===== **DETERMINATE SYSTEMS** =====

Superstable Linear Control Systems.
II. Design[1]

B. T. Polyak and P. S. Shcherbakov

Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia

Received April 24, 2002

Abstract—The notion of superstability introduced in [1] is applied to the design of stabilizing and optimal controllers. It is shown that a static output feedback controller which ensures superstability of the closed-loop system can be found (provided it exists) by means of linear programming (LP) techniques; finding a superstable matrix in the given affine family is a generalization of this problem. The ideology of superstability is also shown to be fruitful in optimal and robust control. This is exemplified by the problems of rejection of bounded disturbances, optimization of the integral performance index which involves absolute values (rather than squares) of the control and state, and by stabilization of an interval matrix family and simultaneous stabilization.

1. INTRODUCTION

In the first part of the paper, [1], the notion of superstability was introduced and the analysis of superstable systems was performed (the behavior of solutions was examined both in the absence and in the presence of exogenous disturbances, the location of the spectrum was analyzed, the robust properties were studied, etc.). However, the most important feature of this notion is its applicability to design. Namely, it turns out that many hard classical problems such as static output stabilizaton, fixed-order controller design, uniform rejection of exogenous disturbances, etc., admit simple solutions provided that the stability requirement for the closed-loop system is replaced with superstability. Thus, all the above-mentioned problems can be solved using LP techniques. Furthermore, some nonstandard problems of optimal control are made tractable in this framework such as the linear-linear (rather than linear-quadratic) regulator problem discussed below.

Clearly, this approach has its own drawbacks. For instance, a controllable linear system is not necessarily superstabilizable using state feedback; also, in optimal control problems, estimates of performance indices are used rather than exact values.

We start with superstabilization. Said another way, we seek a (state or output) feedback which ensures superstability of the closed-loop system.

Similarly to the first part of the paper, the ∞-norm for vectors, $\|x\| = \max\limits_{1 \leq i \leq n} |x_i|$, $x \in \mathbb{R}^n$, and the induced 1-norm for matrices, $\|A\| = \max\limits_{1 \leq i \leq n} \left(\sum\limits_{j=1}^{n} |a_{ij}| \right)$, $A = ((a_{ij})) \in \mathbb{R}^{n \times n}$, are used unless otherwise indicated.

2. SUPERSTABILIZATION OF CONTINUOUS-TIME SYSTEMS

Let us consider the following continuous-time system:

$$\dot{x} = Ax + Bu, \qquad y = Cx, \tag{1}$$

---

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^l$ is the output, and $u \in \mathbb{R}^m$ is control which we seek in the form of the output feedback

$$u = Ky. \tag{2}$$

The closed-loop system takes the form

$$\dot{x} = A_c x, \qquad A_c = A + BKC, \qquad A_c = ((m_{ij})), \tag{3}$$

and the matrix K is referred to as stabilizing if A_c is stable ($\operatorname{Re} \lambda_i(A_c) < 0$ for all eigenvalues λ_i of A_c), and *superstablizing* if A_c is superstable, i.e., if

$$\sigma = \sigma(A_c) \doteq \min_i \left(-m_{ii} - \sum_{j \neq i} |m_{ij}| \right) > 0. \tag{4}$$

In the first part of the paper, [1], the quantity σ was referred to as the degree of superstability.

The problem of output stabilizability is extremely complicated as compared to state stabilizability where the controller is taken in the form $u = Kx$. This problem is the subject of many papers, however, it still remains open [2–4]. In contrast, the output superstabilization problem admits a very simple solution.

Indeed, the entries m_{ij}, $i, j = 1, \ldots, n$, of the matrix A_c are affine linear functions of the entries of $K = ((k_{ij}))$:

$$m_{ij} \doteq m_{ij}(K) = a_{ij} + (BKC)_{ij} = a_{ij} + b_i K c_j,$$

where b_i is the ith row of B and c_j is the jth column of C. For instance, in the state feedback stabilization problem for the simplest system with scalar control, we have $y = x$, $m = 1$, i.e., $C = I$, $B = (b_1, \ldots, b_n)^{\mathrm{T}}$ is a column vector, and $K = (k_1, \ldots, k_n)$ is a row vector, therefore, the entries of the closed-loop system matrix $A + BK$ are equal to $a_{ij} + b_i k_j$, where b_i is the ith element of B and k_j is the jth element of K (BK is a rank-one matrix).

Hence, the superstability condition (4) for the matrix A_c has the form

$$-m_{ii} > \sum_{j \neq i} |m_{ij}|, \qquad i = 1, \ldots, n.$$

Introducing slack variables σ, n_{ij}, $i, j = 1, \ldots, n$, we can re-write the superstability condition for A_c in the following form:

$$\sigma > 0,$$

$$-m_{ii}(K) - \sum_{j \neq i} n_{ij} \geq \sigma, \qquad i = 1, \ldots, n, \tag{5}$$

$$-n_{ij} \leq m_{ij}(K) \leq n_{ij}, \qquad i, j = 1, \ldots, n, \quad i \neq j.$$

If these linear inequalities possess a solution k_{ij}, n_{ij}, $i, j = i, \ldots, n$, for some $\sigma > 0$, then the system is superstable. To test if a solution exists, the following LP with respect to the matrix variables K and $N \doteq ((n_{ij}))$ and the scalar variable σ can be considered:

$$\max \sigma,$$

$$-m_{ii}(K) - \sum_{j \neq i} n_{ij} \geq \sigma, \qquad i = 1, \ldots, n, \tag{6}$$

$$-n_{ij} \leq m_{ij}(K) \leq n_{ij}, \qquad i, j = 1, \ldots, n, \quad i \neq j.$$

Theorem 2.1. *Assume that K, σ is a solution of* (6) *and $\sigma > 0$; then the feedback $u = Ky$ superstabilizes the closed-loop system. If $\sigma \leq 0$, then the system cannot be superstabilized using controllers of the form $u = Ky$.*

This result provides a complete solution to the problem of static output superstabilization (in particular, with $C = I$ we arrive at state superstabilization).

Note that in (6), we seek the maximal value of σ over all K; if it is positive, then this means that a superstabilizing controller is found which maximizes the degree of superstability of the closed-loop system. Recall that in the first part of the paper (see [1]), the following estimate was derived for the state of the superstable system $\dot{x} = A_c x$:

$$\|x(t)\| \leq \|x(0)\|e^{-\sigma t}, \qquad \sigma = \sigma(A_c). \tag{7}$$

Therefore, if $\sigma > 0$ is obtained as solution of problem (6), then it gives the best estimate (7) over all possible ones. In other words, in addition to constructing a superstabilizing controller, we satisfy the most stringent available constraints on system's state.

