Automation and Remote Control, Vol. 63, No. 8, 2002, pp. 1239–1254. Translated from Avtomatika i Telemekhanika, No. 8, 2002, pp. 37–53.

DETERMINATE SYSTEMS

Superstable Linear Control Systems. I. Analysis[1]

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Received April 23, 2002

Abstract—The notion of superstability of linear control systems was introduced. Superstability which is a sufficient condition for stability was formulated in terms of linear constraints on the entries of a matrix or the coefficients of a characteristic polynomial. In the first part of the paper, the properties of superstable systems were studied. The norms of solutions were proved to decrease exponentially monotonically in the absence of perturbations, and the solutions were proved to be uniformly bounded in the presence of bounded perturbations. A generalization to nonlinear and time varying systems was discussed. Spectral properties of superstable systems were studied. For interval matrices, a complete solution was given to the problem of robust superstability.

1. INTRODUCTION

Stability is a key notion of the control theory. Any controllable system must above all be stable and only in this case satisfy various specifications and optimize a performance index. The conventional notion of stability, however, is not convenient for designing the linear control systems. First, stability is an asymptotic property; "peaks" or sharp increases in the trajectory can occur at the initial time instants. Second, in the parameter space the set of stable systems is nonconvex, and equally the set of stabilizing controllers is nonconvex as well. Design of controllers of a given structure (for instance, low-order controllers) is quite difficult. Finally, the linear time invariant systems can easily lose their stability in the presence of time varying and nonlinear perturbations.

Passage to another, narrower class of the so-called *superstable* systems which have property of convexity offers a possible way to surmount these difficulties. In the space of the controller coefficients, the problem of stabilization also becomes convex and can be easily handled by means of linear programming. Moreover, numerous problems presenting serious difficulties within the framework of the standard theory such as static output stabilization, simultaneous stabilization of more than one system, robust stabilization under matrix uncertainty, etc., are solved easily for this class of systems. Additionally, it is possible to formulate new problems of optimal control such as minimization of the integral functional of absolute value (and not the quadratic functional) of the phase variables. These convenient properties arise thanks to formulating superstability as linear conditions for the matrix entries, rather than in terms of the eigenvalues.

Understandably, the superstability-based approach has its drawbacks. First, it is more difficult to secure superstability than stability, and one cannot guarantee this property for an arbitrary controlled system with scalar control. Second, the derived estimates of the performance indices are their upper bounds; therefore, the controls obtained are nothing more than suboptimal.

Sufficient conditions for stability as expressed by the inequalities of the matrix entries or the coefficients of its characteristic polynomial were considered in some publications [1–11]; more de-

[1] This work was supported by the Russian Foundation for Basic Research, projects nos. 00-15-96018 and 02-01-00127, and carried out within the framework of the complex program of the Presidium of the Russian Academy of Sciences.

tailed references will be given in what follows. The main innovation of the present publication is the use of these conditions not only in analysis of the control systems but also in their design. We note that the term itself "superstability" was first introduced for the discrete-time case in [12] and described in brief in [13]. The present paper consists of two parts. The first part analyzes superstable systems; the second one discusses the superstability-based design of controllers.

2. SUPERSTABILITY OF THE CONTINUOUS-TIME LINEAR SYSTEMS

This section presents the definitions for superstability of the continuous-time systems and presents their main characteristics. If not stated explicitly otherwise, the ∞-norm for the vectors $x \in \mathbb{R}^n$: $\|x\| = \max\limits_{1\le i\le n} |x_i|$ and the 1-norm induced by it for the matrices

$$A = ((a_{ij})) \in \mathbb{R}^{n\times n}: \ \|A\| = \max_{1\le i\le n}\left(\sum_{j=1}^{n}|a_{ij}|\right)$$

are used throughout the paper unless otherwise indicated.

A matrix $A = ((a_{ij})) \in \mathbb{R}^{n\times n}$ is said to be *superstable* if on the diagonal it has negative numbers each of which exceeds in absolute value the sum of the absolute values of the off-diagonal terms along the row:

$$\sigma(A) \ = \ \sigma \doteq \min_i \left(-a_{ii} - \sum_{j\ne i}|a_{ij}|\right) \ > \ 0. \tag{1}$$

The quantity $\sigma(A)$ is called the *superstability degree* of A. Most frequently, these matrices are called the matrices with negative diagonal dominants, and sometimes, A are called the Hadamard matrices. As follows immediately from the Gershgorin theorem [5], superstable matrices are stable, that is, $\max\limits_i\{\mathrm{Re}\,\lambda_i\} < 0$, where λ_i are the eigenvalues of A, but not *vice versa*. For example, the matrix

$$A = \begin{pmatrix} -1 & 5 \\ 0 & -1 \end{pmatrix} \tag{2}$$

is stable ($\lambda_1 = \lambda_2 = -1$), but not superstable ($\sigma = -4$).

Let us consider a linear continuous-time invariant dynamic system in the state space:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $u(t) \in \mathbb{R}^m$ is the external perturbation. If the system matrix A is superstable, then this system also is said to be *superstable*. The main property of superstable systems obeys the following theorem.

Theorem A.1. *If system* (3) *is superstable, then*

(a) *for* $u(t) \equiv 0$*, the estimate*

$$\|x(t)\| \le \|x_0\|\, e^{-\sigma t}, \quad t \ge 0 \tag{4}$$

is valid;

(b) *for* $\|u(t)\| \le 1,\ t \ge 0$*, and any initial* $\|x_0\| \le \gamma \doteq \|B\|/\sigma$*, we get*

$$\|x(t)\| \le \gamma, \quad t \ge 0; \tag{5}$$

(c) *for* $\|u(t)\| \le 1,\ t \ge 0,$ *and any initial* x_0,

$$\|x(t)\| \le \gamma + e^{-\sigma t}\,(\|x_0\| - \gamma)_+, \quad t \ge 0, \tag{6}$$

with $\alpha_+ = \max\{0,\ \alpha\}$.

This and the following assertions are proved in the Appendix.

It seems that Lozinskii [1] was the first to establish estimates of the kind of (4)–(6) which later on were rediscovered time and again and used in many works [2–8].

