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REVIEWS

Hard Problems in Linear Control Theory: Possible Approaches to Solution[1]

B. T. Polyak and P. S. Shcherbakov

Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia
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Abstract—A number of challenging problems in linear control theory are considered which admit simple formulation and yet lack efficient solution methods. These problems relate to the classical theory of linear systems as well as to the robust theory where the system description contains uncertainty. Various solution methods are discussed and the results of numerical simulations are given.

1. INTRODUCTION

The modern linear control theory is a well-developed branch of science having its own rigorous apparatus with a wide variety of applications and extensive problem area. The major techniques and results available by early eighties are collected in handbook [1] and the brilliant textbook [2]; earlier progress is summarized in [3], and the modern state-of-the-art is given in [4].

However, it would be unfair to conclude that linear control theory is close to completion; we mention [5–7] among the works concentrated on classification and formalization of open problems. The goal of the present survey is to sketch certain classes of problems in this subject area which admit transparent formulation and yet lack simple and efficient solution methods. The basic distinctive features that hinder systematic analysis and solution of these problems are nonconvexity and NP-hardness [6, 8]. It is these problems that we refer to as "hard" in the sense that efficient methods for their solution are not known and cannot be devised in principle. By efficient we mean a method which is guaranteed to yield a solution (provided it exists) with arbitrary high precision in "reasonable" time. As a result, the methods that we discuss here are either based on sufficient conditions or lean upon numerical procedures. This way or another, there is no guarantee to reach a solution even if it exists, or it turns out to be far from the optimal one, etc. At the same time, the ideas behind these approaches and the solution techniques are fruitful in a wide variety of problems, and in practice the methods demonstrate acceptable performance, which is illustrated below by numerous examples.

Needless to say, this survey does not pretend to be an encyclopaedic collection of all hard problems in linear control theory. First of all, in the exposition to follow, we opt to restrict ourselves to problems formulated in state space and in terms of matrices and polynomials. Moreover, certain conceptual identity is peculiar to all these problems as well as to the solution methods discussed in this paper. By way of future research, it would be of interest to extend the list of such problems.

2. HARD PROBLEMS

In this section we pose a number of hard problems and provide their rigorous mathematical formulations; we then demonstrate generic difficulties that emerge when solving these problems and consider specific particular cases where solutions can be found with no effort.

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2.1. Stabilization by Regulators of Fixed Structure

Let a linear SISO plant be specified by a transfer function of the form

$$G(s) = \frac{A(s)}{B(s)},$$

where

$$A(s) = a_0 + a_1 s + \cdots + a_m s^m, \quad B(s) = b_0 + b_1 s + \cdots + b_n s^n, \quad m \le n.$$

Of our interest is the following question: Can it be made stable by introducing a regulator of the form

$$C(s) = \frac{N(s)}{D(s)}$$

in the feedback loop (see Fig. 1), provided that the degrees of the polynomials in the numerator and denominator do not exceed given numbers? From the engineering viewpoint, such a requirement can be thought of as a constraint on the complexity of a regulator device that realizes the given transfer function.

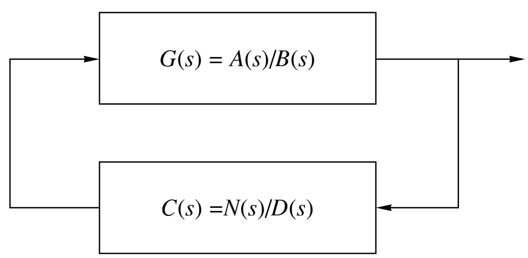

Fig. 1. A stabilizing regulator of fixed structure.

Mathematically, the problem formulates as follows: Determine if there exist polynomials $N(s)$, $D(s)$ of given degrees such that the characteristic polynomial

$$P(s) = A(s)N(s) + B(s)D(s)$$

of the closed-loop system is Hurwitz stable. If we omit the boundedness requirement for the regulator order, a complete solution can be obtained using Youla parametrization (e.g., see [4]) under additional assumption that $A(s)$ and $B(s)$ have no common unstable roots. On the other hand, a plant is always stabilizable by a scalar gain $C(s) = k$ provided that at least one of the polynomials $A(s), B(s)$ is stable. A fruitful solution technique for the two-parameter case, D-decomposition is described in Section 3.

Let us now consider the general case where $\ell > 2$ parameters are involved in the regulator description. After representing the polynomials $N(s), D(s)$ in the form

$$N(s) = q_1 + \cdots + q_k s^{k-1}, \qquad D(s) = 1 + q_{k+1} s + \cdots + q_\ell s^{\ell-k},$$

where k, ℓ are given, the problem reduces to checking if the affine polynomial family

$$\mathcal{P}(s,Q) = \left\{ P(s,q) = P_0(s) + \sum_{i=1}^{\ell} q_i P_i(s), \quad q \in \mathbb{R}^\ell \right\} \tag{1}$$

contains a stable polynomial. Here, we denoted

$$P_0(s) = B(s),$$
$$P_i(s) = s^{i-1}A(s), \;\; i = 1, \dots, k, \qquad P_i(s) = s^{i-k}B(s), \;\; i = k+1, \dots, \ell.$$

If the controller structure is specified in a different form (e.g., PID controller of the form $C(s) = q_1 + q_2/s + q_3 s$), the problem also reduces to the same model (1). Therefore, the issue of *stabilizability by fixed-structure regulators* can be analyzed within the following general framework.

Problem 1. Does affine family (1) contain at least one stable polynomial?

It should be noted that similar problems have been stated at the very dawn of the modern theory of automatic control; by that time, this research direction was referred to as "structural stability." M.A. Aizerman and F.R. Gantmakher [9, 10] were the first to obtain the rigorous pioneering results in the area; however, they relate to polynomial families that differ from those given by (1).

We might as well impose certain constraints on the controller parameters:

$$q \in Q. \tag{2}$$

For example, they can have interval form

$$Q \doteq \left\{ q \in \mathbb{R}^\ell : \;\; \underline{q}_i \le q_i \le \overline{q}_i, \;\; i = 1, \dots, \ell \right\}, \tag{3}$$

alternatively, these constraints can be formulated in terms of proximity of q to the nominal value q^0 in some norm:

$$Q \doteq \left\{ q \in \mathbb{R}^\ell : \;\; \|q - q^0\| \le \gamma \right\}. \tag{4}$$

In that case, we have the constrained version of Problem 1.

Problem 2. Does family (1), (2) contain at least one stable polynomial?

To the authors' knowledge, the first rigorous formulation of Problem 2 is due to [11], where the related results are also given; some sufficient conditions are presented in [12]. This problem still attracts considerable attention, e.g., see [13]; however, no essential progress has been achieved.

At the first glance, Problem 2 resembles very much the well-known problem of robust stability of polynomial families for which a number of efficient solution methods have been elaborated, e.g., see [4]. The only difference is that the robust stability setup deals with the existence of at least one *unstable* polynomial in family (1), (2), while in Problem 2 we are concerned with finding a *stable* polynomial in the same family. This seemingly unimportant difference in statements of the problem leads to a dramatic increase in complexity—Problem 2 turns out to be NP-hard [6, 8] even in its simplest formulations such as finding a stable polynomial in an interval family. As a result, this problem cannot be solved with a method having polynomial complexity, that is, the one in which the number of operations is expressed by a polynomial function of the degree of $P(s, q)$. Moreover, the Kharitonov Theorem, the Edge Theorem (see [4]) and the like are not valid for such problems. In other words, although all vertices and edges of the polyhedron Q may be unstable, there still might be a point $q \in Q$ associated with a stable polynomial. In summary, efficient methods producing exact solutions to generic Problems 1 and 2 are not known at present.

We also note that in the particular case where the q_is have the meaning of the coefficients of $P(s, q)$, Problem 2 can be written in the following form.

Problem 3. Given an unstable polynomial $P_0(s)$, find the nearest stable polynomial of the same degree.

Here, the distance between the polynomials $A(s) = a_0 + a_1 s + \cdots + a_n s^n$ and $B(s) = b_0 + b_1 s + \cdots + b_n s^n$ is to be understood as $\operatorname{dist}(A, B) = \|a - b\|$, where $a, b \in \mathbb{R}^{n+1}$ are the coefficient vectors of $A(s)$ and $B(s)$, and $\|\cdot\|$ is a vector norm in $\mathbb{R}^{n+1}$. Hence, if we obtain

$$\min_{P(s,q)\in\mathcal{P}_y} \operatorname{dist}\Big(P_0(s),\, P(s,q)\Big) = \gamma^* \le \gamma,$$

where $\mathcal{P}_y$ denotes the set of all stable polynomials, then Problem 2 with Q of the form (4) and $P_0(s) = q_0^0 + q_1^0 s + \cdots + q_n^0 s^n$, $P(s,q) = q_0 + q_1 s + \cdots + q_n s^n$ admits a solution; otherwise, there is no solution. Again, Problem 3 arises from the simple problem of finding the robustness margin for polynomial families by alternating the terms "stable" and "unstable," however, such a replacement leads to a sharp increase in complexity.

An explicit formulation of Problem 3 together with an approximate solution method have been first given in [14] for the discrete-time case.

2.2. Static Output Stabilization

In the previous subsection our attention was focused on SISO systems specified by scalar transfer functions and mathematically, the problem was to find a stable element in a family of polynomials. A range of similar hard matrix problems arise when the state space description is considered. Typical is the so-called *static output feedback* design problem: Find out if the system

$$\dot{x} = Ax + Bu, \qquad y = Cx \tag{5}$$

can be stabilized by a feedback of the form

$$u = Ky. \tag{6}$$

A complete solution of this important problem is not known, although in a number of particular cases it can be analyzed in detail. An overview of the results in this subject area is given in [15]; for the most recent progress see [16]. Generic NP-hardness of the problem was proved in [8]. Note that static state feedback stabilization (i.e., finding a stabilizing controller in the form $u = Kx$) is much simpler. This problem is also known as the pole placement problem, since the roots of the characteristic polynomial can be assigned any prescribed location. A sufficient condition for a solution to exist is that the system be controllable; the solution itself is easy to find.

The problem of pole placement by static output feedback has been the subject of intensive study since 1975 [17, 18], also see [19–21]; the most recent results are collected in [22]. It is clear that certain relationship between the dimensions of the state, n, controls, m and output, p, need to be satisfied in order that the problem be solvable not only for the given matrices A, B, C, but also for their small perturbations. In [19], the generalized pole placement problem was shown to have a solution for $n < mp$; the case $n = mp$ was analyzed in [22].

The mathematical formulation of problem (5), (6) is the following: Does there exist a matrix K such that $A_c \doteq A + BKC$ is stable for the given A, B, C? Treating the entries of the matrix K as parameters q, we conclude that the static output feedback stabilization problem is a special case of a more general matrix problem below.

Problem 4. Consider the affine matrix family

$$A(q) = A_0 + \sum_{i=1}^{\ell} q_i A_i, \qquad q \in \mathbb{R}^{\ell}, \tag{7}$$

where $A_0, \ldots, A_\ell$ are given $n \times n$ matrices. Does the family contain at least one stable matrix?

This formulation covers various problems of stabilization by means of fixed-structure controllers, different from the static output control problem. Similar problems for discrete-time systems also fall into this framework with the only modification that the notion of stable matrix is to be understood as Schur stable.

Theoretical solution of Problem 4 is not known, and it is by no means simple, since Problem 1 for polynomials is a special case of the matrix Problem 4 (this becomes clear by considering companion form matrices A_i). The following matrix versions of Problems 2 and 3 are also immediate.

Problem 5. Determine if there is at least one stable matrix in the family $A(q)$ (7), $q \in Q$ (e.g., with Q having form (3)).

Problem 6. Given an unstable matrix A, find the nearest stable one.

All these problems admit simple solutions in a number of particular cases. For instance, let all $A_0, \dots, A_\ell$ be symmetric; then all matrices $A(q)$ are also symmetric. Stability of a symmetric matrix is equivalent to negative definiteness, hence, Problem 4 reduces to the following linear matrix inequality: Find $q \in \mathbb{R}^\ell$ such that the condition

$$A_0 + \sum_{i=1}^{\ell} q_i A_i < 0$$

holds. For this convex problem, well-developed solution tools based on the modern techniques of linear matrix inequalities (LMI) are available (see [23]), for instance, the LMI Control Toolbox in Matlab. Thus, Problem 4 and the related Problem 5 admit efficient solutions as far as symmetric matrices are concerned.

Problem 6 is also solvable in closed form in the class of symmetric matrices. Namely, let A be a symmetric matrix and we are interested in finding its distance to the set of stable symmetric matrices. We diagonalize A by a nonsingular linear transformation T (this is always doable for symmetric matrices) to obtain $TAT^{-1} = \Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, where λ_i are the eigenvalues of A, which are all real. We next compose a matrix Λ_- by replacing positive eigenvalues with zeros:

$$\Lambda_- = \operatorname{diag}(\lambda_1^-, \dots, \lambda_n^-), \qquad \lambda_i^- = \min\{0,\ \lambda_i\}.$$

Then the matrix $A_- \doteq T^{-1}\Lambda_- T$ is the nearest (in the Frobenius norm) to A in the class of symmetric negative-semidefinite matrices, and the distance between A and the set of stable symmetric matrices is given by $\gamma^* = \|A - A_-\|_F$. In other words, by restricting our analysis to the symmetric matrix case, we gain considerable simplification.

In view of the matrix Problems 5 and 6, we mention the problems of robust stability for matrices.

Problem 5′. Determine if all the matrices $A(q)$ (7), $q \in Q$, are stable (e.g., with Q of the form (3)).

Problem 6′. Given a stable matrix A, find the nearest unstable one.

The latter problem relates to finding the robust stability radius for a matrix family.

It should be mentioned that analogous robust stability problems for polynomials are well studied and admit efficient solution; also see Subsection 3.2 below. In particular, the celebrated Kharitonov Theorem [24] reduces the robust stability problem for interval polynomial families to checking the stability of the four distinguished polynomials; the Tsypkin–Polyak plot [25] computes the distance to the nearest unstable polynomial in various norms; the Edge Theorem [26] provides in principle a solution to the robust stability problem for affine polynomial families of the form (1), (3), etc.