Example 1. We consider the following simple problem of state superstabilization. Let $n = 2$, $m = 1$, i.e.,

$$\dot{x} = Ax + Bu, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad u = Kx, \quad K = k = (k_1 \ k_2). \tag{8}$$

Then

$$A_c = A + BK = \begin{pmatrix} a_{11} + b_1 k_1 & a_{12} + b_1 k_2 \\ a_{21} + b_2 k_1 & a_{22} + b_2 k_2 \end{pmatrix},$$

and the superstability condition takes the form

$$a_{11} + b_1 k_1 < -|a_{12} + b_1 k_2|, \qquad a_{22} + b_2 k_2 < -|a_{21} + b_2 k_1|.$$

For simplicity, let $b_1 = b_2 = 1$; we are interested in finding k_1, k_2 which satisfy the inequalities

$$a_{11} + k_1 < -|a_{12} + k_2|, \qquad a_{22} + k_2 < -|a_{21} + k_1|.$$

Each of these inequalities specifies a right-angle sector on the $\{k_1, k_2\}$ plane, see Fig. 1. These angles have a common point (i.e., a solution exists) if and only if the vertex of one of the angles
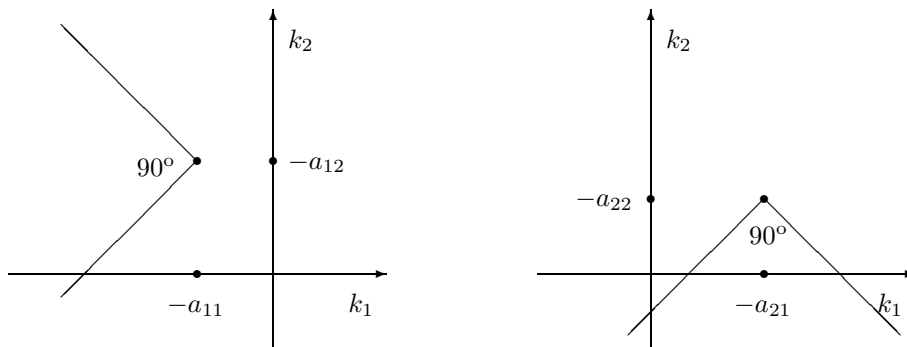
Fig. 1. The coefficients of a superstabilizing controller.

belongs to another angle, i.e., one of the following inequalities holds:

$$a_{11} - a_{21} < -|a_{12} - a_{22}|, \qquad a_{22} - a_{12} < -|a_{21} - a_{11}|,$$

which is possible if and only if

$$\tau \doteq a_{11} - a_{21} + a_{22} - a_{12} < 0.$$

Hence, if $\tau < 0$, then the system is superstabilizable; otherwise, superstabilization is not possible. In other words, superstability cannot be gained for an arbitrary matrix; however, for any matrix A such that $b = (1 \quad 1)^{\mathrm{T}}$ is not its eigenvector, the pair (A, b) is controllable, and (conventional) stabilization is made possible using feedback $u = kx$,—this is the classical solution of the pole placement problem.

The situations where superstabilization is inherently impossible are also easily detectable. Assume that

$$\dot{x} = Ax + bu, \quad u = kx, \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^1,$$

and some of the components of the vector b are zeros, e.g., $b_1 = 0$. Then the first row of $A_c = A + bk$ is the same as that of A, and if this latter does not satisfy the superstability condition (i.e. $-a_{11} \leq \sum_{j \neq 1} |a_{1j}|$), then the matrix A_c is not superstable for all k. In particular, from these considerations it follows that for a system in the controllable canonical form (i.e., with the matrix A in companion form and $b = (0 \ \ldots \ 0 \ 1)^{\mathrm{T}}$), superstability can never be achieved using state feedback. Indeed, the last row of the closed-loop matrix is the only one that is subject to changes, while the first $n-1$ rows have the form $(0 \ \ldots \ 0 \ 1 \ 0 \ \ldots \ 0)$, i.e., the superstability condition is not met. Loosely speaking, in order for a system to be superstabilizable, it is required that the number of control inputs be large enough so that any row of the closed-loop matrix can be affected by the coefficients of the controller.

In summary, superstability is not necessarily attainable (even using state feedback); however, a solution can be easily found if it exists. On the other hand, the simplicity of solution is gained at the expense of restricting the class of stabilizable systems to the smaller class of superstabilizable systems.

In the feedback stabilization, it is natural to impose limitations on the magnitude of the matrix gain in order to bound the magnitude of the control input. For example, let us consider interval constraints on the gain matrix $K = ((k_{ij}))$:

$$\underline{k}_{ij} \leq k_{ij} \leq \overline{k}_{ij}, \qquad i, j = 1, \ldots, n. \tag{9}$$

To check if system (1)–(2) is superstabilizable under these additional conditions, it suffices to incorporate linear inequalities (9) into the set of constraints in LP (6) and apply Theorem 2.1.

Similar additional linear inequalities come into play when accounting for limitations on the magnitude of K in the matrix 1-norm or ∞-norm.

3. SUPERSTABILIZATION OF DISCRETE-TIME SYSTEMS

The results in the previous section relate to continuous-time systems; the discrete-time case is analyzed equally easily. Indeed, for the system

$$\begin{aligned} x_k &= Ax_{k-1} + Bu_{k-1}, \\ y_k &= Cx_k, \end{aligned}$$

with feedback

$$u_k = Ky_k,$$

the superstability condition for the matrix $A_c = A + BKC$ writes $\|A_c\| < 1$, i.e.

$$\sum_{j=1}^{n} |m_{ij}(K)| < 1, \qquad i = 1, \ldots, n.$$

These conditions can be expanded as linear inequalities with respect to the entries of the matrix K. Finding $\min_{K} \|A + BKC\|$ reduces to an LP similar to (6); it has the form

$$\min q,$$

$$\sum_{j} n_{ij} \leq q, \qquad i = 1, \ldots, n, \tag{10}$$

$$-n_{ij} \leq m_{ij}(K) \leq n_{ij}, \qquad i, j = 1, \ldots, n,$$

and the conditions of superstabilizability are given by the following theorem.

Theorem 3.1. *Assume that K, q is a solution to LP (10). If $q < 1$, then the controller $u = Ky$ ensures superstability of the closed-loop matrix $A_c = A + BKC$, i.e., $\|A_c\| < 1$; if $q \geq 1$, then A_c is not superstable for all K.*

In some cases, the lack of superstabilizability can be detected without solving LP (10), but rather using the following result on duality. Along with the problem

$$\text{Find} \quad q^* = \min_{K} \|A + BKC\|_1, \tag{11}$$

we consider its dual:

$$\max \operatorname{tr} A^{\mathrm{T}} Y,$$
$$\|Y\|_{\infty} \leq 1, \tag{12}$$
$$CY^{\mathrm{T}} B = 0,$$

where $Y = ((y_{ij})) \in \mathbb{R}^{n \times n}$ is the matrix to be found, $\|Y\|_{\infty} = \max\limits_{1 \leq j \leq n} \left(\sum\limits_{i=1}^{n} |y_{ij}| \right)$, and $\operatorname{tr} A^{\mathrm{T}} Y$ denotes the trace of the matrix $A^{\mathrm{T}} Y$, i.e., $\operatorname{tr} A^{\mathrm{T}} Y = \sum\limits_{i,j} a_{ij} y_{ij}$.