Property (a) is system stability with respect to the initial conditions. As follows from (4), the superstable system has the nonquadratic Lyapunov function

$$V(x) = \|x\|. \tag{7}$$

This function grows linearly in any direction: $V(\lambda x) = \lambda V(x)$ for any x and any $\lambda \ge 0$, it is piecewise-linear and nondifferentiable. At the same time, it has also the properties of the conventional Lyapunov functions: $V(x) \ge 0$ with $V(x) = 0$ only for $x = 0$, it is convex, and grows on infinity. The function $v(t) = V(x(t))$, where $x(t)$ is the solution of the system $\dot{x} = Ax,\ x(0) = x_0$, decreases monotonically. Generally speaking, it is nondifferentiable, but has the left and right derivatives $\dot{v}_-(t)$ and $\dot{v}_+(t)$; and at that

$$\dot{v}_- \le -\sigma v, \quad \dot{v}_+ \le -\sigma v.$$

We emphasize that it is namely the ∞-norm of the state vector that decreases monotonically, but the coordinate can oscillate. Therefore, the linear function of state ($y(t) = c^{\mathrm{T}}x(t)$, for example) does not necessarily decrease monotonically.

The generic difference from the stable systems is that for the stable matrices equation (4) is replaced by the following estimate:

$$\|x(t)\| \le C(A,\nu)\|x_0\|e^{-\nu t}, \quad 0 < \nu < \min_i\{-\mathrm{Re}\,\lambda_i\},$$

where the constant $C(A,\nu)$ can be quite large. In this case, the norm $x(t)$ does not decrease monotonically with t, but can rather increase for small t. For example, for the same matrix (2) and $x_0 = (1;\ 1)^{\mathrm{T}}$ we get $x(1) \approx (2.207;\ 0.368)^{\mathrm{T}}$, that is, $\|x(1)\|$ is twice as big as $\|x_0\|$. For superstable systems, at the initial part of the trajectory there is no such undesirable peak.

Properties (b) and (c) relate to input-output stability (BIBO stability): bounded inputs correspond to the bounded outputs. At that, the cube

$$\mathcal{Q} = \{x \in \mathbb{R}^n : \|x\| \le \gamma\}$$

is called the *invariant set* for (3), that is, the trajectories originating in this set stay in it for all admissible perturbations u. The reader is referred to [14] for a thorough study of the invariant sets for linear systems.

3. SUPERSTABILITY OF LINEAR DISCRETE-TIME SYSTEMS

The above refers to the continuous case, but a similar notion can be introduced also for the discrete-time systems obeying difference, rather than differential equations. The matrix $A = ((a_{ij})) \in \mathbb{R}^{n\times n}$ is said to be *discrete superstable* if

$$q \doteq \|A\| = \max_{1\le i\le n}\left(\sum_{j=1}^{n}|a_{ij}|\right) < 1, \tag{8}$$

and $1 - q$ is called the *degree of discrete superstability* of A. As in the continuous case, these matrices are stable, that is, $\rho(A) \doteq \max_i |\lambda_i(A)| < 1$ (because $\rho(A) \le \|A\|$ for any matrix norm), but not *vice versa*. For example, the matrix

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

is stable ($\rho = 0$), but not superstable ($q = 2$).

We say that the discrete system

$$x_k = Ax_{k-1} + Bu_{k-1} \tag{9}$$

with the matrix satisfying (8) is *superstable* as well. In what follows, the term "superstability" is applied to matrices and systems satisfying both (1), (3) and (8), (9); it will be clear from the context whether continuous or discrete superstability is meant. The following result is the discrete counterpart of Theorem 2.1.

Theorem A.1. *Let the discrete-time system* (9) *be superstable. Then,*

(a) *for* $u_k \equiv 0$,

$$\|x_k\| \le q^k \|x_0\|, \quad k = 1, 2, \ldots \tag{10}$$

is valid;

(b) *for* $\|u_k\| \le 1,\ k \ge 1$,

$$\|x_k\| \le \gamma, \quad k = 1, 2, \ldots \tag{11}$$

for any $\|x_0\| \le \gamma \doteq \|B\|/(1 - q)$*;*

(c) *for* $\|u_k\| \le 1,\ k \ge 1$,

$$\|x_k\| \le \gamma + q^k \left(\|x_0\| - \gamma\right)_+, \quad k = 1, 2, \ldots \tag{12}$$

for any initial x_0.

As for continuous time, the above estimates suggest that the norm of solution of the superstable system decreases monotonically and there exists the invariant set, the cube

$$\mathcal{Q} = \left\{x \in \mathbb{R}^n : \ \|x\| \le \gamma\right\}.$$

The difference from the stable systems is that estimate (10) is replaced by

$$\|x_k\| \le C(\varepsilon)(\rho + \varepsilon)^k \|x_0\|, \quad \varepsilon > 0, \quad \rho + \varepsilon < 1,$$

where a constant $C(\varepsilon)$ can be very large, that is, $\|x_k\|$ does not decrease monotonically as k goes up, but can rather increase at the initial iterations. We note that in essence Theorem 3.1 is not a novelty; similar results can be found everywhere in the literature on linear algebra, for example, in [5, 6].

As is well known, no efficient methods exist to test matrices for stability. The only known approach is to compose the characteristic polynomial and apply any stability criterion for polynomials. In contrast, matrix testing for superstability presents no problems because these conditions are formulated directly in terms of the entries of a matrix, rather than its eigenvalues.

4. SUPERSTABILITY OF THE SISO DISCRETE-TIME SYSTEMS

We consider the SISO counterpart of superstability. Let instead of the MIMO discrete-time system (9) the following scalar system obeying the difference equation of the nth order be given:

$$x_k + p_1 x_{k-1} + p_2 x_{k-2} + \ldots + p_n x_{k-n} = u_k, \tag{13}$$

where $x_k \in \mathbb{R}^1$, $u_k \in \mathbb{R}^1$. By introducing the back-shift operator $zx_k = x_{k-1}$, we obtain the notation

$$p(z)x_k = u_k, \quad p(z) = 1 + p_1 z + p_2 z^2 + \ldots + p_n z^n, \tag{14}$$

and the system is stable or, in other words, $x_k \to 0$ for any initial conditions $x_{-n}, x_{-n+1}, \ldots, x_{-1}$ and $u_k \equiv 0$ if the polynomial $p(z)$ is stable, that is, its roots lie outside the unit circle, $|\lambda_i| > 1$.