The monographs [4, 27–29] survey the results in this area of research. However, the matrix and the polynomial settings are fundamentally different. It was shown that neither the Kharitonov Theorem, nor the Edge Theorem admit extensions to the matrix case. Furthermore, Problems 5′ and 6′ were shown to be NP-hard in the interval matrix norm [6, 8]. At the same time, in the spectral norm, Problem 6′ is solvable, see [30] for the complex matrix case and [31] for the real case; these results are also given in [4, Ch. 7].

2.3. Simultaneous and Robust Stabilization

The problem of *simultaneous stabilization* arises in many applications. For example, assume that a plant operates under several conditions and switches from one operating mode to another. There is no information available about switching times and this process is beyond control of engineers; for instance, these switchings may be caused by a failure of a certain unit of the plant. The control objective is to construct a regulator which ensures proper functioning of the system (its stability, in the first place) under any of the operating conditions. Such a formulation of this problem has presumably been first given in [32, Ch. 9, p. 540]. In a formal statement, the problem writes as follows.

Problem 7. Given m SISO plants with transfer functions

$$G_i(s) = \frac{A_i(s)}{B_i(s)}, \qquad i = 1, \dots, m,$$

find out if there is a controller

$$C(s) = \frac{N(s)}{D(s)}$$

which simultaneously stabilizes all these plants.

Said another way, can polynomials $N(s), D(s)$ be found such that all polynomials of the form

$$P_i(s) = A_i(s)N(s) + B_i(s)D(s), \qquad i = 1, \dots, m,$$

are Hurwitz? For $m = 1$, it is well known that with A_1, B_1 having no common unstable roots there is always a solution; moreover, using Youla parametrization, all stabilizing controllers can be characterized [33–37].

For the case of two plants, a solution can also be found by reducing the problem to stabilization of a single plant by a stable controller. The latter problem in turn, admits a complete solution in terms of interlacing zeros and poles of the plant. A more general MIMO problem of simultaneous stabilization of two plants can also be analyzed in detail. However, there are no solution methods available for the case of $m = 3$ plants; moreover, certain indirect evidence testify to the absence of a “simple” method [38].

The simultaneous stabilization problem often arises in the matrix formulation.

Problem 8. Given m state space linear systems

$$\dot{x} = A_i x + B_i u, \qquad i = 1, \dots, m, \tag{8}$$

determine if there is a common state feedback regulator

$$u = Kx \tag{9}$$

that stabilizes all these systems.

To put it differently, does there exist a matrix K such that all matrices

$$A_c^i \doteq A_i + B_i K, \qquad i = 1, \ldots, m,$$

are stable? For discrete-time systems the formulation of this problem is completely analogous; the only difference is that stability is understood in the discrete-time sense, i.e., the matrices A_c^i are required to be Schur stable. The general solution method for Problem 8 is not known; however, it is by no means simple as well, since its particular case, Problem 7 is hard.

We also note that the simultaneous stabilization problem is a special case of the *robust stabilization problem* below.

Problem 7′. For a family of polynomials $P(s, q, k)$ which depend on the parameters $q \in Q$ and the controller coefficients k, determine if there is a k^* such that all polynomials $P(s, q, k^*),\ q \in Q$, are stable.

Simultaneous stabilization corresponds to the case where the set Q is finite and no advance constraints are imposed on the dimension of the coefficient vector k.

Similarly to the polynomial problems, robust *matrix* stabilization relates to matrix families.

Problem 8′. Is there a matrix K such that all matrices of the form $A(q) + B(q)K,\ q \in Q$, are stable?

Problem 8 corresponds to the case of finite set Q.

2.4. Optimal Control

So far we discussed problems related to stabilization. Suppose now that stabilization is already accomplished, then it would be natural to impose requirements on the system performance; this leads to problems of *optimal control*.

We first consider the so-called *linear quadratic optimization problem* (linear quadratic regulator problem, LQR), which is also known as the *analytic controller design problem*. Specifically, given the system without exogenous disturbance

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0, \tag{10}$$

find a controller in the form of linear state feedback $u = Kx$ that stabilizes the system and minimizes the quadratic performance index

$$J \doteq \int_0^\infty \Big[(Rx, x) + (Su, u)\Big] dt, \tag{11}$$

where R, S are some positive-definite matrices (so-called matrix weighting coefficients). It is supposed here that the system description is known precisely and the state $x(t)$ is available at any time instant; a possible way for solving this problem is suggested in Section 4. Let us discuss various relaxations of the assumptions above.

First of all, let the system's output $y = Cx$ be available rather than the state; this problem is known as H_2-*optimization*. Then, solution can be found by using dynamic feedback of the form $u = K\widehat{x}$, where $\widehat{x}$ is an estimate of the state vector x obtained with the use of observer. Suppose however that the structure of the controller is given, e.g., we are restricted to static output feedback regulators of the form

$$u = Ky. \tag{12}$$

In other words, a matrix K in (12) is to be found such that it stabilizes the system and provides the minimum value to the cost function (11). We know that even the existence of a stabilizing K

is hard to decide, see Problem 4. Suppose however that a stabilizing controller is already found; alternatively, some sufficient conditions are met which guarantee the existence of stabilizing controllers of the form (12). We then have to choose a regulator which is optimal with respect to the criterion functional (11).

Problem 9. Find a regulator $u = Ky$ which stabilizes the system $\dot{x} = Ax + Bu$, $y = Cx$, and minimizes functional (11).

If the regulator defined by K is stabilizing, the value of $J(K)$ is expressed via a solution of a matrix Lyapunov equation involving the K matrix, and the problem might be solved by direct optimization of $J(K)$. However, the functional $J(K)$ is nonconvex in K and optimization may terminate at a local (rather than the global) minimum. The set of all stabilizing controllers is also noncovex, i.e., the domain of definition of the functional $J(K)$ is nonconvex.

Problem 9 is the optimal control problem for static output regulators. Similar problems arise when solving the LQR problem for fixed-structure controllers having different forms. For example, consider a scalar system $A(s)x = u$ with stable polynomial $A(s)$ of degree n and let the controller be chosen in the form

$$u = k_1 x + k_2 \dot{x}.$$

Then the closed-loop system is certainly stable for small values of k_1, k_2, and the quadratic functional

$$J \doteq \int_0^\infty \sum_{i=0}^{n-1} \alpha_i \left(x^{(i)}\right)^2 dt$$

can be expressed in terms of the coefficients of the polynomial $A(s)$ and the controller coefficients $k \doteq (k_1,\ k_2)$. This construction has been first proposed in the works by A.A. Krasovskii and A.A. Fel'dbaum in the mid-twentieth century, e.g., see [39, Ch. 20]. Again, in such a formulation, both the dependence $J(k)$ and the domain of definition of $J(k)$ are generally speaking nonconvex, which makes minimization with respect to k rather difficult. These difficulties are avoidable in the case when $k \in \mathbb{R}^2$ (e.g., as in problem above), but as the dimension of k grows, the problem becomes intractable.

The following optimal control problem with linear performance index is conceptually close to the linear quadratic problem (10), (11).

Problem 10. Find a linear state feedback $u = Kx$ that stabilizes system (10) and minimizes the linear functional

$$J = \int_0^\infty \Big(\|x\|_\infty + \alpha \|u\|_\infty \Big) dt. \tag{13}$$

Here, $\|x\|_\infty = \max\limits_{1 \le i \le n} |x_i|$ denotes the ∞-norm of the finite-dimensional vector $x \in \mathbb{R}^n$. This problem would be natural to refer to as the *linear linear regulator problem*; it arises in certain topics of l_1-optimization theory and in engineering applications. In the simplest settings, performance indices of this sort were the subject of study at the earliest stage of the automatic regulation theory in the works by V.S. Kulebakin, see [39, Ch. 20]. Systematic methods for solving this problem are not known; a possible approach is presented in Section 5.

Suppose next that the system description in the linear quadratic problem (10), (11) contains uncertainty:

$$\dot{x} = A(q)x + B(q)u, \qquad q \in Q,$$

where the set Q is specified by (3) or (4). Then the problem takes the following form.

Problem 11. Find a robustly stabilizing controller $u = Kx$ that ensures a given value μ of the quadratic performance index (11) for all $q \in Q$.

This is the problem of robust linear quadratic regulator, also known as the *guaranteed quadratic performance index* problem; it will be analyzed in Section 4. In the particular case where the set Q is finite, we arrive at the optimal simultaneous control problem, which is a generalization of Problem 8.

Let us now turn to situations where the system is subjected to exogenous disturbances. We consider a continuous-time MIMO system specified in state space:

$$\begin{aligned} \dot{x} &= Ax + Bu + D_1 w, \qquad && x(0) = x_0, \\ y &= Cx + D_2 w, && \|w(t)\| \le 1, \quad t \ge 0. \end{aligned} \tag{14}$$

Here, $x \in \mathbb{R}^n$ stands for the state, $y \in \mathbb{R}^l$ is the output, $u \in \mathbb{R}^m$ is the control input, and $w \in \mathbb{R}^{m_1}$ is the exogenous disturbance which is only supposed to be bounded at any time instant. The problem is formulated as follows.

Problem 12. Find a static output regulator $u = Ky$ which stabilizes system (14) and minimizes the performance index

$$J = \sup_{\|w\| \le 1} \sup_t \|x(t)\|.$$

This problem is referred to as *optimal rejection of bounded disturbances*; in a more general setting, certain linear functions of the state $x(t)$, so-called controlled input $z(t)$, can be used in the functional above rather than the state itself. In discrete time, such kind of problems are the subject of the so-called l_1-optimization theory [40–42]. They are extremely hard even for scalar systems, the main difficulty being a very high order of the resulting optimal controller. Quite often, this order turns out to be much higher than the order of the plant; moreover, it cannot be evaluated in advance. For continuous-time systems (L_1-optimization theory) the situation is even worse—the optimal controller $u = C(s)y$ may turn out to be infinite-dimensional; i.e., the optimal transfer function $C(s)$ is not rational fractional. Satisfactory solutions are available only in rare special cases. A general approach to rejection of bounded disturbances is proposed in Section 5.

3. METHODS BASED ON D-DECOMPOSITION

In the rest of the paper we describe possible approaches to solution of the problems formulated in the previous section.

3.1. D-decomposition

Let us consider Problem 1. In the simplest situation where the affine family depends on a single parameter, the problem can be solved with the *root locus* method [43, 44] by analyzing graphically the root behavior for the one-parameter family $P(s) = P_0(s) + kP_1(s)$. This case corresponds to stabilization by means of *proportional regulators* (P regulators) of the form $C(s) = k$; however, their potential is limited, and stabilization is enabled only in exceptional situations. For instance, this is the case for stable plants (the polynomial $B(s)$ is stable) or minimum-phase plants (the polynomial $A(s)$ is stable).

A more complicated *proportional integral regulator* (PI regulator) depends on two parameters:

$$C(s) = k_1 + k_2/s; \tag{15}$$

sometimes other similar representations for PI regulators are considered, e.g.,

$$C(s) = \frac{k_1}{s + k_2}, \qquad C(s) = \frac{k_1}{1 + k_2 s},$$

etc. All of them depend on the two parameters $k_1,\ k_2$ so that the characteristic polynomial of the closed-loop system also depends affinely on these two parameters:

$$P(s,k) = P_0(s) + k_1 P_1(s) + k_2 P_2(s), \qquad k = (k_1;\ k_2); \tag{16}$$

e.g., for the PI regulator (15), the three polynomials above are given by $P_0(s) = sB(s),\ P_1(s) = sA(s),\ P_2(s) = A(s)$. For the characteristic polynomial of the form (16), specific domains on the two-dimensional plane $\{k_1,\ k_2\}$ can be indicated such that $P(s,k)$ has a fixed number of left (respectively, right) half-plane roots for each of these domains. In particular, the domain associated with all left half-plane roots corresponds to stable systems. Such an approach is known as *D-decomposition* of the plane of parameters; it is based on the following idea.

Assume that for some fixed k the polynomial $P(s,k)$ of degree n has exactly $m \le n$ roots in the left half-plane and $n - m$ roots in the right half-plane. Clearly, as k varies, the root location may alter only in the following three cases:

(a) the degree of the polynomial $P(s,k)$ changes;

(b) a real root of $P(s,k)$ moves from one half-plane to another, i.e., it becomes zero;

(c) a pair of complex conjugate roots moves from one half-plane to another, i.e., the polynomial $P(s,k)$ acquires a pair of pure imaginary roots $\pm j\omega$.

Hence, the boundaries of the domains of D-decomposition are defined by the parametric equation

$$P(j\omega, k) = 0 \tag{17}$$

in cases (b) and (c), and by the equation

$$a_n(k) = 0 \tag{18}$$

in case (a), where $a_n(k)$ is the leading coefficient of $P(s,k)$. For any fixed ω, Eq. (17) splits into two linear equations (associated with the real and imaginary parts of $P(j\omega,k)$) in the two variables k_1, k_2. Generically, its solution defines a point $k(\omega)$ on the parameter plane which sweeps certain curve as ω varies from 0 to ∞. Furthermore, in the degenerate case, the equations in (17) are linearly dependent for some isolated values of ω; each of these values defines a straight line on the plane of parameters, which is referred to as a singular line. Finally, condition (18) also defines a line.

The overall procedure of D-decomposition is as follows. We plot the curve $k(\omega)$ (17), the line defined by (18), and singular lines; altogether they partition the k-plane into a number of simply connected regions. Each of these regions corresponds to a specific location of the roots of $P(s,k)$. One or more regions constitute the stability domain of $P(s,k)$; however, it may happen to be empty, which means that the characteristic polynomial is unstable for all values of k. "Decoding" the root location picture for the domains of D-decomposition can be executed in several ways. For instance, we can start with an arbitrary polynomial $P(s,k^0)$ and compute the number of its left and right half-plane roots. We then move from within the region associated with $P(s,k^0)$ towards the neighboring regions, keeping in mind that crossing the curve $k(\omega)$ corresponds to the situation where a pair of roots crosses the imaginary axis, while crossing the singular lines corresponds to the transition of one root via the origin. Alternatively, we can take a point inside each region and calculate the roots of the related polynomials; then, the root location is retained for all polynomials in the same region.