Theorem 3.2. *The optimal values in problems (11) and (12) are equal, and for any feasible Y_0 in (12) the following condition holds*: $\operatorname{tr} A^{\mathrm{T}} Y_0 \leq q^*$.

Hence, if a matrix Y_0 satisfying the conditions $\|Y_0\|_{\infty} \leq 1$ and $CY_0^{\mathrm{T}} B = 0$ is found such that $\operatorname{tr} A^{\mathrm{T}} Y_0 \geq 1$, then $q^* \geq 1$, and therefore, superstabilization is not possible.

Example 2. We consider an example similar to Example 1 for continuous-time systems:

$$x_k = A x_{k-1} + B u_{k-1}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \ B = b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ u = Kx, \ K = k = (k_1 \ k_2).$$

Let us take

$$Y_0 = \begin{pmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{pmatrix};$$

then we obtain $\|Y_0\|_\infty = 1$, $CY_0^{\mathrm{T}}B = Y_0^{\mathrm{T}}b = 0$, and $\operatorname{tr} A^{\mathrm{T}}Y_0 = 0.5(a_{11} - a_{12} + a_{22} - a_{21})$. Hence, for $a_{11} - a_{12} + a_{22} - a_{21} \geq 2$, superstabilization is not possible. Moreover, using nearly the same reasonings as in Example 1 (the only difference is that instead of two right angles we deal with two rhombuses), it can be shown that the condition

$$|a_{11} - a_{12}| + |a_{22} - a_{21}| < 2$$

is necessary and sufficient for the system to be superstabilizable.

We note that the superstabilization problem for discrete-time systems was first discussed in [5].

4. SUPERSTABILIZATION OF SISO SYSTEMS

We now discuss the peculiarities of superstabilization in the SISO case. Let a SISO plant be specified by the transfer function

$$G(z) = \frac{a(z)}{b(z)},$$

where

$$a(z) = a_0 + a_1 z + \cdots + a_n z^n, \qquad b(z) = b_0 + b_1 z + \cdots + b_m z^m.$$

We are aimed at putting the controller

$$C(z) = \frac{f(z)}{g(z)}$$

in the feedback path (see Fig. 2) in order to make the characteristic polynomial

$$p(z) = a(z)f(z) + b(z)g(z)$$

of the closed-loop system superstable. Recall that in [1], superstability of the discrete polynomial $p(z) = p_0 + p_1 z + \cdots + p_n z^n$ was defined as

$$\sum_{i=1}^{n} |p_i| < |p_0|.$$

Similarly to the conventional stabilization, the following approach yields a solution: we take $p(z)$ as an arbitrary superstable polynomial, solve the Bezout identity

$$a(z)f(z) + b(z)g(z) = 1 \tag{13}$$

(a solution always exists under the assumption that $a(z)$ and $b(z)$ are coprime), and let

$$f(z) = p(z)f^0(z) + b(z)r(z), \qquad g(z) = p(z)g^0(z) - a(z)r(z),$$

where $f^0(z)$, $g^0(z)$ is the minimal-degree solution to (13), $\deg f^0 \leq m-1$, $\deg g^0 \leq n-1$, and $r(z)$ is an arbitrary polynomial. However, this approach often leads to high-order controllers (the degrees of $f(z)$ and $g(z)$ are high).
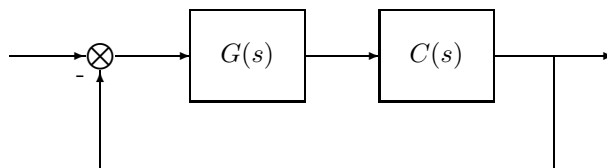

Fig. 2. Stabilizing controller.

Instead, we fix the degrees F and G of the polynomials $f(z)$ and $g(z)$:

$$f(z) = f_0 + f_1 z + \cdots + f_F z^F, \qquad g(z) = g_0 + g_1 z + \cdots + g_G z^G;$$

then the coefficients p_i of the characteristic polynomial are linear functions of $f_0, f_1, \ldots, f_F$, $g_0, g_1, \ldots, g_G$, and the superstability condition takes the form

$$|p_0(f,g)| > \sum_{i=1}^{l} |p_i(f,g)|, \qquad f = (f_0, \ldots, f_F), \quad g = (g_0, \ldots, g_G),$$

where $l = \max\{n + F, \ m + G\}$ is the degree of $p(z)$. The feasibility of this system of linear inequalities can be checked by solving an LP similar to (10). For given F and G this LP may have no solution, in which case the degrees F and G are to be increased and the corresponding linear program of higher dimensionality is to be solved. By the Bezout theorem, such an LP has a guaranteed solution for $G = n - 1$ and $F = m - 1$.

Hence, a discrete SISO system can always be superstabilized, and the order of the superstabilizing controller appears to be considerably lower than that of a stabilizing controller.

Superstabilization of discrete-time SISO systems was studied in detail in [5], which also contains numerical examples.

5. FINDING A SUPERSTABLE ELEMENT IN MATRIX AND POLYNOMIAL FAMILIES

Let us consider the problem of superstabilization from a different point of view. Given a matrix $A \in \mathbb{R}^{n \times n}$, find a stable matrix X which is closest to A. The distance is understood as

$$\mathrm{dist}\,(A, X) \doteq \|A - X\|,$$

where $\| \cdot \|$ is some matrix norm, and stability can be understood either in the discrete-time or continuous-time sense. Another formulation of this problem is as follows. Given the affine matrix family

$$A(q) = A_0 + \sum_{k=1}^{\ell} q_k A_k, \qquad q \in Q \subseteq \mathbb{R}^\ell, \tag{14}$$

where $A_0, \ldots, A_\ell \in \mathbb{R}^{n \times n}$ are known, find if it contains a stable matrix. The set Q is either the whole space $\mathbb{R}^\ell$ or a convex bounded set; usually, it is a ball in some norm:

$$Q = \{q \in \mathbb{R}^\ell \colon \|q - q^0\| \leq \gamma\}. \tag{15}$$

The origin of this problem is static output stabilization of state space MIMO systems. Indeed, stabilization of the system $\dot{x} = Ax + Bu$, $y = Cx$, by means of a controller $u = Ky$ reduces to finding a matrix K such that the matrix $A_c = A + BKC$ is stable, where A, B, C are given. Viewing the entries of K as the parameters q, we arrive at a problem in the form (14), which is, therefore, a generalization of the output stabilization problem.

In a number of specific cases (for instance, when the matrices $A_0, \ldots, A_\ell$ are symmetric) all these problems admit a simple solution. In the general case, such problems are NP-hard (see [6]), and there are no efficient methods for finding exact solutions. Some numerical methods are presented in [7]; however, they do not guarantee that the exact solution will be found.