The polynomial $p(z)$ is said to be *superstable* if

$$\sum_{i=1}^{n} |p_i| < 1; \tag{15}$$

for a polynomial defined in a more general form $p(z) = p_0 + p_1 z + \ldots + p_n z^n$, the condition for superstability is $\sum_{i=1}^{n} |p_i| < |p_0|$. These polynomials were introduced by Cohn in 1922 [11]. We note that (15) is the well-known sufficient condition for discrete stability of polynomials. Superstability was studied and applied to the control problems namely in this form [12, 15]. If the 1-norm of polynomial is defined as the sum of absolute values of its coefficients, then condition (15) is representable as

$$\|p(z) - 1\|_1 < 1. \tag{16}$$

Results similar to Theorem 3.1 are valid for SISO systems with superstable $p(z)$ [12].

Theorem A.1. *Consider the scalar system*

$$p(z)x_k = g(z)u_k,$$

where $g(z) = g_1 z + \ldots + g_m z^m$ *and the polynomial* $p(z) = 1 + p_1 z + \ldots + p_n z^n$ *is superstable. Then,*

(a) *for* $u_k \equiv 0$, *the estimate*

$$|x_k| \le q^{k/n+1} \max_{-n \le i \le -1} |x_i|, \quad k = 0, 1, \ldots, \quad q \doteq \|p(z) - 1\|_1$$

is valid;

(b) *for* $|u_k| \le 1$, $k = 0, 1, \ldots,$ *and any initial* $|x_{-n}| \le \gamma, \ldots, |x_{-1}| \le \gamma$ *such that*

$$\gamma = \frac{\|g(z)\|_1}{1 - q},$$

we get

$$|x_k| \le \gamma, \quad k = 0, 1, \ldots;$$

(c) *for* $|u_k| \le 1$, $k = 0, 1, \ldots,$ *and any initial* $x_{-n}, \ldots, x_{-1}$, *we get*

$$|x_k| \le \gamma + q^{k/n+1} \left(\max_{-n \le i \le -1} |x_i| - \gamma \right)_+, \quad k = 0, 1, \ldots$$

We note that the standard transformation of the nth order SISO system (13) into the equivalent state-space form, that is, to the canonical controllable form, does not lead to a superstable matrix. The question of the SISO analog of superstability for continuous-time systems remains open. It seems that there exist no meaningful variants of superstable polynomial with roots in the left half-plane.

5. SUPERSTABILITY OF THE TIME VARYING AND NONLINEAR SYSTEMS

An important property of superstability, as distinct from stability, is the fact that it is retained in the time varying case, as well in the presence of time varying and nonlinear perturbations. Let us consider a system more general than (9):

$$x_{k+1} = A_k x_k + f_k(x_k), \tag{17}$$

where the matrices A_k can depend on time, and the perturbations $f_k(x_k)$, both on time k and state.

Theorem A.1. *Let*

$$\|A_k\| \le r < 1, \quad \|f_k(x_k)\| \le \alpha + \beta\|x_k\|, \quad 0 \le \beta < 1 - r$$

be satisfied for all k*. Then, for system* (17)

(a) *for* $\alpha = 0$,

$$\|x_k\| \le q^k\|x_0\|, \quad q \doteq r + \beta < 1, \quad k = 1, 2, \dots$$

is valid;

(b) *for* $\alpha > 0$ *and* $\|x_0\| \le \gamma \doteq \alpha/(1-q)$,

$$\|x_k\| \le \gamma, \quad k = 1, 2, \dots$$

is valid;

(c) *for* $\alpha > 0$ *and any* x_0,

$$\|x_k\| \le \gamma + q^k(\|x_0\| - \gamma)_+, \quad k = 1, 2, \dots$$

is valid.

The proof follows literally the same lines as that of Theorem 3.1.

Importantly, a similar theorem does not hold for stable systems. In particular, property (a) is not satisfied: solutions of the system $x_{k+1} = A_k x_k$ do not necessarily tend to zero even if all matrices A_k are stable. For example,

$$A_0 = A_2 = \ldots = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad A_1 = A_3 = \ldots = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix};$$

then $x_{2k} = (0;\ 2^{2k})^{\mathrm{T}} \to \infty$ for $x_0 = (0;\ 1)^{\mathrm{T}}$, although all matrices A_k are stable, $\rho(A_k) = 0$.

The following result is a continuous-time counterpart of Theorem 5.1.

Theorem A.2. *Let the system*

$$\dot{x}(t) = A(t)x(t) + f(t, x(t)), \quad x(0) = x_0,$$

satisfy the condition

$$\sigma(A(t)) \ge \sigma > 0, \quad \|f(t,x)\| \le \alpha + \beta\|x(t)\|, \quad 0 \le \beta < \sigma$$

for all $t > 0$*. Then,*

(a) *for* $\alpha = 0$, *the estimate*

$$\|x(t)\| \le e^{-(\sigma-\beta)t}\|x_0\|, \quad t \ge 0$$

is valid;

(b) *for* $\alpha > 0$ *and any* $\|x_0\| \le \gamma \doteq \alpha/(\sigma-\beta)$,

$$\|x(t)\| \le \gamma, \quad t \ge 0$$

is valid;

(c) *for* $\alpha > 0$ *and any initial* x_0,

$$\|x(t)\| \le \gamma + e^{-(\sigma-\beta)t}(\|x_0\| - \gamma)_+, \quad t \ge 0$$

is valid.

We note that similar results can be found in the aforementioned publications [1–8].

6. SPECTRAL PROPERTIES OF SUPERSTABLE SYSTEMS

Superstable matrices constitute a subset of stable matrices. Does superstability impose constraints on the location of eigenvalues? Conversely, is it possible to conclude from the location of the eigenvalues that a matrix is superstable or similar to a superstable one? Some answers to such questions are discussed below.

