (CPC^T)^{-1} y \leq 1\}$, for the output vector $y(t) \in \mathbf{R}^l$.

System (3) can be represented as

$$\begin{cases} \dot{x} = (A + B_1 K)x + Dw, \\ y = (C + B_2 K)x, \end{cases}$$

that is, in the setup of model (1). Then the problem is reduced to (2) after replacing matrix A by $A + B_1 K$ and matrix C by $C + B_2 K$. It gives us the next optimization

$$\min \mathrm{Tr}[(C + B_2 K) P (C + B_2 K)^T]$$

under

$$(A + B_1 K)P + P(A + B_1 K)^T + \alpha P + \frac{1}{\alpha} DD^T = 0.$$

The last relation is a bilinear matrix equation with respect to P and K. Introduce the new variable $Y = KP$. Since $B_2^T C = 0$, the objective function to be minimized is rewritten as $\text{Tr}[CPC^T + B_2 YP^{-1}Y^T B_2^T]$ subject to the linear equality constraint

$$AP + PA^T + \alpha P + B_1 Y + Y^T B_1^T + \frac{1}{\alpha} DD^T = 0.$$

The function $f(P,Y) = \text{Tr}[YP^{-1}Y^T]$ is convex in matrix variables $P > 0$ and Y. This reduces the problem to multi-dimensional convex optimization. But on the other hand, we can write it in terms of LMIs. Indeed, consider the matrix

$$H = \begin{pmatrix} Z & Y \\ Y^T & P \end{pmatrix},$$

where Z is an auxiliary matrix variable. Then, $H \geq 0$ implies $Z \geq YP^{-1}Y^T \geq 0$. Therefore, minimization of $\text{Tr}[B_2 YP^{-1}Y^T B_2^T]$ is equivalent to minimization of $\text{Tr}[B_2 Z B_2^T]$, and the following result can now be validated.

Theorem 3. The original problem is equivalent to

$$\text{Tr}[CPC^T + B_2 Z B_2^T] \to \min \tag{10}$$

subject to LMI constraints

$$\begin{pmatrix} Z & Y \\ Y^T & P \end{pmatrix} \geq 0, \quad P > 0, \qquad AP + PA^T + \alpha P + B_1 Y + Y^T B_1^T + \frac{1}{\alpha} DD^T = 0.$$

Note that this is a semidefinite programming (SDP) problem for any fixed $\alpha > 0$, and we can exploit standard LMI Toolbox for its numerical solution. Optimization over α is a convex optimization problem.

The state-feedback controller K obtained from Theorem 3 returns the minimum of invariant ellipsoid size (in the sense of trace criterion) for the output of the closed-loop system. However, it optimally reduces the influence of disturbances only for the case when trajectories are initiated within the smallest invariant ellipsoid itself, which can be quite restrictive in many situations. In order to overcome this difficulty, some prior conditions on invariant ellipsoid size can be considered. For instance, let

$$P \geq P_0,$$

where $P_0 > 0$ represents the matrix of lower-bound ellipsoid containing all system trajectories of interest. Inequality (11) can be introduced to the LMI problem of Theorem 3.

In addition, the constraint on the control action can also be taken into account. Consider $u \in \mathbf{R}^p$, $\mu > 0$, and assume

$$\| u \| \leq \mu. \tag{4}$$

The following lemma reduces the constraint (4) to the equivalent LMI.

Lemma 1. Let $P > 0$ be the matrix of invariant ellipsoid for system (3) with $\| w \| \leq 1$. Let $Y = KP$. The inequality (4) holds, if and only if matrices P

and Y satisfy

$$\begin{pmatrix} P & Y^T \\ Y & \mu^2 I \end{pmatrix} \geq 0,$$

where I is the identity matrix of appropriate dimension.

Thus, the original problem with additional constraints on invariant ellipsoid size and on the control action is equivalent to the SDP and 1D convex optimization.

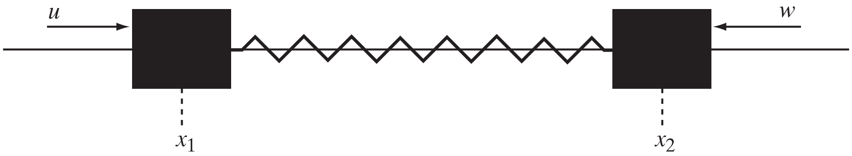

Fig. 2. Double mass-spring system

4. EXAMPLE. Consider an example with oscillation of two unit masses connected by elastic spring and sliding without friction along a fixed horizontal rod (see Fig. 2). The control input $u \in \mathbf{R}$ is applied to the left mass in order to compensate the external disturbance $w \in \mathbf{R}$ exerted on the right one. The state vector of the system is $x = (x_1, \nu_1, x_2, \nu_2)^T$, where x_1, ν_1 and x_2, ν_2 are the values of the coordinate and velocity for the left and right bodies, respectively. The output vector is assumed to be $y = (u,\ x_2)^T$. Then the continuous-time model of this double pendulum is described by