Replacing stability with superstability considerably simplifies the problem, since it reduces to linear or quadratic program, depending on the underlying matrix norm. For example, let us consider the problem of finding the closest superstable matrix in the continuous-time case and the distance

being specified in the Frobenius norm. Assume that a_{ij} are the entries of the given matrix A, $\min_i \left(-a_{ii} - \sum_{j \neq i} |a_{ij}| \right) \leq 0$, and x_{ij} are the entries of the matrix X to be found, i.e., the one that belongs to the closure of the set of superstable matrices and which is closest to A. Then X is a solution of the quadratic programming problem

$$\min \sum_{i,j} (a_{ij} - x_{ij})^2,$$

$$x_{ii} + \sum_{j \neq i} |x_{ij}| \leq 0, \qquad i = 1, \ldots, n. \tag{16}$$

Note that if we denote the minimum value in this problem by γ^*, then obviously, $\gamma \leq \gamma^*$, where γ is the distance between A and the set of stable matrices. Hence, the distance to the closest superstable matrix is an estimate from above for the distance to the closest stable matrix.

If the distance is specified in the interval norm, $\|A - X\| = \max_{ij} |a_{ij} - x_{ij}|$, then the quadratic function (16) under minimization is replaced with a linear one, and we arrive at the problem

$$\min \max_{i,j} |a_{ij} - x_{ij}|,$$

$$x_{ii} + \sum_{j \neq i} |x_{ij}| \leq 0, \qquad i = 1, \ldots, n, \tag{17}$$

which reduces to an LP in a standard way. A similar LP problem arises if the ∞-norm, $\|A - X\|_\infty = \max_j \sum_i |a_{ij} - x_{ij}|$, is considered.

In the discrete-time case, the only difference is that the constraints on the x_{ij} in (16) or (17) are changed for

$$\sum_j |x_{ij}| \leq 1, \qquad i = 1, \ldots, n;$$

they as well reduce to a system of linear inequalities with respect to the x_{ij}.

Sometimes the problem admits a closed-form solution; the simplest case is discrete-time superstability and the 1-norm; namely, given a matrix A, $\|A\|_1 \geq 1$, find X, $\|X\|_1 \leq 1$, which minimizes $\|A - X\|_1$. The solution is attained with the matrix

$$X^* \doteq \arg \min_{\|X\|_1 \leq 1} \|A - X\|_1 = \frac{A}{\|A\|_1},$$

and we have

$$\min \|A - X\|_1 \; = \; \|A - X^*\|_1 \; = \; \|A\|_1 \left(1 - \frac{1}{\|A\|_1} \right) \; = \; \|A\|_1 - 1.$$

Moreover, additional constraints can be imposed on the entries of the matrix X to be found; for instance, they may have the form $\underline{x}_{ij} \leq x_{ij} \leq \overline{x}_{ij}$ or $|x_{ij} - x_{ij}^0| \leq \gamma$, $i, j = 1, \ldots, n$, where $X^0 = ((x_{ij}^0))$ is the nominal value. This will only lead to the incorporation of additional linear constraints on the variables x_{ij} into (16) or (17). In such a formulation, the problem is stated as finding a superstable matrix in a matrix family; if the entries of X are viewed as parameters, we arrive at affine family (14) and finding a superstable matrix in it.

This problem can be solved as shown above, namely, by finding the superstable matrix which is closest to some *a priori* chosen matrix $A = ((a_{ij}))$. For example, in the continuous-time case, the

distance being specified in the interval norm, and the constraints in the form $\|q - q^0\|_\infty \le \gamma$, we arrive at the problem

$$\min_q \max_{i,j} \left| a_{ij} - a_{ij}^0 - \sum_{k=1}^\ell q_k a_{ij}^k \right|,$$

$$a_{ii}^0 + \sum_{k=1}^\ell q_k a_{ii}^k + \sum_{j\neq i} \left| a_{ij}^0 + \sum_{k=1}^\ell q_k a_{ij}^k \right| \le 0, \qquad i = 1, \ldots, n$$

(here, $A_k = ((a_{ij}^k))$, $k = 0, 1, \ldots, \ell$), which reduces to an LP problem with respect to q. Let q^* denote the solution of this problem and assume that $\|q^* - q^0\|_\infty = \gamma^* \le \gamma$; then the family (14), (15) contains the superstable matrix $A(q^*)$, and it is the closest one (in the interval norm) to the given matrix A.

An alternative approach is to test feasibility of the set of simultaneous inequalities. For instance, for the same continuous-time case and with the constraints $\|q - q^0\|_\infty \le \gamma$ on the parameters, the entries $a_{ij}(q)$ of $A(q)$ (14) are affine linear in q, and the set of inequalities has the form

$$-a_{ii}(q) - \sum_{j\neq i} |a_{ij}(q)| > 0, \qquad i = 1, \ldots, n,$$

$$|q_i - q_i^0| \le \gamma, \qquad i = 1, \ldots, \ell.$$

If it is feasible (this can be checked by reducing to an LP problem), then the family (14), (15) contains a superstable matrix. Every superstable matrix is stable, therefore, if a solution exists, it is also a solution to the problem of finding a stable matrix in the affine family.

The cases of discrete-time superstability and other norms are treated similarly.

All the matrix problems considered above can be formulated for polynomials. Given the affine family

$$\mathcal{P}(z, Q) = \left\{ p(z, q) = p^0(z) + \sum_{i=1}^\ell q_i p^i(z), \quad \deg p^i(z) < \deg p^0(z), \quad q \in Q \subseteq \mathbb{R}^\ell \right\}, \qquad (18)$$

where $p^i(z)$, $i = 0, \ldots, \ell$, are known, check if it contains at least one stable polynomial. Analogous to the matrix case, the set Q may coincide with $\mathbb{R}^\ell$, or it may as well be specified by either norm constraints (15) or interval constraints

$$Q = \{q \in \mathbb{R}^\ell : \underline{q}_i \le q_i \le \overline{q}_i, \ i = 1, \ldots, \ell\}. \qquad (19)$$

This problem is a generalization of the problem of stabilizability by means of fixed-order controllers.

In the particular case where the coefficients of the polynomial $p(z, q)$ are considered as uncertain parameters q_i, the problem can be formulated in the following way. Given an unstable polynomial $p^0(z)$, find the closest stable polynomial of the same degree. The distance between the polynomials $a(z) = a_0 + a_1 z + \cdots + a_n z^n$ and $b(z) = b_0 + b_1 z + \cdots + b_n z^n$ is specified by $\mathrm{dist}\,(a(z), b(z)) = \|a - b\|$, where $a, b \in \mathbb{R}^{n+1}$ are the vectors composed of the coefficients of $a(z)$ and $b(z)$, and $\|\cdot\|$ is some norm in $\mathbb{R}^{n+1}$. Likewise the matrix case, if we have

$$\min_{p(z,q)\in\mathcal{P}_s} \mathrm{dist}\left(p^0(z),\, p(z, q)\right) = \gamma^* \le \gamma,$$

where $\mathcal{P}_s$ is the set of stable polynomials, then the problem of finding a stable polynomial in family (18) with $p(z, q) = q_0 + q_1 z + \cdots + q_n z^n$ and constraints (15) admits a solution; otherwise there is no solution.