Theorem A.1. *If the matrix* $A \in \mathbb{R}^{n\times n}$ *of a continuous-time system is superstable, then its eigenvalues lie within the sector*

$$\mathcal{S}_n = \{\lambda \in \mathbb{C} : \ |\arg\lambda - \pi| < (1-n^{-1})\pi/2\}.$$

Conversely, each point in this sector is an eigenvalue of a superstable matrix.

In particular, for $n = 2$ the eigenvalues lie inside the right angle whose bisecting line coincides with the negative semiaxis:

$$\lambda_i \in \mathcal{S}_2 = \{\lambda \in \mathbb{C} : \ \mathrm{Re}\,\lambda < 0, \ -\mathrm{Re}\,\lambda > |\mathrm{Im}\,\lambda|\}. \tag{18}$$

As n grows, the sector tends to the complete left half-plane. The proof is given in the Appendix. It is based on the results of Bobyleva and Pyatnitskii [16] on the spectrum of matrices with a piecewise-linear Lyapunov function.

If a matrix A is discretely superstable, then for $n = 2$ one can directly demonstrate that its eigenvalues belong to and fill up the diamond

$$\lambda_i \in \mathcal{R}_2 = \{\lambda \in \mathbb{C} : \ |\mathrm{Re}\,\lambda| + |\mathrm{Im}\,\lambda| < 1\}. \tag{19}$$

For $n > 2$, the characterization of the eigenvalue location is less complete.

Theorem A.2. *The set* $\mathcal{R}_n$ *of all eigenvalues of the discretely superstable matrices* $A \in \mathbb{R}^{n\times n}$ *comprises all interior points of the regular polygons with* $2k$, $k = 1, \ldots, n$, *sides inscribed into the unit circle and having one of their vertices at the point* $+1$.

Therefore, if $|\lambda| < \cos(\pi/(2n))$, then $\lambda \in \mathcal{R}_n$, that is, for a sufficiently large n any point inside the unit circle belongs to $\mathcal{R}_n$.

We note that the problem of describing the set $\mathcal{R}_n$ is close to the well-known Kolmogorov problem of localizing the spectrum of stochastic matrices that was posed in 1938 and solved in part by Dmitriev and Dynkin [17] and completely, by Karpilevich [18].

Since for any nondegenerate matrix T the matrices A and TAT^{-1} have the same eigenvalues, stability is invariant to the linear transformation of the coordinates. On the contrary, since superstability is formulated in terms of the entries of a matrix, rather than its eigenvalues, this property can be lost or, what is more important, acquired upon passing to other coordinates. One of the simplest situations where in new coordinates a stable matrix becomes superstable is described by the following lemma.

Lemma A.1. *Let a matrix $A \in \mathbb{R}^{n\times n}$ of a discrete-time system have distinct eigenvalues belonging to the diamond $\mathcal{R}_2$* (19)*. Then, it can be made superstable by a nondegenerate real linear transformation of the coordinates.*

An absolutely similar result is valid in the continuous-time case.

Lemma A.2. *Let a matrix $A \in \mathbb{R}^{n\times n}$ of a continuous-time system have distinct eigenvalues belonging to the sector $\mathcal{S}_2$* (18)*. Then, it can be made superstable by a nondegenerate real linear transformation of the coordinates.*

The proof coincides with that of Lemma 6.1.

Now, we turn to the SISO systems and consider location of the roots of the superstable polynomial. To this end, it would be more convenient to consider a discrete-time polynomial in the form

$$p(z) = z^n + p_1 z^{n-1} + \ldots + p_n \tag{20}$$

and to understand by its stability the location of roots *inside* the unit circle. Then, its superstability obeys as before the condition

$$\sum_{i=1}^{n} |p_i| < 1, \tag{21}$$

that is, it follows from (21) that the roots of polynomial (20) lie inside the unit circle. Note that the roots of the polynomial $\overline{p}(z)$ with the inverse order of the coefficients are mutually inverse to the roots of $p(z)$ and lie outside the unit circle.

We set down the point $z \in \mathbb{C}$ in the polar coordinates $z = \rho e^{j\theta}$ and introduce the function

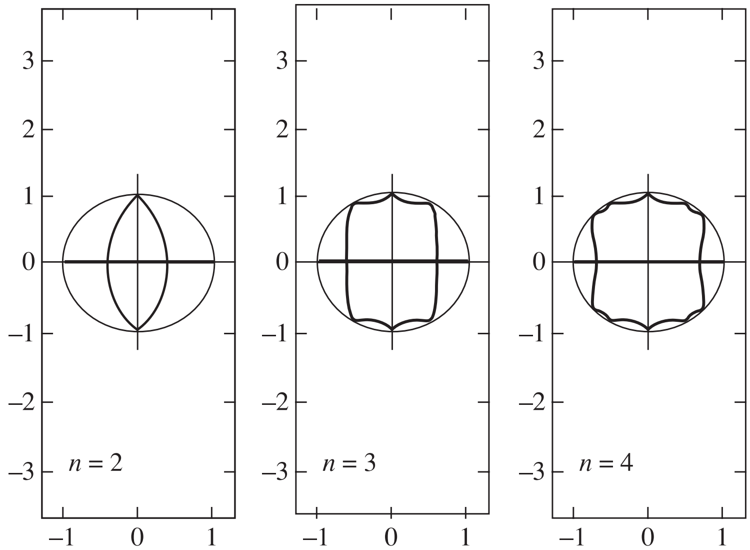
$$\varphi_n(\rho,\theta) = \begin{cases} \min\limits_{1\le i<k\le n} \dfrac{\rho^k|\sin i\theta| + \rho^i|\sin k\theta|}{|\sin(k-i)\theta|} & \text{for} \quad \theta \ne 0,\ \theta \ne \pi \\ \rho & \text{for} \quad \theta = 0,\ \theta = \pi. \end{cases} \tag{22}$$