$$\begin{aligned} \dot{x}_1 &= \nu_1 \\ \dot{\nu}_1 &= -x_1 + x_2 + u \\ \dot{x}_2 &= \nu_2 \\ \dot{\nu}_2 &= x_1 - x_2 - w \\ y &= (u \ \ x_2)^T \end{aligned} \quad \Leftrightarrow \quad \begin{cases} \dot{x} = Ax + B_1 u + Dw, \\ y = Cx + B_2 u, \end{cases}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Matrix pair $(A,\ B_1)$ is controllable, and $B_2^T C = 0$. Let $u = Kx$. Theorem 3 then provides an optimal static state-feedback controller K as a solution of SDP problem and one-parameter minimization; applying LMI Toolbox we get

$$\hat{K} \approx (-2.2724 \ \ -2.3341 \ \ 0.6420 \ \ -1.6564).$$

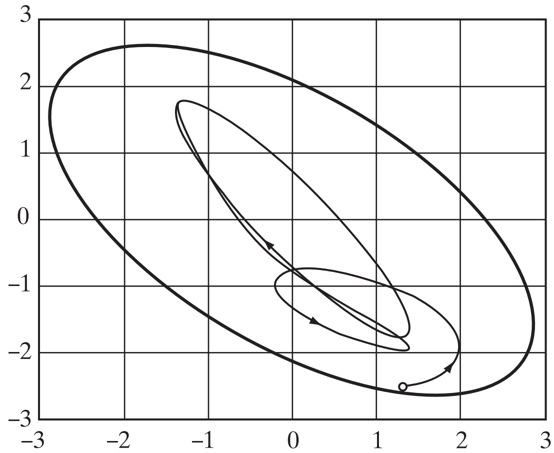

Fig. 3. Trajectory and minimal invariant ellipsoid

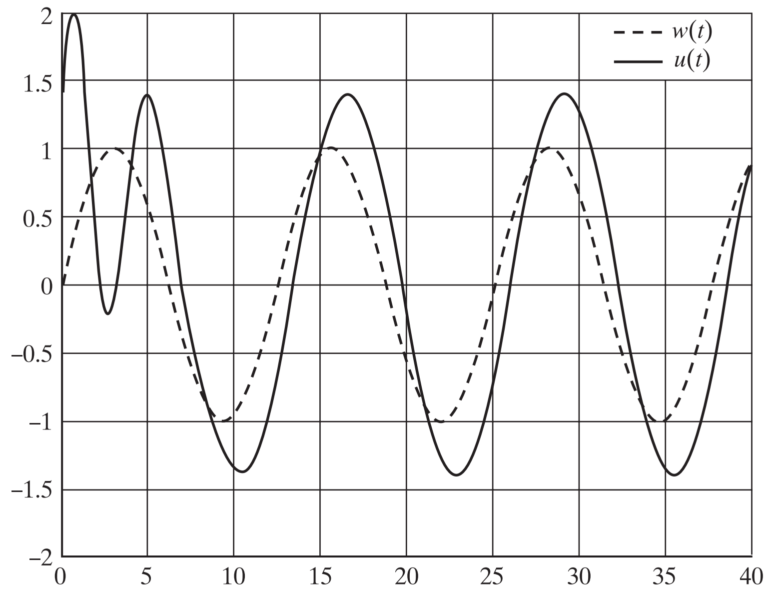

Fig. 4. Disturbance $w(t)$ and control $u(t)$

This controller minimizes the output invariant ellipsoid of the closed-loop system with respect to the trace criterion. Fig. 3 illustrates this optimal invariant ellipsoid $E_{\min}$ (thick line) and, as an example, a trajectory of the output oscillations $y(t)$ of the pendulum affected by the disturbance $w(t) = \sin\left(\frac{t}{2}\right)$ and initiated at a certain point x_0 inside the boundary of the output invariant ellipsoid. This trajectory remains inside this invariant ellipsoid. The disturbance behavior $w(t)$ and the control law $u(t)$ are shown in Fig. 4.

Notice finally, that the overshooting effect often appears when someone looks for a stabilizing controller. This may cause serious troubles. Design of the stabilizing controller that minimizes the invariant ellipsoid of the closed-loop system enables overshooting to be avoided.

5. DISCUSSION. It is of interest to compare the above results with the known ones, mainly with those of [10]. First, in [10] $\|P\|$ ais considered as the objective function, while we deal with $\mathrm{Tr}\,P$, which enables us to transform the problems to standard SDP and to simplify the results. Second, the number of parameters in our results is smaller (the single parameter α, while there are two parameters in [10]). From the technical point of view, we rely on the more advanced result (S-procedure with two constraints versus standard version of S-procedure with one constraint in [10]).

6. CONCLUSION. The simple method based on invariant ellipsoids technique is proposed for the optimal rejection of unknown-but-bounded exogenous disturbances. The static state-feedback controller that returns a minimum of invariant ellipsoid of the closed-loop system is founded via SDP and 1D convex optimization.

Acknowledgments. This study was supported in part by the Russian Foundation for Basic Research (grants RFBR 05-01-00114 and 05-08-01177).

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