Example 1. The goal is to stabilize the second-order plant

$$G(s) = \frac{s-1}{s^2+1}$$

by a PI regulator of the form

$$C(s) = k_1 + \frac{k_2}{s}.$$

The plant is neither stable nor minimum-phase, and it cannot be stabilized by P regulators. The closed-loop characteristic polynomial writes

$$P(s,k) = s(s^2+1) + (s-1)(k_1 s + k_2) = s^3 + k_1 s^2 + (1 - k_1 + k_2)s - k_2;$$

its leading coefficient does not depend on k, hence, the boundary of D-decomposition does not contain line (18). The equality $P(j\omega, k) = 0$ takes the form

$$\begin{aligned} -k_2 - k_1\omega^2 &= 0, \\ \omega(1 - k_1 + k_2 - \omega^2) &= 0. \end{aligned}$$

For $\omega = 0$, we have the singular line

$$k_2 = 0.$$

For any $\omega \neq 0$, the point $k(\omega)$ is defined uniquely. Note that the explicit dependence on ω need not be found; instead, by eliminating $\omega^2 = 1 - k_1 + k_2$ from the second equation and substituting into the first one, the equation of the curve appears in closed form, rather than parametrically:

$$k_2 + k_1(1 - k_1 + k_2) = 0, \qquad k_2 = \frac{k_1^2 - k_1}{1 + k_1}.$$

This equation defines a hyperbola, and the condition $k_2/k_1 = -\omega^2 < 0$ distinguishes the part of it which belongs to the II and IV quadrants. This curve, together with the line $k_2 = 0$ perform the D-decomposition of the k-plane into the four regions depicted in Fig. 2. The symbols D_0 through D_3 characterize the number of the left half-plane roots for polynomials having the parameters k_1, k_2 inside the corresponding regions. The small region D_3 corresponds to stable polynomials. Any point inside D_3 is associated with the coefficients of a PI controller that makes the closed-loop system stable.

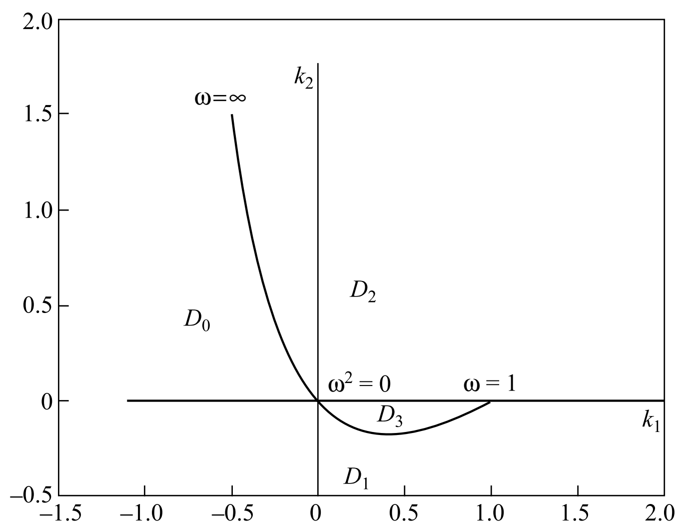

Fig. 2. D-decomposition in Example 1.

For the particular case of third-order polynomials, the D-decomposition concept has been first proposed by I.A. Vyshnegradskii as early as in 1876; today it is known as the "Vyshnegrasdskii diagram," see [45]; in full generality the D-decomposition technique was developed in [46]. These methods have gained wide reputation due to simplicity and ease of application, although they are less popular in the West, with the monographs [28, 47, 48] being a few exceptions. Recently, the structure of D-decomposition was shown to have quite a complicated geometry. For instance, an example demonstrating that the stability domain may consist of $n-1$ simply connected components is given in [49]; the total number of regions in D-decomposition for the case of one and two parameters is evaluated in [50], where the related examples are also presented. For one of such specific examples, the D-decomposition contains $n^2 - 2n + 2$ regions.

It is noted that the D-decomposition methods admit a straightforward generalization to robust stabilization by means of PI regulators, i.e., Problem $7'$ can be solved along these lines, see [51].

3.2. The Zero Exclusion Principle and Frequency-Domain Methods

In this subsection we consider a frequency-domain approach to the robustness analysis for families of polynomials, which is conceptually similar to D-decomposition.

When $\ell > 2$ uncertain parameters enter the description of an uncertain system, complete characterization of the D_i domains by means of D-decomposition cannot be implemented in practice. However, two-dimensional cross-sections of these domains with respect to any pair of the parameters can be obtained and the distance between a point in the parameter space and the boundary of the domain (in particular, the stability domain) can be evaluated. An approach of this subsection leans on the concept of *value set* for an uncertain polynomial, which facilitates use of the frequency-domain description instead of the original parameter space description. Then the so-called *zero exclusion principle* can often be exploited to distinguish explicitly a sub-domain of a regular shape (a ball in some norm) inside each of the D_is such that it hits the boundary of the D-decomposition. We first formulate the zero exclusion principle.

Theorem 1. *Let us consider the following family of polynomials*:

$$\mathcal{P}(s,Q) \doteq \Big\{P(s,q) = a_0(q) + a_1(q)s + \cdots + a_n(q)s^n, \quad q \in Q\Big\}, \tag{19}$$

whose coefficients $a_i(q)$ depend continuously on the parameters $q \in \mathbb{R}^\ell$ which belong to the pathwise connected feasible set $Q \subset \mathbb{R}^\ell$. Assume that there is no degree dropping, i.e., all polynomials in (19) *have the same degree*: $a_n(q) \neq 0$ *for all $q \in Q$. Finally, assume that there exists a $q^0 \in Q$ such that the polynomial $P(s,q^0)$ has exactly m stable roots. Then, in order that all polynomials in family* (19) *possess this property, it is necessary and sufficient that the following condition hold:*

$$0 \notin \mathcal{S}(\omega) \doteq \Big\{P(j\omega,q)\colon\ q \in Q\Big\}, \qquad \forall\, 0 \le \omega < \infty. \tag{20}$$

The domain $\mathcal{S}(\omega)$ on the complex plane is referred to as the *value set* of the polynomial family (19). Clearly, it is the two-dimensional image of the set Q under the mapping $P(j\omega,\cdot)$. The theorem requires that the origin be excluded from the value set for all ω. Similarly to the D-decomposition approach, the idea behind this technique is that the number of stable roots may only alter (as the parameter q varies in Q) if one of them crosses the imaginary axis, and condition (20) violates. In contrast to D-decomposition, the intention here is not to describe the whole domain D_m, but rather determine if the given set Q lies entirely inside D_m. With this approach, large number of parameters can be dealt with.

The zero exclusion principle has been first formulated and applied to the stability analysis of systems in [52]; a detailed study of the potential of this technique is performed in [27].

In order to efficiently exploit Theorem 1, the sets $\mathcal{S}(\omega)$ are to be constructed and condition (20) is then should be checked. Both goals can be attained in a number of important particular cases; this is illustrated below via the example of an nth degree interval polynomial specified in the form

$$\mathcal{P}(s) = \Big\{P(s) = a_0 + a_1 s + \cdots + a_n s^n, \quad |a_i - a_i^0| \le \gamma\alpha_i, \quad i = 0, 1, \ldots, n\Big\}. \tag{21}$$

We assume that $\alpha_0, \alpha_1 > 0$ and introduce the following notation:

$$P_0(j\omega) \doteq U_0(\omega) + j\omega V_0(\omega);$$
$$U_0(\omega) \doteq a_0^0 - a_2^0\omega^2 + a_4^0\omega^4 - \ldots, \qquad V_0(\omega) \doteq a_1^0 - a_3^0\omega^2 + a_5^0\omega^4 - \ldots;$$
$$R(\omega) \doteq \alpha_0 + \alpha_2\omega^2 + \alpha_4\omega^4 + \ldots, \qquad T(\omega) \doteq \alpha_1 + \alpha_3\omega^2 + \alpha_5\omega^4 + \ldots.$$

We next assume that the nominal polynomial $P_0(s)$ has exactly m stable roots and introduce the *Tsypkin–Polyak plot*

$$\begin{gathered} z(\omega) = x(\omega) + jy(\omega), \qquad 0 \le \omega \le \infty, \\ x(\omega) = \frac{U_0(\omega)}{R(\omega)}, \qquad y(\omega) = \frac{V_0(\omega)}{T(\omega)}. \end{gathered} \tag{22}$$

Theorem 2. *All polynomials in family* (21) *have the same number of stable roots if and only if*

$$a_0^0 > \gamma\alpha_0, \qquad a_n^0 > \gamma\alpha_n, \tag{23}$$

and the plot $z(\omega)$ *does not intersect the square with vertices* $(\pm\gamma, \pm\gamma)$ *as* ω *varies from* 0 *to* ∞.

Theorem 2 is nothing but the zero exclusion principle for family (21) because its value set is the rectangle

$$\mathcal{S}(\omega) = \Big\{z = x + jy\colon\ |x - U_0(\omega)| \le \gamma R(\omega),\ \ |y - \omega V_0(\omega)| \le \gamma\omega T(\omega)\Big\},$$

and the condition $0 \in \mathcal{S}(\omega)$ is equivalent to the requirement $|U_0(\omega)| \le \gamma R(\omega),\ |V_0(\omega)| \le \gamma T(\omega)$, i.e., $|x(\omega)| \le \gamma,\ |y(\omega)| \le \gamma$. Hence, the plot $z(\omega)$ does not intersect the square $\{z = x + jy : |x| \le \gamma,\ |y| \le \gamma\}$ if and only if $0 \notin \mathcal{S}(\omega)$. Condition (23) on the initial ($\omega = 0$) and the terminal ($\omega = \infty$) points of the plot (their projections must not fall inside the γ-square) precludes any root from crossing the imaginary axis at the origin or at infinity.

Therefore, under the conditions of the theorem, the rectangle

$$Q_\gamma = \Big\{a \in \mathbb{R}^{n+1}\colon\ |a_i - a_i^0| \le \gamma\alpha_i, \quad i = 0, 1, \ldots, n\Big\}$$

lies entirely in the domain D_m associated with the polynomial $P_0(s)$. In particular, if $P_0(s)$ is stable, the above rectangle lies in the stability domain D_n. It is easily verified that the plot $z(\omega)$ makes n rotations around the origin in the counter-clockwise direction. This is in agreement with the Mikhailov criterion, since scaling by $R(\omega)$ and $T(\omega)$ retains the direction of rotation and the number of turns. It is therefore appropriate to consider the theorem above as a robust modification of the classical Mikhailov test. Another salient feature of the approach is that a single curve is only need to be plotted for the whole family.

The plot (22) differs substantially from the Mikhailov plot in that it begins at a point inside the first quadrant (rather than at the real axis) and ends at a finite point $z(\infty)$ without going to infinity. Moreover, using the plot $z(\omega)$ not only can we check for robust stability for a fixed $\gamma > 0$, but also determine the maximal value of γ retaining stability. This *robustness radius* $\gamma_{\max}$ can be found from the formula

$$\gamma_{\max} = \min\{\gamma^*,\, \gamma_0,\, \gamma_\infty\}, \tag{24}$$

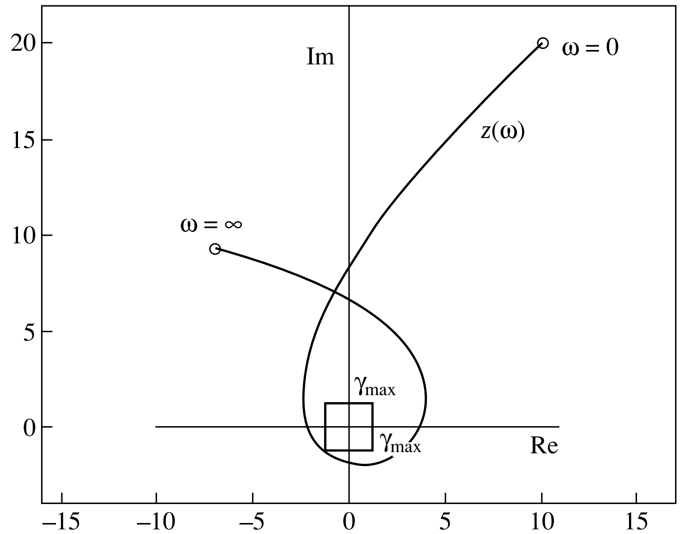

Fig. 3. The Tsypkin–Polyak plot.

where γ^* is the "size" of the largest square $\{|x| \le \gamma^*,\ |y| \le \gamma^*\}$ "inscribed" into the plot $z(\omega)$, $\gamma_0 = a_0^0/\alpha_0$, and $\gamma_\infty = a_n^0/\alpha_n$. The associated set $Q_{\gamma_{\max}}$ contacts the boundary of the stability domain in the parameter space. Figure 3 depicts a typical shape of the plot $z(\omega)$ for a robustly stable family of sixth degree together with the maximal possible square inscribed in it.

Theorem 2 was proved in [53]; it generalizes to other ℓ^p-norm constraints on the uncertain coefficients, discrete-time stability, other root location regions (the so-called *D-stability*), etc.; for more details also see [4, 25].

4. QUADRATIC STABILIZATION AND LINEAR MATRIX INEQUALITIES

The approach described in this section is based on sufficient conditions of stability, namely, on the existence of a quadratic Lyapunov function for the system or a common function for the family of systems; it is therefore referred to as the quadratic stability (quadratic stabilization) approach. The solution of this problem involves use of linear matrix inequalities techniques [23].