Theorem A.3. *The roots of all superstable polynomials* (20), (21) *fill up the domain*

$$\mathcal{P}_n = \{z = \rho e^{j\theta} :\ \varphi_n(\rho,\theta) < 1\}.$$

Let us consider the qualitative form of the domain $\mathcal{P}_n$. First of all, for $\theta = 0$ or $\theta = \pi$ the condition $\varphi_n(\rho,\theta) < 1$ means that $\rho < 1$, that is, the interval $(-1,\ 1)$ belongs to $\mathcal{P}_n$ for any n. Then, we obtain for $\rho = 1$ and $\theta \ne 0, \pi$ that

$$\varphi_n(1,\theta) = \min_{1\le i<k\le n} \frac{|\sin i\theta| + |\sin k\theta|}{|\sin(k-i)\theta|},$$



The domain $\mathcal{P}_n$ for $n = 2$, $n = 3$, and $n = 4$.

and $\varphi_n(1,\theta) \geq 1$ because $|\sin(\alpha-\beta)| = |\sin\alpha\cos\beta - \cos\alpha\sin\beta| \leq |\sin\alpha| + |\sin\beta|$. Correspondingly, $\varphi_n(\rho,\theta) > 1$ for $\rho > 1$, $\theta \neq 0, \pi$. Therefore, $\mathcal{P}_n$ lies within the unit circle, which confirms the fact that superstability implies stability. Here, if $\theta = \pm\ell\pi/m,\ m = 2,\dots,n,\ \ell < m$, then $\varphi(1,\theta) = 1$. Indeed, it suffices to take $i = 1,\ k = m$, and then

$$\varphi_n(1,\theta) \leq \frac{|\sin\theta| + |\sin m\theta|}{|\sin(m-1)\theta|} = \frac{|\sin\frac{\ell}{m}\pi| + |\sin\ell\pi|}{|\sin\frac{m-1}{m}\ell\pi|} = \frac{|\sin\frac{\ell}{m}\pi|}{|\sin(\ell\pi - \frac{\ell}{m}\pi)|} = 1.$$

As was shown above, $\varphi(1,\theta) \geq 1$ with $\theta \neq 0,\pi$, that is, the minimum is attained for $i = 1$, $k = m$, and in doing so $\varphi_n(1,\theta) = 1$. The general form of the domain $\mathcal{P}_n$ is shown in the figure for $n = 2, 3, 4$.

For $n = 2$, in particular, $\mathcal{P}_n$ has a simple analytic description. Indeed, since

$$\varphi_2(\rho,\theta) = \frac{\rho^2|\sin\theta| + \rho|\sin 2\theta|}{|\sin\theta|} = \rho^2 + 2\rho|\cos\theta|,$$

the set $\varphi_2(\rho,\theta) < 1$ follows the inequality $\rho^2 + 2\rho|\cos\theta| < 1$ which is the intersection of two circles centered at ±1 and having radii $\sqrt{2}$. Additionally, $\mathcal{P}_2$ (as any $\mathcal{P}_n$) comprises the real interval $(-1,\ 1)$.

As n grows up, the set $\mathcal{P}_n$ fills up entire interior of the unit circle. For this purpose, let us fix some point $z = \rho e^{j\theta},\ |z| = \rho < 1$, and prove that there exists n such that $\varphi_n(\rho,\theta) < 1$. Indeed, with $i = 1,\ k = n$ in (22) we have

$$\varphi_n(\rho,\theta) \leq \frac{\rho^n|\sin\theta| + \rho|\sin n\theta|}{|\sin(n-1)\theta|}. \qquad (23)$$

If θ/π is rational, $\theta = \ell\pi/m$, then by taking $n = m$ we obtain $\sin\theta = 0$ and

$$\varphi_n(\rho,\theta) \leq \frac{\rho^n|\sin\theta|}{|\sin(n-1)\theta|} = \rho^n < 1.$$

If θ/π is irrational, then as n grows $n\theta/\pi$ approaches an integer number arbitrarily closely and $|\sin n\theta|/|\sin(n-1)\theta|$ can be made arbitrarily small by choosing n large enough. We have also

$\rho^n \to 0$ as $n \to \infty$; hence, the right-hand side of (23) can be made smaller than unity by a proper choice of n. Quite a different proof of this fact (any z, $|z| < 1$, is a root of a certain superstable polynomial) follows from Lemma 7 of [12].

We also note that $\mathcal{P}_n \subset \mathcal{S}_n$, that is, the set of roots of superstable polynomials is contained in the set of eigenvalues of discrete superstable matrices. Indeed, for $|a| < 1$, $\sum\limits_{i=1}^{n} |p_i| < 1$ the matrix

$$\begin{pmatrix} 0 & a & 0 & \dots & 0 \\ 0 & 0 & a & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a \\ -p_1 & -p_2 & -p_3 & \dots & -p_n \end{pmatrix}$$

is discretely superstable, and its eigenvalues tend to the roots of polynomial (20) with $a \to 1$. Sometimes, inverse conclusions are possible for polynomials: location of the roots can guarantee superstability.

Lemma A.3. *If the condition*

$$|z_i| < 2^{1/n} - 1, \quad i = 1, \dots, n, \tag{24}$$

is satisfied for all roots z_i of the polynomial $p(z)$ (20), then $p(z)$ is superstable, that is, $\sum\limits_{i=1}^{n} |p_i| < 1.$

It is clear that estimate (24) is very stringent because the quantity in the right-hand side decreases rapidly as n grows. Nevertheless, there are no counterparts to this result in the matrix case because a matrix can have all zero eigenvalues and at the same time lack discrete superstability.

7. ACCURACY OF ESTIMATES

Another problem lies in how conservative are the estimates of Theorems 2.1 and 3.1 which just upper-bound the corresponding quantities. It is not quite clear how large is their deviation from the true values. One can easily construct examples showing that this difference can be very large. For example, for the system

$$x_{k+1} = Ax_k, \quad A = \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix}, \quad |q| < 1,$$

$x_k = 0$, $k \ge 2$, for any x_0, whereas (10) provides $\|x_k\| \le |q|^k \|x_0\|$. In the nonhomogeneous system $x_{k+1} = Ax_k + u_k$, $\|u_k\| \le 1$, however, it follows from (11) for the same matrix that $\|x_k\| \le 1/(1-|q|)$, whereas $\sup_k \|x_k\| = 1 + |q|$, that is, the difference is not so dramatically large if $|q|$ is not too close to unity.