At the first glance, this problem is quite similar to the problem of robust stability of polynomials, which can be efficiently solved by means of well-known techniques (e.g., see [8]). However, the difference between these two problems is that in the latter, we are aimed at detecting if there is at least one unstable polynomial in family (18), (19) (or (18), (15)), while in the former, the goal is to find at least one stable polynomial in the same family. Likewise the matrix case, this difference leads to a dramatic increase in complexity, and the problem becomes NP-hard (see [6, 9]). Approximate solution methods for such problems can be found in [7].

In contrast, finding the closest superstable polynomial is immediate (recall that for polynomials, superstability was defined only in discrete time). Indeed, assume that $p = (p_0, p_1, \ldots, p_n)$ are the coefficients of the given unstable polynomial $p(z)$ and $q = (q_0, q_1, \ldots, q_n)$ are the coefficients of the polynomial $q(z)$ to be found, i.e., the one that belongs to the closure of the set of superstable polynomials. Minimization of the distance between $p(z)$ and $q(z)$ (specifically, the Euclidean norm is considered) leads to the quadratic programming problem

$$\min \sum_{i=0}^{n} (p_i - q_i)^2,$$

$$-|q_0| + \sum_{i=1}^{n} |q_i| \leq 0$$

(linear constraints on the absolute values of the variables reduce in the standard way to the linear constraints on the variables themselves). For other norms such as the 1-norm or ∞-norm, the formulation of the problem is the same, and we arrive at a linear programming problem.

Sometimes solutions can be obtained in closed form, for instance, if the 1-norm is used (similarly to the matrix case). If we restrict ourselves to monic polynomials having $p_0 = q_0 = 1$, then the problem reduces to finding $\min \|p(z) - q(z)\|_1$ under the conditions $\|q - 1\|_1 \leq 1$, $q_0 = 1$. Here, the solution is attained with the polynomial $q^*(z) = 1 + (p(z) - 1)/\|(p - 1)\|_1$, and the minimum value is equal to

$$\min \|p(z) - q(z)\|_1 = \|p - 1\|_1 - 1.$$

6. OPTIMAL CONTROL

If a system can be stabilized by means of output or state feedback, then it might be worth trying to optimize its performance. We consider the following two problems. In the first problem we deal with the optimal rejection of bounded disturbances; in the second problem, a specific integral performance index is minimized. However, instead of stability of the closed-loop system, we consider superstability and study the consequences that result.

6.1. Rejection of Bounded Disturbances

Let the state-space continuous-time MIMO system

$$\dot{x} = Ax + Bu + D_1 w, \qquad x(0) = x_0,$$

$$y = Cx + D_2 w, \qquad \|w(t)\| \leq 1, \quad t \geq 0,$$

$$(20)$$

be given, where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^l$ is the output, $u \in \mathbb{R}^m$ is control, and $w \in \mathbb{R}^{m_1}$ is an exogenous disturbance such that it is only assumed to be bounded at all time instants. We seek an output regulator $u = Ky$ and first of all require that it superstabilize the closed-loop system

$$\dot{x} = A_c x + Dw, \qquad A_c \doteq A + BKC, \quad D \doteq D_1 + BKD_2, \tag{21}$$

i.e.,

$$\sigma(A_c) > 0,$$

where $\sigma(A_c)$ is defined by (4). Among such regulators, of interest is the one that minimizes the performance index

$$J = \sup_{\|w\| \le 1} \sup_t \|x(t)\|,$$

i.e., provides the maximal reduction of the effect of bounded disturbances (in a more general setting, a linear function of the state $x(t)$ can be considered rather than the state itself). Such kind of classical problems where stability (not superstability) of the closed-loop system is required are the subject of the so-called theory of l_1-optimization [10–12]. They are known to be extremely hard even for SISO systems (and especially in the continuous-time case), and reasonable solutions can be obtained only in particular cases. We show that replacing stability with superstability considerably simplifies the problem of rejection of bounded disturbances so that a generalization to MIMO systems is made possible and the continuous-time and discrete-time cases can be treated equally easily.

First of all, we remind that for continuous-time superstable system (21) with $\|w(t)\| \le 1$, the estimate

$$\|x(t)\| \le \frac{\|D\|}{\sigma(A_c)} = \frac{\|D_1 + BKD_2\|}{\sigma(A + BKC)}$$

was obtained in [1] provided that $\|x(0)\| \le \frac{\|D\|}{\sigma(A_c)}$. We make use of this estimate and minimize it over all K under the condition that K is a superstabilizing controller. Introducing the parameter $\sigma > 0$, we arrive at the problem

$$\min_{K, \sigma} \frac{\|D_1 + BKD_2\|}{\sigma}, \tag{22}$$

$$\sigma(A + BKC) \ge \sigma > 0. \tag{23}$$

As above, for fixed σ this problem can be recast into a set of linear inequalities with respect to the entries of the controller matrix K.

Theorem 6.1. *If parametric LP* (22), (23) *has a solution* K, σ *with the optimal value* J^* *of performance index* (22), *then the controller* $u = Ky$ *superstabilizes system* (20), *and for any initial conditions* $\|x(0)\| \le J^*$, *the relation* $\|x(t)\| \le J^*$, $t > 0$, *holds.*

Hence, the regulator obtained minimizes the norm of the state vector uniformly in t, which prevents undesired peak effects.

It was already mentioned that not every system can be superstabilized. Therefore, problem (22), (23) may not have a solution, and even if a solution exists, the quantity J^* is just an upper bound for the optimal value of J.

In discrete time, the problem is formulated and solved in the same way; namely, we are dealing with the system

$$x_k = Ax_{k-1} + Bu_{k-1} + D_1 w_{k-1},$$

$$y_k = Cx_k + D_2 w_k, \qquad \|w_k\| \le 1, \quad k = 0, 1, \dots, \tag{24}$$

and are aimed at finding a controller $u_k = Ky_k$ which superstabilizes the closed-loop system

$$x_k = A_c x_{k-1} + D w_{k-1}, \qquad A_c \doteq A + BKC, \quad D \doteq D_1 + BKD_2, \tag{25}$$

i.e., the one that assures $\|A_c\| < 1$. Similarly to the continuous-time case, we make use of the estimate

$$\|x_k\| \leq \frac{\|D\|}{1 - \|A_c\|} = \frac{\|D_1 + BKD_2\|}{1 - \|A + BKC\|}$$

obtained in [1] for the discrete superstable system (25) with $\|w_k\| \leq 1$. Then, introducing the parameter $0 \leq q < 1$ reduces the minimization of this estimate to the parametric LP problem

$$\min_{K,q} \frac{\|D_1 + BKD_2\|}{1 - q},$$
$$\|A + BKC\| \leq q.$$

Let us now consider a discrete-time SISO system specified by the following scalar difference equation:

$$a(z)y_k = b(z)u_k + w_k, \tag{26}$$

where

$$a(z) = 1 + a_1 z + \cdots + a_n z^n, \qquad b(z) = b_1 z + \cdots + b_m z^m \tag{27}$$

are polynomials and z is the backward shift operator, i.e., $z^i y_k = y_{k-i}$. Note that $b(0) = 0$, i.e., y_k depends on $u_{k-1}, \ldots, u_{k-m}, y_{k-1}, \ldots, y_{k-n}$, and the polynomial $a(z)$ can always be normalized to have $a(0) = 1$. As far as the disturbance w is concerned, it is only assumed to be bounded at all time instants k, i.e.,

$$|w_k| \leq r, \qquad k = 0, 1, \ldots, \tag{28}$$

where $r > 0$ is a constant. We seek a stabilizing feedback control u in the form

$$u_k = -\frac{f(z)}{g(z)} y_k = -C(z)y_k \tag{29}$$

which minimizes the performance index

$$J \doteq \sup_w \sup_k |y_k|. \tag{30}$$

In other words, we want the output to be as small as possible under the worst-case disturbance.