4.1. Quadratic Stability

We begin with analysis problems. It is well known that the existence of a positive-definite solution $P > 0$ of the linear matrix inequality

$$A^{\mathrm{T}}P + PA < 0$$

is equivalent to the stability of A, with $V(x) = x^{\mathrm{T}}Px$ being a Lyapunov function for the system $\dot{x} = Ax$. Let us now consider the robust stability problem (Problem 5′). We require that the family of systems $\dot{x} = A(q)x,\ q \in Q$, possess a *common* quadratic Lyapunov function $V(x) = x^{\mathrm{T}}Px$, $P > 0$, or differently, that the *system* of linear matrix inequalities

$$A^{\mathrm{T}}(q)P + PA(q) < 0, \qquad P > 0, \quad q \in Q, \tag{25}$$

be feasible. Then the robust stability of the matrix family $A(q),\ q \in Q$, is guaranteed; in this case it is said to be *quadratically stable*. A systematic study of such a concept was originated in the works by B. Barmish and J. Leitmann (e.g., see [54]); later, it was developed in the works on linear matrix inequalities [23]. We stress that the condition above is only sufficient for robust stability—although the system of linear matrix inequalities (25) may be infeasible, this does not mean the absence of robust stability, i.e., there exist robustly stable families having no common quadratic Lyapunov functions.

When $A(q)$ is linear in q and Q is a polyhedron, inequalities (25) are to be solved only for the vertices of the set Q (we denote them by $\mathcal{V}$). Indeed, since any $q \in Q$ can be represented as a convex combination of the vertices, the satisfaction of condition (25) for the vertices implies its satisfaction (with the same P) for any other $q \in Q$. Said another way, it is sufficient to solve a finite number of the matrix inequalities

$$A^{\mathrm{T}}(q^v)P + PA(q^v) < 0, \qquad P > 0, \quad q^v \in \mathcal{V}, \tag{26}$$

where $A(q^v)$ are the *vertex matrices*. In the case of interval family

$$A = (a_{ij}), \qquad \underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}, \quad i, j = 1, \ldots, n, \tag{27}$$

these are the matrices whose elements a_{ij} take extreme values ($\underline{a}_{ij}$ or $\overline{a}_{ij}$), and if an affine family

$$A(q) = A_0 + \sum_{i=1}^{\ell} q_i A_i, \quad |q_i| \le \gamma, \quad i = 1, \ldots, \ell, \tag{28}$$

is concerned, then its vertex matrices are those with $|q_i| = \gamma,\ i = 1, \ldots, \ell$. There exist powerful numerical solvers for finite systems of linear matrix inequalities, some of them are represented in the LMI Control Toolbox in Matlab. However, the number of inequalities in (26) may turn out to be quite large even if n and ℓ are relatively small. Thus, for an $n \times n$ interval matrix family, the number of inequalities to be solved is equal to 2^{n^2} so that with n as low as $n = 5$, implementation of these methods faces serious computational difficulties. In such situations, simpler iterative numerical methods can be applied which process one randomly generated inequality at each iteration step; moreover, these methods can work with infinite number of inequalities. Such a technique devised in [55, 56] has been proved to be quite efficient.

4.2. Quadratic Stabilization

We now proceed to controller design problems and apply the same approach based on quadratic Lyapunov functions. We consider the system

$$\dot{x} = Ax + Bu \tag{29}$$

and seek a state feedback

$$u = Kx. \tag{30}$$

The closed-loop system takes the form

$$\dot{x} = A_c x, \qquad A_c = A + BK,$$

so that $V(x) = x^{\mathrm{T}}Px$ is a Lyapunov function for the system if and only if

$$A_c^{\mathrm{T}}P + PA_c < 0,$$

i.e., if there exist K and $P > 0$ such that

$$(A + BK)^{\mathrm{T}}P + P(A + BK) < 0. \tag{31}$$

The two matrix variables P and K enter this inequality in a nonlinear way. Changing the variables

$$X \doteq P^{-1}, \qquad Y \doteq KP^{-1},$$

we premultiply and postmultiply (31) by X to arrive at the inequality

$$XA^{\mathrm{T}} + AX + Y^{\mathrm{T}}B^{\mathrm{T}} + BY < 0, \qquad X > 0, \tag{32}$$

which is seen to be linear in the new variables Y, X. Now, the matrix Y can be eliminated from (32). Indeed, the quadratic form

$$f(x) \doteq x^{\mathrm{T}}(Y^{\mathrm{T}}B^{\mathrm{T}} + BY)x = (B^{\mathrm{T}}x, Yx)$$

vanishes on the subspace $B^{\mathrm{T}}x = 0$. By Finsler's lemma, there exists $\gamma > 0$ such that $f(x) + \gamma\|B^{\mathrm{T}}x\|^2 \geq 0$ for all x, i.e.,

$$Y^{\mathrm{T}}B^{\mathrm{T}} + BY \geq -\gamma BB^{\mathrm{T}}. \tag{33}$$

Hence, the satisfaction of inequality (32) implies

$$XA^{\mathrm{T}} + AX - \gamma BB^{\mathrm{T}} < 0. \tag{34}$$

Conversely, solving the linear matrix inequality (34) and letting $Y \doteq -\frac{\gamma}{2}B^{\mathrm{T}}$ we obtain equality in (33), and thereby inequality (32) is satisfied. Since the product $K = YX^{-1}$ remains unchanged under the choice of γ, we take $\gamma = 2$ and arrive at the following result.

Theorem 3. *Let X be a solution of the matrix Lyapunov inequality*

$$XA^{\mathrm{T}} + AX - 2BB^{\mathrm{T}} < 0, \qquad X > 0. \tag{35}$$

Then controller (30) *with matrix*

$$K = -B^{\mathrm{T}}X^{-1}$$

stabilizes system (29), *and the quadratic form*

$$V(x) = x^{\mathrm{T}}X^{-1}x$$

is a Lyapunov function for the closed-loop system.

At the first glance, the above approach to system stabilization does not look quite natural. Indeed, under the condition that inequality (35) is feasible (for instance, the pair (A, B) is controllable), the stabilizing K can be obtained in closed form—this is the so-called pole placement problem. Instead, Theorem 3 reduces the problem to solving a linear matrix inequality. However, this technique easily generalizes to robust stabilization problems.

The presence of uncertainty. Let us now consider the same problem (29)–(30) in the robust setting. We have a family of systems

$$\dot{x} = A(q)x + Bu, \qquad q \in Q, \tag{36}$$

and seek a common regulator of the form $u = Kx$ such that all closed-loop systems

$$\dot{x} = A_c(q)x, \qquad A_c(q) = A(q) + BK, \quad q \in Q,$$

possess the common quadratic Lyapunov function

$$V(x) = x^{\mathrm{T}}Px, \qquad P > 0.$$

In Theorem 3, the solution of the design problem for a fixed matrix A was determined via a solution of one linear matrix inequality. For an uncertain matrix, a *set* of inequalities associated with all possible values of the parameter q comes into the picture. The theorem below is an immediate generalization of Theorem 3 to this situation.

Theorem 4. *Let X provide a solution to the system of linear matrix inequalities*

$$XA^{\mathrm{T}}(q) + A(q)X - 2BB^{\mathrm{T}} < 0, \qquad q \in Q, \quad X > 0. \tag{37}$$

Then the controller with matrix

$$K = -B^{\mathrm{T}}X^{-1}$$

robustly stabilizes system (36)*, and the quadratic form*

$$V(x) = x^{\mathrm{T}}X^{-1}x$$

is a common Lyapunov function for the closed-loop system for all $q \in Q$.

Again, the problem is reduced to solving analogous linear matrix inequalities; sometimes their number may be finite (see remarks at the end of the previous subsection). The case of a finite set Q in Theorem 4 corresponds to the simultaneous stabilization problem (for $B_i \equiv B$), and the result reformulates in the following way: If the set of linear matrix inequalities

$$XA_i^{\mathrm{T}} + A_iX - 2BB^{\mathrm{T}} < 0, \qquad i = 1, \dots, m, \qquad X > 0,$$

has solution X, then the feedback of the form (9) with $K = -B^{\mathrm{T}}X^{-1}$ simultaneously stabilizes m systems in (8).

It should be stressed that the results obtained are based on a sufficient condition, and robust (or simultaneous) stabilization can still be made possible even in the absence of a common quadratic Lyapunov function. Perhaps it was A.M. Meilakhs who first proposed such an approach to robust stabilization as early as in 1975 [57]; a systematic development of this ideology is due to S. Boyd and co-authors [23].

4.3. The LQR Problem

In the previous subsection we showed how stabilization can be performed by constructing quadratic Lyapunov functions. This approach generalizes to the presence of uncertainty in the system's description; however, it yields only sufficient conditions. Technically, the problem reduces to solving a linear matrix inequality or a system of such inequalities. A generalization is also possible if a certain quadratic performance index is to be optimized on top of stabilization. This problem also can be solved by seeking a common Lyapunov function, but in this case quadratic matrix inequalities come into play. We now consider such optimal control problems.

Assume that the pair (A, B) in problem (10), (11) is controllable and the control $u = Kx$ is stabilizing for some K. Then the value of the quadratic functional (11) along the trajectories of a stable closed-loop system with matrix $A_c \doteq A + BK$ is equal to

$$J(K) = \int_0^\infty \Big[x^{\mathrm{T}}Rx + u^{\mathrm{T}}Su\Big]dt = x_0^{\mathrm{T}}Px_0,$$

where $P > 0$ is the solution of the matrix Lyapunov equation

$$(A + BK)^{\mathrm{T}}P + P(A + BK) = -(R + K^{\mathrm{T}}SK); \tag{38}$$

e.g., see Lemma П.14 in [4]. Let us minimize the magnitude of $J(K)$ over all stabilizing K. To this end, we introduce the new variables

$$X \doteq P^{-1}, \qquad Y \doteq KP^{-1}, \tag{39}$$

much in the same way as above. Then for the value of the functional we obtain

$$J = x_0^{\mathrm{T}} X^{-1} x_0, \tag{40}$$

and Eq. (38) takes the form

$$XA^{\mathrm{T}} + AX + XRX + L = 0, \qquad L = (Y^{\mathrm{T}}B^{\mathrm{T}} + BY + Y^{\mathrm{T}}SY),$$

which is a quadratic *matrix Riccati equation* in the variable X. For any $R \geq 0$ it is known to have the following monotonicity property (e.g., see [4, Lemma II.23]): If X_1 and X_2 denote the solutions of this equation for $L = L_1$ and $L = L_2 \geq L_1$, respectively, then $X_1 \geq X_2$. With this property, for the associated values of the functional (40) we have $J_1 \leq J_2$. Therefore, the value of L is to be estimated from below. Using the standard trick for extracting the perfect square, for the matrix $L = L(Y)$ above we obtain

$$L(Y) = \left(S^{1/2}Y + S^{-1/2}B^{\mathrm{T}}\right)^{\mathrm{T}}\left(S^{1/2}Y + S^{-1/2}B^{\mathrm{T}}\right) - BS^{-1}B^{\mathrm{T}},$$

whence it follows that for all Y the relation $L(Y) \geq L(Y_*) = -BS^{-1}B^{\mathrm{T}}$ is valid, where it is denoted $Y_* = -S^{-1}B^{\mathrm{T}}$. In other words, the value of the functional J is minimal for $Y = Y_*$; with the account for (39) this gives

$$K = -S^{-1}B^{\mathrm{T}}X^{-1}$$

and the stabilizing control $u = Kx$. We finally obtain the following result.

Theorem 5. *Let the pair (A, B) be controllable and let $R > 0,\ S > 0$. Then there exists a unique solution $X > 0$ of the Riccati equation*

$$XA^{\mathrm{T}} + AX + XRX - BS^{-1}B^{\mathrm{T}} = 0, \tag{41}$$

and the control

$$u = Kx, \qquad K = -S^{-1}B^{\mathrm{T}}X^{-1},$$

stabilizes the system

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0,$$

and minimizes the quadratic performance index

$$J = \int_0^\infty \left[x^{\mathrm{T}}Rx + u^{\mathrm{T}}Su\right]dt$$

so that

$$J = x_0^{\mathrm{T}}X^{-1}x_0.$$

It is worth noting that the function $V(x) = x^{\mathrm{T}}X^{-1}x$ is a Lyapunov function for the optimal closed-loop system $\dot{x} = (A - BS^{-1}B^{\mathrm{T}}X^{-1})x$.

The presence of uncertainty. We now consider the same LQR problem with the added assumption on the presence of uncertainty (having the same form as in the preceding sections). Such a *robust linear quadratic regulator problem* was formulated above as Problem 11. For the sake of simplicity, we restrict our attention to the situation where the matrices $B, R > 0$, and $S > 0$ are known precisely, while the uncertainty is concentrated in the A matrix:

$$\dot{x} = A(q)x + Bu, \qquad q \in Q, \quad x(0) = x_0. \tag{42}$$

For a fixed system, the Lyapunov function was designed by solving one Riccati equation (41). At this point, let us try to find a common Lyapunov function for the family of systems (42). The whole set of Eqs. (41) associated with all possible values of q cannot be satisfied by a single matrix X; therefore, we seek to satisfy inequalities, i.e., we assume that there exists a common solution $X_o > 0$ of the set of quadratic inequalities

$$XA^{\mathrm{T}}(q) + A(q)X + XRX - BS^{-1}B^{\mathrm{T}} \le 0, \qquad q \in Q. \tag{43}$$

Then the control $u = Kx$ with matrix $K = -S^{-1}B^{\mathrm{T}}X_o^{-1}$ is robustly stabilizing, and the function $V(x) = x^{\mathrm{T}}X_o^{-1}x$ is a common Lyapunov function for all closed-loop systems $\dot{x} = A_c(q)x,\ A_c(q) = A(q) - BK$. Indeed, the derivative of the function $V(x)$ along the trajectories of the systems is equal to

$$\dot{V}(x) = x^{\mathrm{T}}\Big(A_c^{\mathrm{T}}(q)X_o^{-1} + X_o^{-1}A_c(q)\Big)x,$$

and hence, $\dot{V} < 0$ provided that

$$A_c^{\mathrm{T}}(q)X_o^{-1} + X_o^{-1}A_c(q) < 0, \qquad q \in Q.$$

Substituting the expression for $A_c(q)$ into the last relation, we premultiply and postmultiply it by X_o to obtain an equivalent inequality

$$X_oA^{\mathrm{T}}(q) + A(q)X_o - 2BS^{-1}B^{\mathrm{T}} < 0, \qquad q \in Q,$$

whose validity follows from (43).