Numerical modelling based on random generation of superstable systems (matrices) and comparison of $\sup\limits_k \|x_k\|$ with the estimates obtained can provide some insight into conservatism of estimates (10) and (11). We demonstrate first how for discrete time one can randomly generate superstable matrices from $\mathbb{R}^{n\times n}$. We need to generate a vector $x \in \mathbb{R}^n$ uniformly distributed over the unit simplex $\left\{\sum\limits_{i=1}^{n} x_i \le 1,\ x_i \ge 0\right\}$. This distribution obeys the n-dimensional Dirichlet distribution with all parameters equal to 1 (see [19]). In turn, it can be obtained as follows:

$$x = \left(\frac{\xi_1}{\sum\limits_{k=1}^{n+1} \xi_k}, \quad \frac{\xi_2}{\sum\limits_{k=1}^{n+1} \xi_k}, \dots, \frac{\xi_n}{\sum\limits_{k=1}^{n+1} \xi_k} \right),$$

where $\xi_k,\ k = 1,\ldots,n+1$, are independent random variables having exponential distribution with the density $f(x) = e^{-x}$ [19]. Finally, the exponentially distributed variable ξ is generated as $\xi = -\ln u$, where u is uniformly distributed over $[0,\ 1]$ [19]. We describe an algorithm for random generation of the uniformly distributed discrete superstable matrices.

Algorithm.

(1) Generate a random vector $s \in \mathbb{R}^n$ that is uniformly distributed over $[0,\ 1]^n$.

(2) Generate for each $i = 1,\ldots,n$ an n-dimensional vector a_i that is uniformly distributed over the unit simplex $\left\{\sum\limits_{j=1}^{n} a_{ij} \le 1,\ a_{ij} > 0\right\}$ and normalize it to the surface of the simplex $\left\{\sum\limits_{j=1}^{n} a_{ij} \le s_i,\right.$ $\left. a_{ij} > 0\right\}$, that is, $a_i \longrightarrow \dfrac{s_i}{\sum\limits_{i=1}^{n} a_{ij}} a_i$.

(3) Arrange randomly the signs + and − of the numbers a_{ij}.

Step 1 provides the vector of row sums. Step 2 provides the superstable matrix $A = ((a_{ij}))$ with positive entries and $\|A\| = \max\limits_i s_i$. Step 3 makes the signs of a_{ij} arbitrary.

Test 1. In a homogeneous system $x_k = Ax_k$, conservatism of estimate (10) is due to the replacement of $\|A^k\|$ by its upper bound $\|A\|^k$. It is known that $\rho(A) = \lim\limits_{k\to\infty} \|A^k\|^{1/k}$, where $\rho(A) = \max\limits_i |\lambda_i(A)|$ is the spectral radius of A, that is, the quantity $\|x_k\|$ behaves asymptotically as ρ^k, rather than as $\|A\|^k$. For randomly generated superstable matrices, the ratio $\|A\|/\rho$ was calculated and averaged over $N = 1000$ realizations. For $n = 2$, 5, 10, and 20, this ratio was, respectively, 1.90, 2.76, 3.73, and 5.17.

Test 2. Consideration was given to the system with perturbations

$$x_{k+1} = Ax_k + u_k, \quad \|u_k\| \le 1,$$

for which

$$x_k = A^{k-1}x_0 + A^{k-2}u_0 + A^{k-2}u_1 + \ldots + u_{k-1}.$$

Generated were the superstable matrices with positive entries for which the maximum $\|x_k\|$ for all admissible perturbations and all initial conditions $\|x_0\| \le 1$ is attained for $u_i \equiv e$, $x_0 = e$, where e is the vector consisting of ones. Then, $x_k = (I + A + \ldots + A^{k-1})e$, where I is the identity matrix, the supremum in k is $\sup\limits_k \sup\limits_u \|x_k\| = \|(I - A)^{-1}e\| = \|(I - A)^{-1}\|$, and estimate (11) provides $\|x_k\| \le 1/(1 - \|A\|)$ (we recall that $\|x\| = \max\limits_i |x_i|$ and $\|A\| = \max\limits_i \sum\limits_{i=1}^{n} |a_{ij}|$). Modelling was limited to the matrices having the degree of superstability $1 - q = 1 - \|A\| \ge 0.05$. The ratio of $1/(1 - \|A\|)$ and $\|(I - A)^{-1}\|$ was averaged over $N = 1000$ realizations and was equal to 1.53, 2.51, 3.41, and 4.32, respectively, for $n = 2$, 5, 10, and 20.

8. ROBUST SUPERSTABILITY

Until now, we gave consideration to the problems with exact description of the plant, that is, to the case where the matrices A and B are given. Descriptions of plants in applications unavoidably are not free of uncertainties. However, the systems must be designed so as to be operable and, in particular, stable in the presence of uncertainties, that is, to be robust. An extensive literature is devoted to the problems of robustness (see, for instance, [20]), but many of them are very difficult and have not been solved until now. One of them is the problem of robust stability of interval matrix families.

Let us consider an interval matrix family

$$A = ((a_{ij})), \quad a_{ij} = a_{ij}^0 + \gamma\Delta_{ij}, \quad |\Delta_{ij}| \le m_{ij}, \quad i, j = 1, \ldots, n, \tag{25}$$

where $A_0 \doteq ((a_{ij}^0))$ is the nominal matrix; $\gamma \ge 0$ is a numerical parameter; Δ_{ij} are uncertainties; and $m_{ij} \ge 0$ are the given numbers which are the entries of the matrix $M = ((m_{ij}))$. Let A_0 Hurwitz. Our goal is to determine the stability radius, that is, the largest $\gamma_{\max}$ such that the family is robust stable for all $\gamma < \gamma_{\max}$. This problem is known [21] to be NP-hard and have no efficient method of solution (additional references and discussion can be found in [22]). We set out to demonstrate that the problem of *robust superstability* of the interval matrices is extremely simple.