The problem (26)–(30) is known to be hard (as well as its MIMO counterpart discussed above). The solution methods available in the literature are based on the parameterization of all stabilizing controllers followed by the optimization with respect to the parameter so that the problem reduces to infinite-dimensional linear program. The solution is attained with a finite-dimensional vector, and the optimal regulator is reproduced from this vector. The basic difficulty along this way is high orders of the optimal regulators which result (often, this order appears to be higher than that of the plant itself); moreover, they cannot be evaluated in advance. The continuous-time case (L_1-optimization) is even more complicated, since the optimal regulator $u = C(s)y$ may happen to be infinite-dimensional, i.e., the optimal function $C(s)$ may not be rational fractional.

We take the superstability point of view and show that this approach to the rejection of bounded disturbances simplifies the problem considerably; in particular, an optimal controller of prespecified order can be designed. The characteristic polynomial of the closed-loop system is

$$p(z) = a(z)g(z) + b(z)f(z),$$

and the system takes the form

$$p(z)y_k = g(z)w_k, \qquad |w_k| \leq r, \quad k = 0, 1, \ldots. \tag{31}$$

Similarly to $a(z)$, the polynomial $g(z)$ can be normalized to have the property $g(0) = 1$; in combination with the conditions $a(0) = 1$ and $b(0) = 0$, this gives $p(0) = 1$, i.e., the characteristic polynomial is equal to

$$p(z) = 1 + p_1 z + \cdots + p_n z^n.$$

We require it to be superstable, i.e., $\sum_{i=1}^{n} |p_i| < 1$, or, equivalently,

$$\|p(z) - 1\|_1 < 1,$$

where the 1-norm of a polynomial is defined as the sum of the absolute values of its coefficients. In [1], the following estimate was obtained for the output of system (31) with superstable polynomial $p(z)$:

$$|y_k| \leq \frac{\|g\|_1}{1 - \|p - 1\|_1} r \qquad \text{for all } k.$$

Hence, in order to minimize the maximum of the absolute value of the output, $\sup_k |y_k|$, the quantity

$$\gamma(f, g) \doteq \frac{\|g\|_1}{1 - \|p - 1\|_1} r \tag{32}$$

can be minimized (with respect to $f(z)$ and $g(z)$) under the assumption that the denominator of this expression is positive. Introducing the parameter q, $0 \leq q < 1$, we arrive at the problem

$$\min_{f,g} \|g\|_1/(1 - q),$$
$$\|ag + bf - 1\|_1 \leq q, \qquad g(0) = 1. \tag{33}$$

Let us now fix the degrees F and G of the polynomials $f(z)$ and $g(z)$:

$$f(z) = f_0 + f_1 z + \cdots + f_F z^F, \qquad g(z) = 1 + g_1 z + \cdots + g_G z^G,$$

and the value of the parameter q. Then using standard tools (see Section 3), problem (33) can be reformulated as an LP with respect to the variables $f_0, f_1, \ldots, f_F, g_1, \ldots, g_G$. We next solve this LP for various $0 \leq q < 1$ and optimize with respect to q thus finding the minimum value γ^* of quantity (32) so that the controller $C(z) = f(z)/g(z)$ provides the estimate

$$\sup_k |y_k| \leq r\gamma^* \tag{34}$$

for the closed-loop system. It may turn out that for small values of F and G problem (33) admits no solution even for $q \approx 1$, i.e., the polynomial $p(z)$ cannot be made superstable; in that case, the degrees F and G of the polynomials of the controller should be increased. It is important to note that with this approach we do not face any problems associated with high orders. Indeed, by the Bezout theorem, the equation $ag + bf = 1$ possesses a solution with $G \leq n - 1$ and $F \leq m - 1$ (provided that the polynomials $a(z)$ and $b(z)$ are coprime), and the constraints in (33) are feasible for $q = 0$, which means that problem (33) admits a guaranteed solution for $G = n - 1$, $F = m - 1$.

The method described above is simpler than l_1-optimization; moreover, optimal controllers having prespecified orders can be designed within this framework. However, estimate (34) is just an upper bound for $\sup_k |y_k|$, therefore, l_1-optimization may result in better values of this criterion.

The superstability-based approach to the problem of rejection of bounded disturbances was first proposed in [5]. A similar approach to the design of fixed-order optimal regulators for SISO systems was first studied in [13], also see [14]. Related problems were the subject of [15, 16].

6.2. Linear Linear Regulator

The following Linear Quadratic Regulator (LQR) problem is considered classical in control theory. Given a linear system

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0,$$

find a stabilizing control $u = Kx$ which minimizes the integral quadratic performance index

$$J = \int_0^\infty \left(x^{\mathrm{T}} R x + u^{\mathrm{T}} S u \right) dt.$$

Solution to this problem is well known, e.g., see [17]).

However, sometimes it makes sense to consider the optimal control problem with the linear performance functional:

$$J = \int_0^\infty \left(\|x\| + \alpha\|u\| \right) dt.$$

It is natural to refer to this problem as *Linear Linear Regulator* (LLR) problem. In particular, it arises in some specific problems in l_1-optimization, [18], and in control engineering applications (see [19]), but systematic methods for solving it are not known. However, similarly to many problems discussed in the previous sections, a solution can be easily found by replacing stability with superstability. Indeed, if we require that the closed-loop matrix $A_c = A + BK$ be superstable and make use of estimate (7), then straightforward algebra leads to the relationship

$$J \leq \frac{(1 + \alpha\|K\|)\,\|x_0\|}{\sigma(A + BK)}.$$

Now the right-hand side of this inequality, being an upper bound for J, can be minimized with respect to K. In turn, this problem reduces to the parametric LP

$$\min_{K,\sigma} \frac{1}{\sigma} \left(1 + \alpha\|K\| \right),$$
$$\sigma(A + BK) \geq \sigma > 0$$

using standard tools, see above.