Let us now fix $q \in Q$ arbitrarily and estimate the value of J_q for any of the systems in (42) under the chosen control $u = -S^{-1}B^{\mathrm{T}}X_o^{-1}x$. By the same lemma on the value of the quadratic functional (see beginning of the section) we have

$$J_q = x_0^{\mathrm{T}}P_qx_0,$$

where $P_q > 0$ is the solution of the Lyapunov equation for the stable matrix $A_c(q)$:

$$A_c^{\mathrm{T}}(q)P + PA_c(q) = -(R + X_o^{-1}BS^{-1}B^{\mathrm{T}}X_o^{-1}). \tag{44}$$

Let us now perform certain manipulations with inequality (43) for given q and $X = X_o$; namely, we premultiply and postmultiply it by X_o^{-1} and then subtract and add the term $X_o^{-1}BS^{-1}B^{\mathrm{T}}X_o^{-1}$. We have:

$$\begin{aligned} &A^{\mathrm{T}}(q)X_o^{-1} + X_o^{-1}A(q) + R - X_o^{-1}BS^{-1}B^{\mathrm{T}}X_o^{-1} \\ &\qquad - X_o^{-1}BS^{-1}B^{\mathrm{T}}X_o^{-1} + X_o^{-1}BS^{-1}B^{\mathrm{T}}X_o^{-1} \le 0, \end{aligned}$$

or, equivalently,

$$A_c^{\mathrm{T}}(q)X_o^{-1} + X_o^{-1}A_c(q) \le -(R + X_o^{-1}BS^{-1}B^{\mathrm{T}}X_o^{-1}).$$

Now consider equality (44) with $P = P_q$ and subtract it from the last relation. We obtain that the stable matrix $A_c(q)$ satisfies the inequality

$$A_c^{\mathrm{T}}(q)\Big(X_o^{-1} - P_q\Big) + \Big(X_o^{-1} - P_q\Big)A_c(q) \le 0,$$

whence it follows that $X_o^{-1} - P_q \ge 0$, i.e., $P_q \le X_o^{-1}$ and $x_0^{\mathrm{T}}P_qx_0 \le x_0^{\mathrm{T}}X_o^{-1}x_0$. Therefore, for any solution $X_o > 0$ of the set of inequalities (43), the control $u = -S^{-1}B^{\mathrm{T}}X_o^{-1}$ robustly stabilizes the uncertain system, and the value of the performance index is guaranteed to be

$$J \le x_0^{\mathrm{T}}X_o^{-1}x_0.$$

To simplify testing the system of quadratic inequalities (43) for feasibility, the Schur lemma can be exploited to convert (43) into the following equivalent linear inequalities:

$$\begin{pmatrix} XA^{\mathrm{T}}(q) + A(q)X - BS^{-1}B^{\mathrm{T}} & XR^{1/2} \\ R^{1/2}X & -I \end{pmatrix} \leq 0, \qquad X > 0, \qquad q \in Q. \tag{45}$$

This form is suitable for numerical solution, e.g., by using the LMI Control Toolbox in Matlab. We arrive at the following robust counterpart of Theorem 5 (see [23, 58, 59]).

Theorem 6. *Let X denote a solution of the system of linear matrix inequalities* (45). *Then the control*

$$u = Kx, \qquad K = -S^{-1}B^{\mathrm{T}}X^{-1},$$

robustly stabilizes the family of systems

$$\dot{x} = A(q)x + Bu, \quad x(0) = x_0, \qquad q \in Q,$$

and for all $q \in Q$, the following value of the performance index is guaranteed:

$$J \leq x_0^{\mathrm{T}}X^{-1}x_0.$$

Again, the problem is reduced to solving a system of linear matrix inequalities (45) associated with all possible $q \in Q$. As it was already noted, there exist efficient numerical methods for finding solutions to such systems; specifically, the gradient-type iterative method developed in [60] is worth mentioning.

The last step in the overall solution process would be to minimize the guaranteed upper bound $x_0^{\mathrm{T}}X^{-1}x_0$ above over all feasible solutions of the system of linear inequalities (45). The function $J(X) = x_0^{\mathrm{T}}X^{-1}x_0$ is seen to be convex in the matrix variable X so that the problem reduces to the minimization of a convex function subject to convex constraints.

5. SUPERSTABILIZATION

The core of the methods discussed in this section is yet another sufficient condition of stability which has got the name *superstability* in [61]. This property is suitable for control purposes by many reasons. For example, the norm of solutions of superstable systems decays monotonically thus avoiding undesirable peak effects peculiar to stabilization; superstability is retained in the time-varying case and in the presence of nonlinear and time-dependent disturbances; superstable systems possess specific convenient spectral properties, etc. In what follows, we restrict our discussion to illustration of the potential of the superstability concept as applied to system design and various problems of optimal control and robustness. These new capabilities stem from the fact that superstability is formulated in the form of linear constraints on the entries of a system matrix rather than in terms of its eigenvalues. As a result, the reformulated control problems become convex and admit transparent solution by standard linear programming techniques.

5.1. Superstable Linear Systems

We introduce the notion of superstability for state-space continuous-time systems and formulate their major properties. Throughout Section 5 we use the ∞-norm for vectors: $\|x\| = \max\limits_{1\leq i\leq n} |x_i|$, $x \in \mathbb{R}^n$, and the induced 1-norm for matrices: $\|A\| = \max\limits_{1\leq i\leq n} \left(\sum\limits_{j=1}^{n} |a_{ij}| \right)$, $A = (a_{ij}) \in \mathbb{R}^{n\times n}$.

A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is said to be *superstable* if its diagonal entries are negative and their absolute values are greater then the row sums of the absolute values of the off-diagonal entries:

$$\sigma(A) = \sigma \doteq \min_i \left(-a_{ii} - \sum_{j\neq i} |a_{ij}| \right) > 0. \tag{46}$$

The quantity $\sigma(A)$ is referred to as the *degree of superstability* of A; such matrices are also called negative diagonal dominant matrices. Superstable matrices are stable, i.e., $\max_i\{\operatorname{Re}\lambda_i\} < 0$, where λ_i are the eigenvalues of A, but not the reverse.

We consider the following linear continuous-time, time-invariant dynamic system specified in state space:

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \quad A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}. \tag{47}$$

If the system matrix A is superstable, system (47) is also said to be superstable. The fundamental properties of such systems are given by the theorem below [62]; also see [63].

Theorem 7. *Let system* (47) *be superstable. Then,*

(a) *for $u(t) \equiv 0$, the norm of solution decreases monotonically and the following estimate is valid:*

$$\|x(t)\| \le \|x_0\|\, e^{-\sigma t}, \qquad t \ge 0; \tag{48}$$

(b) *for $\|u(t)\| \le 1,\ t \ge 0$, and any initial value $\|x_0\| \le \gamma \doteq \|B\|/\sigma$, we have*

$$\|x(t)\| \le \gamma, \qquad t \ge 0; \tag{49}$$

(c) *for $\|u(t)\| \le 1,\ t \ge 0$, and arbitrary initial point x_0, we have*

$$\|x(t)\| \le \gamma + e^{-\sigma t}\left(\|x_0\| - \gamma\right)_+, \qquad t \ge 0, \tag{50}$$

where $\alpha_+ = \max\{0,\ \alpha\}$.

Property (a) relates to the system's stability with respect to initial conditions. Clearly, stable (but not superstable) systems also possess this property, that is, $x(t) \to 0$ for any $x(0)$; however, instead of (48), the following estimate is only valid:

$$\|x(t)\| \le C(A,\nu)\|x_0\|e^{-\nu t}, \qquad 0 < \nu < \min_i\{-\operatorname{Re}\lambda_i\},$$

where the constant $C(A,\nu)$ may take very large values so that the norm $\|x(t)\|$ may increase for small t. Superstable systems do not experience such an undesirable peak effect at the initial part of the trajectory, i.e., the ∞-norm of solution decreases monotonically as time evolves.

Properties (b), (c) are related to input-output stability of a system; i.e., bounded inputs yield bounded solutions. The hypercube

$$\mathcal{Q} = \{x \in \mathbb{R}^n : \|x\| \le \gamma\}, \qquad \gamma = \sigma^{-1}\|B\|,$$

is referred to as an invariant set for system (47), that is, all trajectories initiating inside this set, remain there for all admissible disturbances u. In combination with linearity of condition (46), properties (a)–(c) constitute the background for resolving many hard problems.

Following the same reasonings, we introduce a similar notion for discrete-time systems. Namely, the matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ of a discrete-time system is said to be (discrete) superstable, if

$$q \doteq \|A\| = \max_{1\le i\le n} \left(\sum_{j=1}^{n} |a_{ij}| \right) < 1;$$

the quantity $1-q$ is referred to as the degree of discrete superstability of A. Likewise the continuous-time case, this condition is sufficient for a system to be stable, i.e., $\rho(A) \doteq \max_i |\lambda_i(A)| < 1$, but not the reverse. Notably, the superstability property is also formulated in terms of linear conditions on the entries of a matrix.

The theorem below is a discrete-time counterpart of Theorem 7.

Theorem 8. *Let the discrete-time system* $x_k = Ax_{k-1} + Bu_{k-1}$ *be superstable. Then,*
(a) *for* $u_k \equiv 0$*, the following estimate is valid:*

$$\|x_k\| \le q^k \|x_0\|, \qquad k = 1, 2, \dots; \tag{51}$$

(b) *for* $\|u_k\| \le 1,\ k \ge 1$*, and any initial* $\|x_0\| \le \gamma \doteq \|B\|/(1-q)$*, we have*

$$\|x_k\| \le \gamma, \qquad k = 1, 2, \dots; \tag{52}$$

(c) *for* $\|u_k\| \le 1,\ k \ge 1$*, and arbitrary initial* x_0*, we have*

$$\|x_k\| \le \gamma + q^k \left(\|x_0\| - \gamma\right)_+, \qquad k = 1, 2, \dots . \tag{53}$$

The difference from stable systems is the same as in the continuous-time case.

It is well known that there exist no efficient stability tests for matrices; the only common practice is to compose the characteristic polynomial and apply stability criteria for polynomials. At the same time, to check if a matrix is superstable is not a serious problem, since the corresponding conditions are formulated directly in terms of the entries of a matrix rather than its eigenvalues. Sufficient conditions of stability formulated in terms of inequalities on the entries of the system matrix, as well as the properties of such matrices have been discussed in the earlier literature, e.g., see [47, 64–67]; however, it is only recently that the notion of superstability has been applied to design, see [61, 68–70]. We also note that a similar superstability condition can be formulated for SISO systems (in discrete time); we however put this issue aside.

5.2. Stabilization Problems

Let us first discuss in detail the output feedback stabilization problem. We consider the continuous-time system

$$\dot{x} = Ax + Bu, \qquad y = Cx, \qquad x \in \mathbb{R}^n, \tag{54}$$

where $u \in \mathbb{R}^m$ is the control input which we seek in the form

$$u = Ky. \tag{55}$$

Then for the closed-loop system we have

$$\dot{x} = A_c x, \qquad A_c = A + BKC, \qquad A_c = (m_{ij}), \tag{56}$$

and the matrix K is said to be *superstabilizing* if A_c is superstable, i.e.,

$$\sigma = \sigma(A_c) \doteq \min_i \left(-m_{ii} - \sum_{j \ne i} |m_{ij}| \right) > 0. \tag{57}$$

We then require that the closed-loop system be superstable rather than stable and exploit the resulting simplifications.

Indeed, the entries $m_{ij},\ i,j=1,\dots,n$, of the matrix A_c are affine linear functions of $K=(k_{ij})$:

$$m_{ij} \doteq m_{ij}(K) = a_{ij} + (BKC)_{ij} = a_{ij} + b_i K c_j,$$

where b_i is the ith row of the matrix B, and c_j is the jth column of the matrix C. The superstability condition (57) for the matrix A_c has the form

$$-m_{ii} > \sum_{j\neq i} |m_{ij}|, \qquad i=1,\dots,n.$$

By introducing slack variables σ, $n_{ij},\ i,j=1,\dots,n$, this condition can be rewritten in the following way:

$$\begin{aligned} &\sigma > 0,\\ &-m_{ii}(K) - \sum_{j\neq i} n_{ij} \geq \sigma, \qquad i=1,\dots,n,\\ &-n_{ij} \leq m_{ij}(K) \leq n_{ij}, \qquad i,j=1,\dots,n, \quad i\neq j. \end{aligned} \tag{58}$$

If the set of linear inequalities above has solution k_{ij}, n_{ij}, $i,j=i,\dots,n$, for some $\sigma>0$, then the system is superstable. To check if (58) is feasible, we formulate the linear program

$$\begin{aligned} &\max \sigma,\\ &-m_{ii}(K) - \sum_{j\neq i} n_{ij} \geq \sigma, \qquad i=1,\dots,n,\\ &-n_{ij} \leq m_{ij}(K) \leq n_{ij}, \qquad i,j=1,\dots,n, \quad i\neq j, \end{aligned} \tag{59}$$

with respect to the matrix variables $K, N \doteq (n_{ij})$ and the scalar σ. We thus arrive at the following result.

Theorem 9. *Let K,σ denote a solution of problem (59) and assume that $\sigma>0$. Then the feedback $u=Ky$ ensures the superstability of the closed-loop system. Otherwise, if $\sigma\leq 0$, then the system cannot be superstabilized by controllers of the form $u=Ky$.*

This results provides a complete solution to the static output feedback superstabilization problem; in particular, for $C=I$, we obtain a superstabilizing state feedback.