Let the nominal matrix A_0 be superstable, that is,

$$\sigma(A_0) \doteq \min_i \left(-a_{ii}^0 - \sum_{j \ne i} |a_{ij}^0| \right) > 0.$$

We require that the superstability condition be satisfied for all matrices of the family:

$$-(a_{ii}^0 + \gamma\Delta_{ii}) - \sum_{j \ne i} |a_{ij}^0 + \gamma\Delta_{ij}| > 0, \quad i = 1, \ldots, n,$$

that is, that the family be robustly superstable. Clearly, this inequality will be satisfied for all admissible Δ_{ij} if and only if

$$-a_{ii}^0 - \gamma m_{ii} - \sum_{j \ne i} \left(|a_{ij}^0| + \gamma m_{ij} \right) > 0, \quad i = 1, \ldots, n,$$

that is, for

$$\gamma < \gamma^* \doteq \min_i \frac{-a_{ii}^0 - \sum\limits_{j \ne i} |a_{ij}^0|}{\sum\limits_j m_{ij}}. \tag{26}$$

In particular, if $m_{ij} \equiv 1$ (the scales of variations of all matrix entries are the same), then

$$\gamma^* = \frac{\sigma(A_0)}{n}. \tag{27}$$

Consequently, we established the explicit *superstability radius* γ^* of the interval matrix family whose value is the lower bound of the stability radius $\gamma_{\max}$.

Similar formulas are valid in the discrete-time case: if $\|A_0\| < 1$, then the matrix family (25) remains superstable for

$$\gamma < \gamma^* \doteq \min_i \frac{1 - \sum\limits_j |a_{ij}^0|}{\sum\limits_j m_{ij}}, \tag{28}$$

and for $m_{ij} \equiv 1$,

$$\gamma^* = \frac{1 - \|A_0\|}{n}. \tag{29}$$

9. CONCLUSIONS

The notion of superstable linear (continuous-time or discrete-time) system was introduced. The superstability condition is formulated by means of linear constraints on the entries of the system matrices; it is more strict than the stability condition. Superstable systems have some useful features. For example, the norm of solution decreases exponentially monotonically in the absence of perturbations and remains (for all t) a bounded quantity admitting an efficient estimate in the presence of constrained perturbations. Superstability is extended to the time varying and nonlinear systems. Spectral characteristics of the superstable systems were studied. Finally, in contrast to the similar hard problem of robust stability, the problem of robust superstability of the interval matrices admits a simple solution.

ACKNOWLEDGMENTS

The authors should like to thank E.S. Pyatnitskii, A.S. Nemirovskii, M. Halpern, M. Sznaier, and D. Siljak for valuable discussions and references.

APPENDIX

Proof of Theorem 2.1. Let us prove that for a superstable A

$$\|e^{At}\| \le e^{-\sigma t}.$$

For small $t = \delta t$, $e^{A\,\delta t} \approx I + A\,\delta t$ is true, that is, we get for the entries m_{ij} of the matrix $e^{A\,\delta t}$ that

$$\begin{aligned} m_{ii} &\approx 1 + a_{ii}\,\delta t > 0, \\ m_{ij} &\approx a_{ij}\,\delta t, \quad i \ne j, \end{aligned}$$

so that

$$\begin{aligned} \|e^{A\delta t}\| &= \max_i \sum_j |m_{ij}| \approx \max_i \left(|1 + a_{ii}\,\delta t| + \delta t \sum_{j \ne i} |a_{ij}| \right) \\ &= \max_i \left(1 + \left(a_{ii} + \sum_{j \ne i} |a_{ij}| \right) \delta t \right) \le 1 - \sigma\,\delta t \le e^{-\sigma\,\delta t}. \end{aligned}$$

Therefore, for an arbitrary $t = N\,\delta t$ with small δt, the following is true:

$$\|e^{At}\| = \|e^{AN\,\delta t}\| \le \|e^{A\,\delta t}\|^N \le \left(e^{-\sigma\,\delta t} \right)^N = e^{-\sigma t}.$$

Hence, by means of the explicit formula

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$$

solving system (3), we get the estimates of the theorem.

Proof of Theorem 3.1. Assertion (a) follows from the fact that

$$\|x_k\| \le \|A\|\,\|x_{k-1}\| = q\|x_{k-1}\|$$

is valid for any $k \ge 1$.

In the general case, for any $k \geq 1$

$$\|x_k\| \leq \|A\| \, \|x_{k-1}\| + \|B\| \, \|u_{k-1}\| \leq q\|x_{k-1}\| + \|B\|,$$

and we obtain by induction

$$\|x_k\| \leq \frac{\|B\|}{1-q} + q^k \left(\|x_0\| - \frac{\|B\|}{1-q} \right),$$

hence, (b) and (c) are valid.

Proof of Theorem 6.1. By Theorem 1 [16], if a matrix A admits a piecewise-linear Lyapunov function with $2n$ faces, then its eigenvalues lie in $\mathcal{S}_n$. Conversely, let $\lambda = -a + jb \in \mathcal{S}_n$, that is, $a > 0$, $b > 0$, $a/b > \tan(\pi/(2n))$ (without loss of generality we can assume that $b > 0$). We take the matrix

$$A = \begin{pmatrix} -\alpha & \beta & 0 & \ldots & 0 \\ 0 & -\alpha & \beta & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{n-1}\beta & 0 & 0 & \ldots & -\alpha \end{pmatrix}$$

whose eigenvalues are equal to $-\alpha + \beta \sqrt[n]{-1}$. We choose the value of the root equal to $\exp(j\pi/n)$. For $\alpha = a + b\dfrac{\cos(\pi/n)}{\sin(\pi/n)}$ and $\beta = \dfrac{b}{\sin(\pi/n)}$, this eigenvalue then will be equal to λ. At that, $\alpha > 0, \beta > 0, \alpha - \beta = a - b \tan(\pi/(2n)) > 0$, that is, the matrix A will be superstable.