It is important to note that removing the term $\alpha\|u\|$ may lead to the degeneration of solutions. For example, if the inequality $\sigma(A + BK) \geq \sigma > 0$ is feasible for all σ (in particular, this is the case for B square and nonsingular), then σ can be made arbitrarily large, and J can be made arbitrarily small. However, in this situation, the matrix K and the control input u will take very large values; the term $\alpha\|u\|$ serves to prevent such effects. On the other hand, the presence of this term does not guarantee the existence of solutions. For instance, taking $u = -kx$ in the SISO system $\dot{x} = u$ gives $J = (\alpha + 1/k)|x_0|$, and the optimum cannot be attained with finite k. The system in Example 1 of Section 2 experiences the same effect.

Similar results in the LLR problem are valid for discrete-time systems.

7. SOME ISSUES IN ROBUST DESIGN

In the first part of the paper (see [1]), robust superstability of the interval matrix family

$$A = ((a_{ij})), \qquad a_{ij} = a_{ij}^0 + \gamma\Delta_{ij}, \quad |\Delta_{ij}| \leq m_{ij}, \quad i, j = 1, \ldots, n, \tag{35}$$

was analyzed and the complete solution of the problem was obtained. Specifically, it was shown that if the nominal matrix $A_0 = ((a_{ij}^0))$ is superstable, then the family is robustly superstable for any

$$\gamma < \gamma^* \doteq \frac{\sigma(A_0)}{n} \tag{36}$$

(in the simplest case $m_{ij} \equiv 1$).

We now turn to design of superstabilizing controllers in the presence of uncertainty in the plant description, i.e., to robust superstabilization. We also study the related problem of so-called simultaneous stabilization when several fixed systems are to be stabilized by means of a single controller.

7.1. Robust Superstabilization

Let us consider an uncertain continuous-time system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned}$$

and assume for simplicity that uncertainty is lumped in the matrix A. Assume next that A is interval and has the form (35), where $A_0 \doteq ((a_{ij}^0))$ is the matrix of the nominal system, $\gamma \geq 0$ is a scalar parameter (range of uncertainty), and $m_{ij} \geq 0$ are given numbers that constitute the matrix $M = ((m_{ij}))$. The problem is to detect if there is a feedback

$$u = Ky$$

which stabilizes the whole family of systems. It is well known (e.g., see [6]) that even checking for robust stability of an interval family (and finding the robustness radius) is hard; the same difficulties arise in robust stabilization. However, the problem becomes nearly trivial if the superstability point of view is taken.

Indeed, we represent the closed-loop matrix $A_c = A + BKC$ in the form

$$A_c = A_0 + BKC + \gamma\Delta = A_c^0(K) + \gamma\Delta, \qquad \Delta = ((\Delta_{ij})),$$

where $A_c^0(K) = ((a_{ij}^0(K))) = A_0 + BKC$ denotes the nominal matrix of the closed-loop system, which affine linearly depends on K. Then the uncertain system can be superstabilized by a feedback $y = Ku$ if and only if the following relation holds for all admissible Δ:

$$-(a_{ii}^0(K) + \gamma\Delta_{ii}) - \sum_{j \neq i} |a_{ij}^0(K) + \gamma\Delta_{ij}| > 0, \qquad i = 1, \ldots, n.$$

The simultaneous satisfaction of these inequalities for all admissible uncertainties Δ_{ij} is equivalent to the feasibility of the following set of linear inequalities with respect to the entries of K:

$$-a_{ii}^0(K) - \gamma m_{ii} - \sum_{j \neq i} \left(|a_{ij}^0(K)| + \gamma m_{ij}\right) > 0, \qquad i = 1, \ldots, n.$$

As usual, checking for feasibility reduces to an LP with respect to K.

Moreover, assume that the nominal system is superstabilizable, i.e., the LP (cf. (6))

$$\max \sigma,$$

$$-a_{ii}^0(K) - \sum_{j \neq i} n_{ij} \geq \sigma, \qquad i = 1, \ldots, n, \tag{37}$$

$$-n_{ij} \leq a_{ij}^0(K) \leq n_{ij}, \qquad i, j = 1, \ldots, n, \quad i \neq j,$$

admits a solution K, $\sigma > 0$ (see Theorem 2.1), and let us denote $\sigma \doteq \sigma(A_c^0)$. Then the radius of *maximal robustness* can easily be found, which is the maximal value of the range γ of uncertainty that allows for robust superstabilization. Indeed, if the matrix $A_c^0(K)$ of the nominal closed-loop system is superstable for some K, then by (36), the perturbed system retains superstability for all

$$\gamma < \gamma_K^* \doteq \frac{\sigma(A_c^0(K))}{n}$$

(for simplicity, the case $m_{ij} \equiv 1$ is considered). The value of $\sigma(A_c^0) > 0$ obtained in the course of optimization in (37) over all superstabilizing controllers K is the maximal possible among all $\sigma(A_c^0(K))$, hence, the radius of maximal robustness is equal to

$$\gamma^* \doteq \frac{\sigma(A_c^0)}{n}.$$

Similar results are valid in the discrete-time case.

7.2. Simultaneous Stabilization

The problem of simultaneous stabilization arises in many practical applications. Assume that a plant is functioning in several operating conditions and it may switch from one mode to another. Information about such transitions may not be available; for instance, they may be caused by faults of the elements of the plant. The objective is to design a controller which ensures proper functioning (first of all, stability) of the system under all operating conditions. The following problem is a mathematical formalization of such a situation. Given m SISO plants with transfer functions

$$G_i(s) = \frac{a_i(s)}{b_i(s)}, \qquad i = 1, \ldots, m, \tag{38}$$

detect if there is a controller

$$C(s) = \frac{f(s)}{g(s)} \tag{39}$$

which simultaneously stabilizes all the plants. To put it differently, can polynomials $N(s)$ and $D(s)$ be found such that all characteristic polynomials

$$p_i(s) = a_i(s)f(s) + b_i(s)g(s), \qquad i = 1, \ldots, m,$$

are Hurwitz?

For $m = 1$ and a_1, b_1 having no common unstable zeros, a solution can always be found using Bezout theorem; furthermore, all stabilizing controllers can be obtained using Youla parameterization. For two plants, a solution can also be found in both the SISO and MIMO cases. However, for $m = 3$, solution methods are not known. Moreover, there is an indirect evidence of the absence of "simple" solutions. The reader is referred to the monograph [20] for a survey of the results in this area of research.

The problem of simultaneous stabilization also arises in the matrix form. Given m linear state space systems

$$\dot{x} = A_i x + B_i u, \qquad i = 1, \ldots, m, \tag{40}$$

find a state feedback

$$u = Kx \tag{41}$$

that stabilizes all of them. In other words, does there exist a matrix K such that all matrices

$$A_c^i \doteq A_i + B_i K, \qquad i = 1, \ldots, m,$$

are stable? The identical formulation is valid in discrete time. General solution methods for this problem are not known, and it is obvious that they cannot be straightforward, since the particular case (38), (39) of this problem is hard.