It is important to note that solution of problem (59) provides the maximal value of σ over all feasible K; if this quantity is positive, the resulting superstabilizing controller maximizes the degree of superstability of the closed-loop system. In accordance with (48) this means that apart from superstabilization, we have found the best uniform estimate for the decay rate of $\|x(t)\|$.

We should stress that superstabilization (even by state feedback) is not necessarily attainable. For example, any controllable canonical-form system $\dot{x}=Ax+bu$ can be stabilized, but not superstabilized by state feedback. On the other hand, if a superstabilizing solution exists, it can be found with no effort, although this simplicity is gained at the expense of contracting the class of stabilizable systems to a much narrower class of superstabilizable systems.

Using superstability instead of stability allows for equally straightforward account for the additional constraints which are usually imposed on the entries of the matrix gain K in (55) (such constraints put an upper bound on the magnitude of the control input). For instance, in order to check if system (54), (55) is superstabilizable under interval constraints

$$\underline{k}_{ij} \leq k_{ij} \leq \overline{k}_{ij}, \qquad i,j=1,\dots,n, \tag{60}$$

imposed on the gain matrix $K = (k_{ij})$, one should merely incorporate linear inequalities (60) into the set of constraints in LP (59) and apply Theorem 9 to this extended problem. If a solution exists, it also provides a solution to the "original" Problem 5 of existence of a stable element in constrained matrix families.

Let us also show how Problem 6 can be solved by replacing stability with superstability. Let a_{ij} denote the entries of A, and x_{ij} be the elements of the desired matrix X that belongs to the closure of the set of superstable matrices and minimizes the distance to A. For example, if the distance is measured in the Frobenius norm, the matrix X is given by solving the following *quadratic programming* problem:

$$\begin{aligned} &\min \sum_{i,j} (a_{ij} - x_{ij})^2, \\ &x_{ii} + \sum_{j \neq i} |x_{ij}| \leq 0, \qquad i = 1, \ldots, n. \end{aligned} \tag{61}$$

It is clear that $\gamma \leq \gamma^*$, where γ^* denotes the minimum in the QP above, and γ stands for the distance between A and the set of stable matrices.

It is interesting to note that sometimes this problem can be solved in closed form. The simplest case is discrete superstability and 1-norm: Given a matrix A, $\|A\|_1 \geq 1$, find X, $\|X\|_1 \leq 1$, that minimizes $\|A - X\|_1$. The solution of this problem is attained with

$$X^* \doteq \arg \min_{\|X\|_1 \leq 1} \|A - X\|_1 = \frac{A}{\|A\|_1},$$

so that

$$\min \|A - X\|_1 = \|A - X^*\|_1 = \|A\|_1 \left(1 - \frac{1}{\|A\|_1}\right) = \|A\|_1 - 1.$$

5.3. Superstability and Robustness

Suppose now that the system description contains uncertainty. We first turn to analysis of robustness and consider the classical problem of robust stability for the matrix family given by

$$A = (a_{ij}), \qquad a_{ij} = a_{ij}^0 + \gamma \Delta_{ij}, \quad |\Delta_{ij}| \leq m_{ij}, \quad i, j = 1, \ldots, n, \tag{62}$$

see Problems $5'$ and $6'$. Here as usual, $A_0 \doteq (a_{ij}^0)$ denotes the nominal matrix, $\gamma \geq 0$ is the common scaling factor, Δ_{ij} represent uncertainty, and $m_{ij} \geq 0$ are given numbers which make up the shaping matrix $M = (m_{ij})$. In contrast to robust stability of interval matrices, the problem of *robust superstability* is seen to be nearly trivial.

Indeed, assuming that the nominal matrix A_0 is superstable, i.e.,

$$\sigma(A_0) \doteq \min_i \left(-a_{ii}^0 - \sum_{j \neq i} |a_{ij}^0| \right) > 0,$$

we require this property to be satisfied for all matrices in the family:

$$-(a_{ii}^0 + \gamma \Delta_{ii}) - \sum_{j \neq i} |a_{ij}^0 + \gamma \Delta_{ij}| > 0, \qquad i = 1, \ldots, n.$$

Clearly, this inequality is valid for all admissible Δ_{ij} if and only if

$$-a_{ii}^0 - \gamma m_{ii} - \sum_{j \neq i} \Big(|a_{ij}^0| + \gamma m_{ij} \Big) > 0, \qquad i = 1, \ldots, n,$$

or, equivalently, with

$$\gamma < \gamma^* \doteq \min_i \frac{-a_{ii}^0 - \sum\limits_{j \neq i} |a_{ij}^0|}{\sum\limits_j m_{ij}}. \tag{63}$$

In the particular case $m_{ij} \equiv 1$ (all matrix entries have equal uncertainty intervals), we have

$$\gamma^* = \frac{\sigma(A_0)}{n}. \tag{64}$$

We thus obtain an explicit expression for the *radius of superstability* γ^* of an interval matrix family; this value γ^* is a lower bound for the stability radius $\gamma_{\max}$. Discrete-time case analysis leads to similar expressions.

Let us now turn to robust controller design and simultaneous stabilization (see Problems 8 and 8′) and show how these problems can be solved by changing the stability requirement for superstability. We consider the uncertain continuous-time system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned}$$

and assume for simplicity that uncertainty is wholly accumulated in the matrix A and has interval form (62). The goal is to synthesize a feedback $u = Ky$ such that it superstabilizes the entire family of systems. A solution to this problem can be found without difficulty (of course, provided it exists). Indeed, we represent the closed-loop system matrix $A_c = A + BKC$ in the form

$$A_c = A_0 + BKC + \gamma\Delta = A_c^0(K) + \gamma\Delta, \qquad \Delta = (\Delta_{ij}),$$

where $A_c^0(K) = \left(a_{ij}^0(K)\right) = A_0 + BKC$ denotes the nominal closed-loop matrix, which is seen to depend affinely on K. Then the feedback $y = Ku$ superstabilizes the uncertain system if and only if there exists K such that the inequalities

$$-(a_{ii}^0(K) + \gamma\Delta_{ii}) - \sum_{j \neq i} |a_{ij}^0(K) + \gamma\Delta_{ij}| > 0, \qquad i = 1, \ldots, n,$$

are valid for all admissible Δ. These inequalities are satisfied for all admissible Δ_{ij} if and only if the following set of linear inequalities in K is feasible:

$$-a_{ii}^0(K) - \gamma m_{ii} - \sum_{j \neq i} \left(|a_{ij}^0(K)| + \gamma m_{ij}\right) > 0, \qquad i = 1, \ldots, n.$$

Checking the feasibility is accomplished through reducing to a linear program with respect to K in much the same way as in the previous subsection.

Moreover, assume that the nominal system is supperstabilizable, i.e., the LP problem

$$\begin{aligned} &\max \sigma, \\ &-a_{ii}^0(K) - \sum_{j \neq i} n_{ij} \geq \sigma, \qquad i = 1, \ldots, n, \\ &-n_{ij} \leq a_{ij}^0(K) \leq n_{ij}, \qquad i, j = 1, \ldots, n, \quad i \neq j \end{aligned} \tag{65}$$

(cf. (59)), has solution $K,\ \sigma > 0$ (see Theorem 9), and denote $\sigma \doteq \sigma(A_c^0)$. We then readily obtain the *radius of maximal robustness*, which is the maximal value of the uncertainty span γ that allows for robust superstabilization. Indeed, if the matrix $A_c^0(K)$ of the nominal closed-loop system is superstable for some K, then, in accordance with (63), the superstability of the perturbed system is retained for all

$$\gamma < \gamma_K^* \doteq \frac{\sigma(A_c^0(K))}{n}$$

(for simplicity, the case $m_{ij} \equiv 1$ is considered). The quantity $\sigma(A_c^0) > 0$ is the result of optimization in (65) over all superstabilizing controllers K so that it is the maximal possible value among all $\sigma(A_c^0(K))$; hence, the radius of maximal robustness is equal to

$$\gamma^* \doteq \frac{\sigma(A_c^0)}{n}.$$

Completely analogous results take place in the discrete-time case.

Equally trivial result can be obtained in the following problem of simultaneous stabilization (8), (9): Does there exist a matrix K such that all matrices

$$A_c^i \doteq A_i + B_iK, \qquad i = 1, \dots, m,$$

are superstable? Indeed, each of the conditions $\sigma(A_i + B_iK) > 0$ is a set of linear inequalities of the form (58) with respect to the entries of K, and we conclude that if the system of linear inequalities

$$\sigma(A_i + B_iK) > 0, \qquad i = 1, \dots, m, \tag{66}$$

admits a solution K, then controller (9) simultaneously superstabilizes all m systems (8). Since superstability implies stability, this also provides a solution to the problem of simultaneous stabilization. We stress once again that the proposed method is based on a sufficient condition so that simultaneous stabilization can still be made possible even when the system of linear inequalities (66) is infeasible.

5.4. Optimal Control

In this subsection, the superstability concept is applied to hard problems arising in optimal control. We consider Problems 10 and 12.

Rejection of bounded disturbances. Let us demonstrate that changing stability for superstability leads to substantial simplifications in Problem 12. Remarkably, the discrete-time and continuous-time cases are treated equally easily on a uniform basis; it is also important to note that we proceed directly to the MIMO case.

System (14) with controller in the feedback loop has the form

$$\dot{x} = A_cx + Dw, \qquad A_c \doteq A + BKC, \quad D \doteq D_1 + BKD_2. \tag{67}$$

First of all we require it to be superstable, i.e., $\sigma(A_c) > 0$, where $\sigma(A_c)$ is defined by (46). Among all such regulators we seek the one that minimizes the norm $\|x(t)\|$. By Theorem 7, the continuous-time superstable system (67) with $\|w(t)\| \le 1$ admits estimate (49):

$$\|x(t)\| \le \frac{\|D\|}{\sigma(A_c)} = \frac{\|D_1 + BKD_2\|}{\sigma(A + BKC)},$$

provided that $\|x(0)\| \le \frac{\|D\|}{\sigma(A_c)}$. Let us now minimize this upper bound with respect to K under the condition that the controller K is superstabilizing. Introducing the parameter $\sigma > 0$ leads to the problem

$$\min_{K,\sigma} \frac{\|D_1 + BKD_2\|}{\sigma}, \tag{68}$$

$$\sigma(A + BKC) \ge \sigma > 0. \tag{69}$$

With σ being fixed, this problem easily reduces to a system of linear inequalities with respect to the elements of the regulator matrix K; this yields the following result.

Theorem 10. *Assume that the parametric linear program* (68), (69) *has solution* K, σ, *and let* J^* *denote the optimal value of performance index* (68). *Then the controller* $u = Ky$ *superstabilizes system* (14), *and for all initial conditions satisfying* $\|x(0)\| \le J^*$, *the estimate* $\|x(t)\| \le J^*$ *holds for all* $t > 0$.

Therefore, if the LP problem (68), (69) is feasible, the resultant regulator minimizes the norm of the state vector uniformly in t, thus evading undesirable peak effects. On the other hand, as it was already mentioned, superstabilization is not necessarily attainable for an arbitrary system so that the LP above may not be feasible; however, even if it admits a solution, the quantity J^* in the theorem is just an upper bound for the optimal value of the functional.

Solution of the problem in the discrete-time case is completely analogous; the only difference is that instead of (49), we make use of estimate (52) from Theorem 8:

$$\|x_k\| \le \frac{\|D\|}{1 - \|A_c\|} = \frac{\|D_1 + BKD_2\|}{1 - \|A + BKC\|}.$$

Then, introducing the parameter $0 \le q < 1$ reduces the minimization of this estimate to the parametric LP of the form

$$\min_{K,q} \frac{\|D_1 + BKD_2\|}{1 - q},$$
$$\|A + BKC\| \le q.$$

<u>The linear-linear regulator.</u> Let us next show how Problem 10 can be solved by using superstability instead of stability. We require superstability of the closed-loop system matrix $A_c = A + BK$ and make use of estimate (48); then, after obvious manipulations over functional (13), we obtain the following upper bound for J:

$$J \le \frac{\Big(1 + \alpha\|K\|\Big)\|x_0\|}{\sigma(A + BK)}.$$

The right-hand side of the inequality above is now to be minimized over K. This optimization problem reduces to the parametric linear program

$$\min_{K,\sigma} \frac{1}{\sigma}\Big(1 + \alpha\|K\|\Big),$$
$$\sigma(A + BK) \ge \sigma > 0,$$

in the same way as it was done earlier in this subsection.

Note that sometimes solution may degenerate if we omit the term $\alpha\|u\|$ in functional (13). For instance, if the inequality $\sigma(A + BK) \ge \sigma > 0$ is solvable for any σ (e.g., square nonsingular B suffices), then σ can be made arbitrarily large, with J being arbitrarily small. However, with such values of K, the magnitude of the control input u will be very large, and the term $\alpha\|u\|$ is intended to prevent this effect to some extent. On the other hand, a finite solution may not be found even in the presence of this term. For example, choosing $u = -kx$ in the scalar system $\dot{x} = u$ leads to $J = (\alpha + 1/k)|x_0|$, and the optimum cannot be attained with finite values of k.

We conclude this section by noting that almost all hard problems formulated in Section 1 can be solved to a certain degree by using the superstability-based approach. This concept, however, suffers a number of inherent drawbacks. First, to gain superstability rather than stability is a more complicated task, since the set of superstable systems is much smaller than the set of stable systems; e.g., there is no guarantee to superstabilize an arbitrary controllable system by scalar controls (the notion of *extended superstability* formulated recently in [70] substantially broadens the applicability of the approach). Second, within this approach, control system performance can only be evaluated from above so that the computed control inputs are suboptimal. Finally, the control problems under discussion may lead to extremely high-dimensional linear programs, and their numerical solution is not immediate, although specialized gradient-type iterative procedures can presumably be elaborated for this purpose.