Proof of Theorem 6.2. Let us consider the matrix

$$A = \begin{pmatrix} A_k & 0 \\ 0 & 0 \end{pmatrix}, \quad A_k = \begin{pmatrix} 0 & \beta & 0 & \ldots & 0 \\ 0 & 0 & \beta & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{k-1}\beta & 0 & 0 & \ldots & 0 \end{pmatrix},$$

where $A_k \in R^{k\times k}$, $A \in R^{n\times n}$, $1 \leq k \leq n$. It is discretely superstable for $|\beta| < 1$, and its eigenvalues are equal to $\beta \sqrt[k]{-1}$ and zero (for $k < n$). For $\beta \to 1$, there exists an eigenvalue tending to $\exp(j\pi/n)$. For the discrete superstable matrix A, a matrix of the form $c_0 + c_1 A + \ldots + c_k A^k$, $\sum_{i=0}^{k} |c_i| \leq 1$ also is superstable and has an eigenvalue close to $\sum_{i=0}^{k} c_i \exp(ji\pi/n)$. For $\sum_{i=0}^{k} |c_i| \leq 1$, points of this kind belong to the interior of the regular $2k$-agon M_k which is inscribed into the unit circle and has one vertex at the point $+1$.

Proof of Lemma 6.1. The matrix with distinct eigenvalues $\lambda_{2i-1} = u_i + jv_i$, $\lambda_{2i} = u_i - jv_i$, $v_i \neq 0$, $i = 1, \ldots, p$, $\lambda_i \in \mathbb{R}$, $i = 2p+1, \ldots, n$, is known (see, for example, [5]) to be real-similar to the block-diagonal matrix

$$T^{-1}AT = \operatorname{diag}(J_1, \ldots, J_p, \lambda_{2p+1}, \ldots, \lambda_n),$$

where the blocks $J_i \in \mathbb{R}^{2\times 2}$, $i = 1, \ldots, p$, are as follows:

$$J_i = \begin{pmatrix} u_i & v_i \\ -v_i & u_i \end{pmatrix}.$$

Since $|u_i| + |v_i| < 1$ for $i = 1, \ldots, p$ and $|\lambda_i| < 1$ for $i = 2p, \ldots, n$, this exactly implies that in the new coordinates the matrix A is discretely superstable.

Proof of Theorem 6.3. Let $z \in \mathbb{C}$, $z \neq 0$, be a root of a superstable polynomial (20) or, equivalently, $q(z) = 0$, where $q(z) = 1 + p_1 z^{-1} + \ldots + p_n z^{-n}$. This means that in the problem

$$\min \|p\|_1,$$
$$q(z) = 0,$$

where $p_1, \ldots, p_n$ are variables and z is fixed, the minimum value is less than unity. In the real domain, the minimization problem is set down as follows:

$$\begin{aligned}&\min \|p\|_1,\\ &(p, u) = -1, \quad u \in \mathbb{R}^n, \quad u_i = \operatorname{Re} z^{-i}, \quad i = 1, \ldots, n,\\ &(p, v) = 0, \qquad v \in \mathbb{R}^n, \quad v_i = \operatorname{Im} z^{-i}, \quad i = 1, \ldots, n.\end{aligned}$$

The duality theorem

$$\min_{(p,u)=-1,(p,v)=0} \|p\|_1 \;=\; 1/\min_{\alpha\in\mathbb{R}} \|u + \alpha v\|_\infty$$

is known [23] to be valid for this optimization problem. The one-dimensional problem of minimization in α can be solved as follows:

$$\min_\alpha \|u + \alpha v\|_\infty = \min_\alpha \max_i |u_i + \alpha v_i|$$
$$= \min_\alpha \max_{i,k} \left\{ \left|\frac{u_i}{v_i} + \alpha\right| |v_i|, \; \left|\frac{u_k}{v_k} + \alpha\right| |v_k| \right\} = \max_{1\le i,k\le n} \frac{|u_i v_k - u_k v_i|}{|v_i| + |v_k|},$$

because the minimum in α is attained where $\left|\dfrac{u_i}{v_i} + \alpha\right| |v_i| = \left|\dfrac{u_k}{v_k} + \alpha\right| |v_k|$.

By taking into account the fact that for $z = \rho^{j\theta}$ the quantities u_i and v_i take the form $u_i = \rho^{-i}\cos i\theta$ and $v_i = \rho^{-i}\sin i\theta$, we get that

$$|u_i v_k - u_k v_i| = \rho^{-i-k} |\sin(k-i)\theta|.$$

Therefore, if z is the root of a superstable polynomial, then

$$1 \;>\; \min_{q(z)=0} \|p\|_1 \;=\; 1/\max_{1\le i<k\le n} \frac{|\sin(k-i)\theta|}{\rho^k |\sin i\theta| + \rho^i |\sin k\theta|}.$$

By introducing the function

$$\varphi_n(\rho, \theta) \;\doteq\; \min_{1\le i<k\le n} \frac{\rho^k |\sin i\theta| + \rho^i |\sin k\theta|}{|\sin(k-i)\theta|},$$

we rearrange the last condition in $\varphi_n(\rho, \theta) < 1$.

Proof of Lemma 6.3. Let z_i, $i = 1, \ldots, n$, be the roots of $p(z)$ and $|z_i| < \alpha$. Then,

$$p(z) = \prod_{i=1}^n (z - z_i) = z^n + p_1 z^{n-1} + \ldots + p_n,$$

where

$$p_1 = -(z_1 + \ldots + z_n), \quad \ldots, \quad p_n = (-1)^n z_1 \cdots z_n.$$

Hence, $|p_1| \le |z_1| + \ldots + |z_n| < n\alpha, \; \ldots, \; |p_n| < \alpha^n$ and

$$|p_1| + \ldots + |p_n| \;<\; n\alpha + \ldots + \alpha^n \;=\; (1+\alpha)^n - 1.$$

The condition $\sum\limits_{i=1}^n |p_i| < 1$ is guaranteed if $(1+\alpha)^n - 1 < 1$, that is, if $(1+\alpha)^n < 2$, $\alpha < 2^{1/n} - 1$.

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This paper was recommended for publication by E.S. Pyatnitskii, a member of the Editorial Board