However, if we require superstability instead of stability, the problem becomes trivial, and solution is made possible using linear programming techniques. Indeed, each of the conditions $\sigma(A_i + B_i K) > 0$ is nothing but a system of linear inequalities of the form (5) with respect to the entries of K, and we conclude that if the cumulative system of linear inequalities

$$\sigma(A_i + B_i K) > 0, \qquad i = 1, \ldots, m, \tag{42}$$

possesses a solution K, then controller (41) simultaneously superstabilizes all m systems (40). Since superstability implies stability, the problem of simultaneous stabilization gains a solution. However, this is just a sufficient solution in the sense that simultaneous stabilization can be performed in spite of infeasibility of the set of linear inequalities (42).

8. CONCLUSIONS

The notion of superstability introduced in [1] appears to be very useful in controller design. Thus, in contrast to output stabilization, the superstabilization problem can be solved using extremely simple tools; however, a solution does not always exist. The problem of optimal rejection of bounded disturbances in MIMO systems also admits a solution using linear programming techniques. It is also possible to solve nonstandard problems of optimal control such as the minimization of the L_1-norm of the output. However, only suboptimal controls can be obtained in these problems, since estimates of solutions are used instead of the exact values.

By and large we can conclude that the superstability-based approach to control problems appears to be very much promising.

APPENDIX

Proof of Theorem 3.2. Let us recast the problem $\min \|A + BKC\|_1$ into the form of constrained minimization

$$\min \|H\|_1,$$
$$H = A + BKC.$$

Compose the Lagrange function with the matrix Lagrange multiplier $Y \in \mathbb{R}^{n \times n}$:

$$L(H, K, Y) = \|H\|_1 + \langle Y, \, H - A - BKC \rangle,$$

where $\langle X, \, Y \rangle$ denotes the inner product in the space of $n \times n$ matrices:

$$\langle X, \, Y \rangle = \operatorname{tr} X^{\mathrm{T}} Y = \operatorname{tr} Y X^{\mathrm{T}}.$$

Minimizing L with respect to K and using the condition $\partial L/\partial K = 0$, we obtain $\partial L/\partial K = B^\mathrm{T} Y C^\mathrm{T} = 0$, i.e., $C Y^\mathrm{T} B = 0$; hence, $\langle Y,\, BKC \rangle = \operatorname{tr} Y^\mathrm{T} BKC = \operatorname{tr} C Y^\mathrm{T} BK = 0$ and

$$L = \|H\|_1 + \langle Y,\, H - A \rangle.$$

The minimum of this expression with respect to H is finite if and only if $\|Y\|_\infty \le 1$ (because $\min_x (\|x\|_p + (x, c)) > -\infty$ if and only if $\|c\|_q \le 1$ for $1/p + 1/q = 1$). Therefore,

$$\psi(Y) \doteq \min_{K,H} L(H, K, Y) = -\langle Y,\, A \rangle, \qquad \psi(Y) > -\infty,$$

if and only if $\|Y\|_\infty \le 1$ and $C Y^\mathrm{T} B = 0$. By the duality theorem for convex programming, we have

$$\min_{H = A + BKC} \|H\|_1 \;=\; \max_{\psi(Y) > -\infty} \psi(Y),$$

and replacing Y with $-Y$ we arrive at the assertion of the theorem.

REFERENCES

1. Polyak, B.T. and Shcherbakov, P.S., Superstable Linear Control Systems. I. Analysis, *Avtom. Telemekh.*, 2002, no. 8, pp. 37–53.

2. Syrmos, V.L., Abdallah, C.T., Dorato, P., and Grigoriadis K., Static Output Feedback: A Survey, *Automatica*, 1997, vol. 33, no. 2, pp. 125–137.

3. Blondel, V., Sontag, E., Vidyasagar, M., and Willems, J., *Open Problems in Mathematical Systems and Control Theory*, London: Springer, 1999.

4. Blondel, V. and Tsitsiklis, J.N., A Survey of Computational Complexity Results in Systems and Control, *Automatica*, 2000, vol. 36, no. 9, pp. 1249–1274.

5. Polyak, B. and Halpern, M., Optimal Design for Discrete-Time Linear Systems via New Performance Index, *Int. J. Adaptive Control Signal Proc.*, 2001, vol. 15, no. 2, pp. 129–152.

6. Nemirovskii, A.A., Several NP-hard Problems Arising in Robust Stability Analysis, *Math. Control, Sign., Syst.*, 1994, vol. 6, pp. 99–105.

7. Polyak, B.T. and Shcherbakov, P.S., Numerical Search of Stable or Unstable Element in Matrix or Polynomial Families: A Unified Approach to Robustness Analysis and Stabilization, *Lect. Notes Control Inf. Sci.*, 1999, vol. 245, pp. 344–358.

8. Bhattacharyya, S., Chapellat, H., and Keel, L., *Robust Control: The Parametric Approach*, Upper Saddle River: Prentice Hall, 1995.

9. Padmanabhan, P. and Hollot, C.V., Complete Instability of a Box of Polynomials, *IEEE Trans. Autom. Control*, 1992, vol. 37, no. 8, pp. 1230–1233.

10. Dahleh, M. and Pearson, J., l_1-optimal Controllers for MIMO Discrete-Time Systems, *IEEE Trans. Autom. Control*, 1987, vol. 32, pp. 314–322.

11. Dahleh, M. and Diaz-Bobillo, I., *Control of Uncertain Systems: A Linear Programming Approach*, Englewood Cliffs: Prentice Hall, 1995.

12. Barabanov, A., *Sintez minimaksnykh regulyatorov* (Minimax Controller Design), St. Petersburg: S.-Peterburg. Gos. Univ., 1996.

13. Blanchini, F. and Sznaier, M., A Convex Optimization Approach for Fixed-Order Controller Design for Disturbance Rejection in SISO Systems, *IEEE Trans. Autom. Control*, 2000, vol. 45, pp. 784–789.

14. Vishnyakov, A.N. and Polyak, B.T., Low-Order Controller Design for Discrete-Time Systems Subjected to Nonrandom Disturbances, *Avtom. Telemekh.*, 2000, no. 9, pp. 112–119.

15. Kiselev, O.N. and Polyak, B.T., Overshoot Minimization in Linear Discrete-Time Systems Using Low-Order Controllers, *Avtom. Telemekh.*, 2001, no. 4, pp. 98–108.

16. Halpern, M. and Polyak, B.T., Optimization-based Design of Fixed-Order Controllers for Command Following, *Automatica*, 2002, vol. 38, no. 9, pp. 1615–1619.

17. Kwakernaak, H. and Sivan, R., *Linear Optimal Control Systems*, New York: Wiley-Intersciene, 1972. Translated under the title: *Lineinye optimal'nye sistemy upravleniya*, Moscow: Mir, 1977.

18. Yu, J. and Sideris, A., Optimal Induced l^1-norm State Feedback Control, *36th CDC*, San-Diego, 1997, pp. 1558–1563.

19. Ogata, K., *Modern Control Engineering*, Upper Saddle River: Prentice Hall, 1990.

20. Blondel, V., *Simultaneous Stabilization of Linear Systems*, London: Springer, 1995.

This paper was recommended for publication by E.S. Pyatnitskii, a member of the Editorial Board