6. NUMERICAL METHODS

For a number of hard problems, efficient numerical procedures can be devised which perform direct optimization in the space of uncertain parameters of a system. Despite the fact that these technically simple methods are applicable to a wide range of problems, they unavoidably suffer the drawbacks peculiar to any method aimed at solving "hard" problems. Namely, the convergence is not easy to establish, a solution may not be found even if it exists, the result that follows is only an approximation to the exact solution, etc. The reason is that the problems under consideration are nonconvex, and the method may get stuck in a local optimum. On the other hand, experimental study of such numerical procedures testify to their practical efficiency.

6.1. Methods Based on the Perturbation Theory

We consider a matrix family $A(q) \in \mathbb{R}^{n\times n}$ which depends on the vector parameter $q \in Q$ and introduce the function

$$\eta(q) \doteq \max_k \operatorname{Re} \lambda_k(q).$$

If $\eta(q) < 0$ for a given q, then $A(q)$ is stable, and vice versa. The methods of this subsection are implemented as iterative numerical procedures for finding a point q^* such that $\eta(q^*) = 0$. The motion from the initial point q^0 with the property $\eta(q^0) < 0$ corresponds to the problem of searching an unstable element in the family (estimation of the robustness radius); conversely, the case $\eta(q^0) > 0$ corresponds to finding a stable element. The methods work directly with eigenvalues and use linear estimates for their perturbations under small variations of the parameters. This technique was proposed in [71, 72]; let us discuss it in more detail.

Assume that $\eta(q^0) < 0$ for some q^0. At the ith step of the method, a linear approximation to every eigenvalue $\lambda_k(q^i + dq_k)$, $k = 1, \dots, n$, is constructed in the neighborhood of the current point and the minimum norm perturbation dq_k is found such that it vanishes this linear approximation. Next, the minimal (over k) among these perturbations is chosen, call it dq^i. If $\eta(q^i + dq^i) < 0$, we perform a step of the method: $q^i \to q^i + dq^i$; otherwise, reduce the step-size until the condition $\eta(q^i + \alpha\, dq^i) = 0$ is satisfied for some $0 < \alpha < 1$ and stop. The point $\widehat{q}$ thus obtained lies on the stability boundary in the space of parameters; the corresponding matrix $A(\widehat{q})$ is unstable, and the quantity $\|\widehat{q} - q^0\|$ is adopted as an estimate of the robustness radius.

In other words, by using linear approximations, we iteratively shift the eigenvalues towards the boundary of stability, i.e., to the right half-plane $\operatorname{Re} s \geq 0$. This procedure, call it Algorithm 1, is aimed at finding the nearest *unstable* matrix in the family. The same idea will be used below to design a similar procedure, call it Algorithm 2, for shifting the eigenvalues of an unstable matrix towards the left half-plane $\operatorname{Re} s \leq 0$, thus finding the nearest *stable* matrix.

A cornerstone of these methods is formation of linear approximations to eigenvalues; it is based on the following fundamental theorem of perturbation theory [73].

Theorem 11. *Assume that a parameter-dependent matrix $A(q) \in \mathbb{R}^{n\times n}$, $q \in \mathbb{R}^{\ell}$, is differentiable at $q = 0$ and denote*

$$D_i \doteq \left.\frac{\partial A(q)}{\partial q_i}\right|_{q=0}, \qquad i = 1, \dots, \ell.$$

Let $\lambda_k \doteq \lambda_k(0)$ be an algebraically simple eigenvalue of $A(0)$ and let x_k and y_k denote the associated right and left eigenvectors, i.e.,

$$A(0)x_k = \lambda_k x_k, \quad y_k^* A(0) = \lambda_k y_k^*, \quad k = 1, \dots, n.$$

Then for q small enough there exists an eigenvalue $\lambda_k(q)$ of the matrix $A(q)$ such that

$$\lambda_k(q) = \lambda_k + \sum_{i=1}^{\ell} \frac{y_k^* D_i x_k}{y_k^* x_k} q_i + o(q).$$

With this result, we can estimate how sensitive are the eigenvalues of the matrix $A(q)$ to variations in the parameter q. In what follows, the linear approximations

$$\widetilde{\lambda}_k(q) = \lambda_k + (w^k,\, q) \tag{70}$$

will be used, where

$$w^k = (w_1^k, \dots, w_\ell^k)^{\mathrm{T}}, \qquad w_i^k = \frac{y_k^* D_i x_k}{y_k^* x_k}, \qquad 1 \le i \le \ell, \quad 1 \le k \le n.$$

Finding the nearest unstable element (robustness analysis). We assume that $A(0)$ is stable, i.e., $\mathrm{Re}\,\lambda_k < 0$ for all eigenvalues, and use some norm $\|q\|$. At every iteration step, the maximum norm perturbation is found such that the real parts of all linear approximations $\widetilde{\lambda}_k(q)$ in (70) are still negative, i.e., the relation

$$\max_k \max_q \mathrm{Re}\,\widetilde{\lambda}_k(q) < 0$$

holds. Equivalently, we find

$$\gamma_{est} = \sup\Big\{\gamma : \max_k \Big(\mathrm{Re}\,\lambda_k + \gamma \max_{\|q\|\le 1} (w^k, q)\Big) < 0\Big\}, \tag{71}$$

denote by q^* the value of q that provides the maximum, and perform the step of the method by $\gamma_{est} q^*$. The quantity γ_{est} is an approximate estimate of the robustness radius, since the underlying relation (70) is only valid under the condition that the perturbation is small enough.

All the quantities above can be computed analytically. First, for various types of dependence $A(q)$, an explicit form of D_i is easy to obtain (together with the eigenvectors x_k, y_k, these matrices define the quantities w_i^k). For instance, for affine family (28) we have $D_i = A_i$. The innermost maximum $\max\limits_{\|q\|\le 1} (w^k, q)$ in (71) can as well be computed in closed form for various vector norms; for example, we have $\|w^k\|_2$ if the euclidean norm is taken and $\|w^k\|_1 = \sum_{i=1}^{\ell} |w_i^k|$ if the ℓ_∞-norm is considered, i.e., interval constraints on q. In the latter case we obtain

$$q^* = (\mathrm{sgn}\, w_1^k, \dots, \mathrm{sgn}\, w_\ell^k)^{\mathrm{T}}$$

that leads to the following estimate of the robustness radius:

$$\gamma_{est} = \min_k \left(-\frac{\mathrm{Re}\,\lambda_k}{\|w^k\|_1}\right).$$

With minor modifications all the reasonings above apply to the Schur stability case.

Theorem 11 particularizes to a similar result for the perturbed roots $s(q)$ of polynomials, which can be equally obtained by straightforward expanding $s(q)$ in Taylor series. Therefore, with the proposed setup, both matrix and polynomial problems can be analyzed on a uniform basis.

We illustrate Algorithm 1 via several simple test problems where solutions can be found by other means. It is noted that, strictly speaking, the destabilizing point $q = \widehat{q}$ obtained with the algorithm does not minimize the distance from q^0 to the boundary of stability (because the problem is nonconvex), and the quantity $\widehat{\gamma} \doteq \|\widehat{q} - q^0\|$ is therefore just an upper estimate for the stability radius.

Example 2 (robust stability margin for polynomial families). We consider an affine family of polynomials which depend on ℓ parameters taking values in a rectangular uncertainty set and estimate the radius of robustness. Firstly, an exact solution in this problem can be found (e.g., by applying the frequency-domain approach of Subsection 3.2) and compared with the outcome of Algorithm 1. Secondly, for the case $\ell = 2$, the entire stability domain of the family can readily be plotted by means of D-decomposition. Let

$$p(s, q) = p_0(s) + q_1 p_1(s) + q_2 p_2(s), \qquad |q_i| \le \gamma,$$

where the fifth-degree stable nominal polynomial $p_0(s)$ and the polynomials $p_1(s), p_2(s)$ have the following numerical form:

$$\begin{aligned} p_0(s) &= s^5 + 1.9282s^4 + 1.3622s^3 + 0.4406s^2 + 0.0637s + 0.0030; \\ p_1(s) &= -0.5626s^4 + 0.1922s^3 + 0.9098s^2 - 0.7358s - 0.0464; \\ p_2(s) &= 0.7214s^3 - 0.0810s^2 - 0.7314s + 0.1206. \end{aligned}$$

The exact value of the robustness radius for this family is equal to $\gamma_{\max} = 0.0181$. Figure 4 depicts the stability domain D_5 for the polynomial $p(s, q)$ in the two-dimensional parameter space $\{q_1,\ q_2\}$ along with the destabilizing $\widehat{q}$ obtained by the algorithm from the initial point $q^0 = (0,\ 0)$. It is seen that among the points on the stability boundary, $\widehat{q}$ is not the closest to the point q^0, and the quantity $\widehat{\gamma} \doteq \|\widehat{q} - q^0\|_\infty = 0.0319$ is therefore an upper estimate of the robustness radius. To refine this estimate, the algorithm is re-started several times from various initial points generated randomly in a small neighborhood of q^0, and the shortest among the distances to the boundary

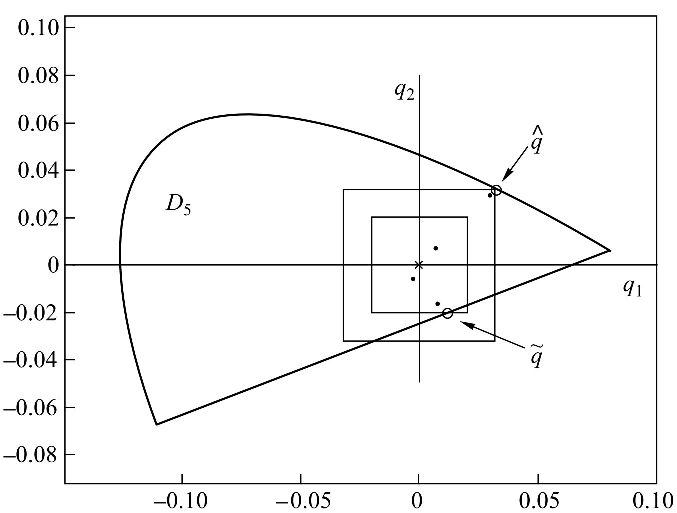

Fig. 4. The stability domain and estimates of $\gamma_{\max}$ in Example 2.

points thus obtained is taken as an estimate of $\gamma_{\max}$. In this example such a refined value (the best over 20 runs) was found to be $\widetilde{\gamma} = 0.0198$.

Example 3 (stability radius of interval matrices). Given an interval matrix family $A = A_0 + \Delta$, $|\Delta_{ij}| \le r$, find the stability radius of A_0, i.e., the maximum value of r retaining the stability ($\operatorname{Re}\lambda_i(A) < 0$) of the entire family; see Problem 6′ of Subsection 2.2. To illustrate the potential of the method, we consider the class of Metzler matrices defined by $a_{ii} < 0$, $a_{ij} > 0$ for $i \ne j$. The value of $r_{\max}$ can be computed numerically by noting that such matrices are destabilized by perturbations of the shape $\Delta_{ij} \equiv \delta$. The experiment was conducted with the matrix $A_0 \in \mathbb{R}^{40\times 40}$ having entries $a_{ii} = -60$, $a_{ij} = 2$ for $i < j$, and $a_{ij} = 1$ for $i > j$; the value $r_{\max} \approx 0.0913$ was found. Using Algorithm 1, we obtain the exact solution $\overline{r} = 0.0913$ after two steps.

Finding the nearest stable element (stabilization). We briefly discuss a similar Algorithm 2 for finding the nearest *stable* matrix, which is based on the same idea. This algorithm also exploits the linear estimates of perturbations from Theorem 11, but this time, the "inverse" problem is solved: Find the minimum norm q such that the real parts of *all* linear approximations are negative. This necessitates taking multiple eigenvalues into account, whereas approximations (70) are only valid under the condition that λ_k is a simple eigenvalue, see Theorem 11. For real eigenvalues of multiplicity 2 (this type of multiplicity is most important), *second order approximations* can be derived which are also linear in the vector of perturbations. Finally, the eigenvalues λ_k with maximal real parts will be referred to as "rightmost," and problem (72) below will contain only those constraints associated with such eigenvalues.

Thus, at every iteration we have the following minimization problem under linear constraints:

$$\text{Find} \quad \min \|dq\| \quad \text{subject to} \quad \lambda_k + (w^k, dq) \le 0, \quad k = 1, \ldots, m, \tag{72}$$

whose solution defines the step of the method; here m is the number of the rightmost eigenvalues. Further manipulations are similar to those prescribed by Algorithm 1. A rigorous derivation of the second order approximations and a detailed description of Algorithm 2 are given in [72].

It should be admitted that such an approach does not necessarily lead to correct conclusions, since the function $\eta(q)$ is nonconvex, and the algorithm may get stuck in a local minimum; moreover, eigenvalues of higher multiplicity are not taken into account. As a result, a desired stable matrix may not be found even if it exists. However, numerous experiments conducted with various specific numerical problems showed that in the large, the method avoids local minima and yields a solution.

We turn to examples. In practical implementations of Algorithm 2 (likewise Algorithm 1), certain numerical constants should be specified. For example, we consider a matrix A stable if $\max\limits_k \operatorname{Re}\lambda_k(A) \le -\delta$ for some small $\delta > 0$, which has the meaning of the degree of stability.

Example 4 (static output stabilization). This example with a 1×2 gain matrix $K = [q_1\ q_2]$ was considered in [74] where the problem was reduced to finding a stable polynomial in the family

$$\mathcal{P}(s, q) \doteq \{P(s, q) = P_0(s) + q_1 P_1(s) + q_2 P_2(s)\}$$

with

$$P_0(s) = s^3 - 13s; \qquad P_1(s) = s^2 - 5s; \qquad P_2(s) = s + 1,$$

i.e., to Problem 1. A very lengthy derivation in [74] based on the decidability algorithms of Tarski and Seidenberg yielded a stable point $\widehat{q} = (2;\ 50)$. We first note that the problem involves two uncertain parameters so that a complete solution can be found by means of the D-decomposition technique described in Subsection 3.1. In particular, the whole stability domain of the family $P(s, q)$

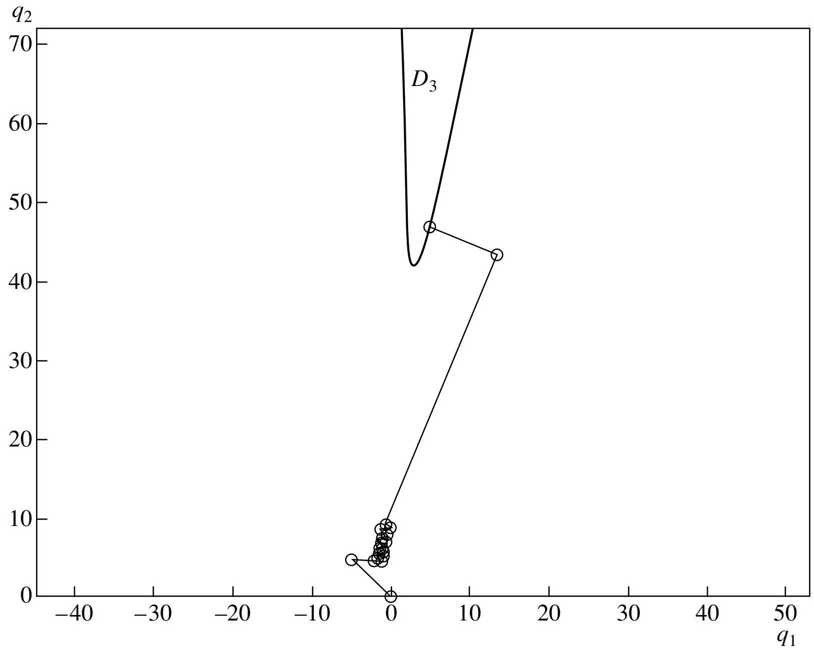

Fig. 5. The trajectory of Algorithm 2 in Example 4.

can be plotted; in Fig. 5 it is denoted by D_3. By way of comparison, we apply Algorithm 2 with initial approximation $q^0 = (0;\ 0)$ to obtain a stabilizing solution $\widehat{q} \approx (4.7631;\ 46.7713)$ after 20 steps.

In a similar manner the method applies to other hard problems. For instance, search for a stable element in an interval polynomial or matrix family is performed by using a slightly modified Algorithm 2 in which the system of inequalities in (72) is supplemented with linear constraints of the form (3). The problem of finding the maximal degree of stability for a family can be solved by repeatedly running Algorithm 2 with successively increased values of δ (this quantity has the meaning of the "depth" of penetration into the stability domain). The robust and simultaneous stabilization problems are also treated in exactly the same way. For example, under simultaneous stabilization, every plant contributes its own set of linear inequalities to the cumulative constraints in problem (72) so that the number of constraints increases whereas the structure of the problems remains the same.

Earlier attempts of applying the optimization approach to hard problems in control theory (including design) are also worth mentioning, e.g., see [75, 76]; similar estimates of stability radii for matrix families were obtained in [77].

6.2. Direct Search Methods

The modern control theory possesses a number of sound algorithmic approaches to optimal controller design; however, one of their common drawbacks is that the orders of the resulting optimal regulators are high. For instance, within the H^∞-optimization framework, certain weighting matrices have to be chosen and included in the extended plant description, which unavoidably leads to increase in dimension. Moreover, all the available methods are quite complicated computationally. These defects can be removed by "separating" the process of system stabilization from the optimization of the performance index. Methods of such type are elaborated in [78]. At every interactive cycle, optimization in the space of the two free parameters of a controller is performed on a computer; the search is carried out inside the stability domain, which is plotted by using D-decomposition techniques with respect to these two parameters. In a sense, such methods lack algorithmic validity; in particular, the above-mentioned search is not formalized, convergence cannot be proved, the value of the objective functional cannot be estimated theoretically, etc. In practice, however, the methods yield low-order controllers with acceptable, or even higher quality.

An outline of the approach is given below.

(1) An initial low-order controller $C_0(s)$ is chosen which stabilizes the nominal plant. In a number of important cases this can be done without difficulty, e.g., see [78, Section 2].

(2) The controller $C_0(s)$ is embedded into a two-parameter family of controllers $C(s,a,b)$ and the stability domain $\mathcal{D}$ is then plotted in the space of the parameters (a,b). If the plant description contains uncertainty, the robust stability domain is plotted.

(3) In the domain of (robust) stability, numerical optimization is performed with respect to the given performance index by picking the points in $\mathcal{D}$ on the computer screen.

(4) If the results of optimization are poor, the optimal controller $C_1(s) \doteq C(s,a^*,b^*)$ thus obtained is re-denoted by $C_0(s)$ and Step 2 is executed.

We illustrate the algorithm above via the following numerical example borrowed from [79, p. 77].

Example 5. Considered is the following *model-matching problem*: For the plant

$$G(s) = \frac{(s-1)(s-2)}{(s+1)(s^2+s+1)},$$

find a controller $C(s)$ such that the *sensitivity*

$$S(s) \doteq \frac{1}{1+G(s)C(s)}$$

of the closed-loop system at low frequencies be as small as possible, i.e., the performance index is taken to be

$$J(C) = \max_{0\le\omega\le 0{,}01} |S(j\omega)|. \tag{73}$$

In [79], this problem is reduced to the H^∞-optimization problem:

$$\min \|WS\|_\infty, \qquad W(s) = \frac{s+1}{10s+1}, \tag{74}$$

where the weighting function W selects low frequencies. The solution obtained in [79] with the Nevannlina–Pick method gives the fourth-order controller

$$C(s) = \frac{0.6114(s+0.3613)(s+1)(s^2+s+1)}{(s+0.004698)(s+0.528)(s^2+5.612s+9.599)}$$

with the associated value of the functional $J(C) = 0.1202$.

Let us now turn to the direct search method. First note that the plant $G(s)$ is stable, hence, it can be stabilized by a P controller. We therefore embed it in a two-parameter family of PI controllers of the form $C(s) = a + b/s$ and plot the two-dimensional stability domain $\mathcal{D}$ for the closed-loop polynomial using the D-decomposition approach. By picking with the mouse one or another numerical value of the pair $(a,b) \in \mathcal{D}$, we compute the associated value of the performance index and take the best result thus obtained. On top of computing the value of the functional, the step response for every chosen controller can be calculated so that other properties of the closed-loop system (e.g., overshoot or oscillation) can be taken into account based on its shape. Such direct search resulted in the regulator

$$C^*(s) = -0.0086 + 0.1381/s, \qquad J^* = 0.0362,$$

with a reasonable step response. It is seen that a simple PI controller is three times as good as the much more complicated controller obtained with the use of H^∞-optimization! Moreover, the value of J^* can be further reduced (down to $J = 0.0229$ at $a = 0.1637$, $b = 0.2186$) by allowing for highly oscillating behavior of the response. This surprising result is due to the fact that straightforward optimization of functional (73) is impossible in H^∞-technique and we are forced to make use of objective (74) instead, whereas the proposed method can work directly with the original performance index.

The procedure applies to other root location regions, different from the left half-plane as in Example 5, and other performance indices. In the latter case, execution of step 3 suggests a different method for computing the value of the objective function (for instance, if the maximal robustness is taken as an objective, then the result of Theorem 2 can be applied). The above methodology was tested over numerous benchmark problems and proved to be highly efficient. Thus, we never faced multiextremality, which might complicate search, and the devised low-order controllers usually yielded almost the same (or better) values of the objective function as higher-order controllers obtained with standard methods.

6.3. Methods Using Linear Programming

Among the approaches to stabilization by fixed-structure controllers, the numerical method proposed in [80, 48] is worth discussing. Given a plant specified by a scalar transfer function $A(s)/B(s)$, stabilize it by means of a controller of the form $C(s) = N(s)/D(s)$, where the degrees of the polynomials $N(s)$ and $D(s)$ are fixed. Let $q_1, \dots, q_\ell$ denote the coefficients of $N(s)$ and $D(s)$, and let d be the degree of the closed-loop characteristic polynomial $P(s,q) = A(s)N(s)+B(s)D(s)$. Its coefficients $p_i(q)$ are seen to be linear functions in the parameters $(q_1, \dots, q_\ell) \doteq q$ of the controller. Instead of solving Problem 1, which is hard, we fix a “desired” characteristic polynomial of the closed-loop system, that is, some stable polynomial $P^0(s)$ of degree d. By the root continuity property, its stability is preserved under small deviations of the coefficients p_i^0 up to some values p_i, e.g., those having form

$$|p_i - p_i^0| \le \alpha_i, \quad \alpha_i > 0, \qquad i = 0, \dots, d. \tag{75}$$

Moreover, the maximal possible range $r_{\max}$ of the deviations $|p_i - p_i^0| \le r_{\max}\alpha_i$ which retain stability can be easily found using robust stability criteria for interval polynomials. This gives a certain rectangular domain

$$Q_p \doteq \{p \in \mathbb{R}^{d+1}\colon\ p_i^- \le p_i \le p_i^+, \quad i = 0, \dots, d\},$$

$$p_i^- = p_i^0 - r_{\max}\alpha_i, \qquad p_i^+ = p_i^0 + r_{\max}\alpha_i,$$

such that its points correspond to stable polynomials of degree d. Standard linear programming routines can now be used to test the feasibility of the system of linear inequalities

$$p_i^- < p_i(q) < p_i^+; \qquad i = 0, \dots, d, \tag{76}$$

with respect to the parameters q. If a feasible point q^* exists, it gives us the coefficients of the desired controller.

Although being simple and natural, this method suffers serious drawbacks. Improper choice of the roots of $P^0(s)$ and the scales α_i for the coefficient variations (as well as the norm in (75)) may lead to “poor” domains Q_p and hence, to infeasibility of the system of inequalities (76).

7. CONCLUSION

It is important to note that the list of hard and open problems in linear control theory is not limited to those discussed in Section 1. This theory is a vivid and dynamic research area with lots of new problems springing up and novel formulations being claimed. Moreover, many problems still remain open in the classical control theory.

First of all, in practice, most of the requirements imposed on control system performance are formulated in the form of specification on the desired engineering properties of the system such as overshoot, settling time, degree of stability, process oscillation, to name just a few, rather than in terms of the modern optimal control theory including linear quadratic optimization, H_∞-theory and the like. At present, a number of non-formalized engineering methods of control design are exploited to approximately attain the desired system performance with respect to these qualitative indices. However, strict and sound analytic solution methods for such problems, e.g., those similar to linear quadratic optimization, yet have to be devised.

Next, in the present paper, we basically dealt with problems where a single control objective such as stabilization, robust stabilization, optimization of one or another performance functional, etc., was declared. In control practice, however, a system is usually assigned with multiple specification so that several performance indices are to be optimized simultaneously. Quite often, problems of this sort are nearly intractable, and closed-form solutions are available in a few exceptional settings.

Finally, linear control problems constitute just a small portion of the massive body of nonlinear problems. As a rule, linear models serve as approximations to real-life problems, which inherently expose deviations from linearity. Control problems that arise in nonlinear system analysis and synthesis are far more complicated; however, they are beyond the scope of this survey.

REFERENCES

1. *Spravochnik po teorii avtomaticheskogo upravleniya* (Handbook in Automatic Control Theory), Krasovskii, A.A., Ed., Moscow: Nauka, 1987.
2. Pervozvanskii, A.A., *Kurs teorii avtomaticheskogo upravleniya* (A Course in Automatic Control Theory), Moscow: Nauka, 1986.
3. Besekerskii, V.A. and Popov, E.P., *Teoriya sistem avtomaticheskogo regulirovaniya* (The Theory of Automatic Regulation Systems), Moscow: Nauka, 1966.
4. Polyak, B.T. and Shcherbakov, P.S., *Robastnaya ustoichivost' i upravlenie* (Robust Stability and Control), Moscow: Nauka, 2002.
5. Bernstein, D.S., Some Open Problems in Matrix Theory Arising in Linear Systems and Control, *Linear Algebra Appl.*, 1992, no. 162–164, pp. 409–432.
6. Blondel, V. and Tsitsiklis, J.N., NP-hardness of Some Linear Control Design Problems, *SIAM J. Control Optimiz.*, 1997, vol. 35, no. 6. pp. 2118–2127.
7. Blondel, V., Sontag, E., Vidyasagar, M., and Willems, J., *Open Problems in Mathematical Systems and Control Theory*, London: Springer, 1999.
8. Nemirovskii, A.A., Several NP-hard Problems Arising in Robust Stability Analysis, *Math. Control Sig. Syst.*, 1994, vol. 6, pp. 99–105.
9. Aizerman, M.A., On Some Structural Conditions of Stability of Automatic Regulation Systems, *Avtom. Telemekh.*, 1948, no. 11.
10. Aizerman, M.A. and Gantmakher, F.R., Conditions of Existence of a Stability Domain for One-Loop Automatic Regulation System, *Prikl. Mat. Mekh.*, 1954, vol. XVIII, no. 1.
11. Padmanabhan, P. and Hollot, C.V., Complete Instability of a Box of Polynomials, *IEEE Trans. Automat. Control*, 1992, vol. 37, no. 8, pp. 1230–1233.

12. Nemirovskii, A.S. and Polyak, B.T., Necessary Conditions for Stability of Polynomials and Their Applications, *Avtom. Telemekh.*, 1994, no. 11, pp. 113–119.

13. Pujara, L.R., Some Necessary and Sufficient Conditions for Low-Order Interval Polytopes to Contain a Hurwitz Polynomial, *Proc. Conf. Decision Control*, Phoenix, USA, 1999, pp. 5024–5029.

14. Moses, R.L. and Liu, D., Determining the Closest Stable Polynomial to an Unstable One, *IEEE Trans. Automat. Control*, 1991, vol. 39, no. 4, pp. 901–907.

15. Syrmos, V.L., Abdallah, C.T., Dorato, P., and Grigoriadis, K., Static Output Feedback: A Survey, *Automatica*, 1997, vol. 33, no. 2, pp. 125–137.

16. Garcia, G., Pradin, B., Tarbouriech, S., and Zeng, F., Robust Stabilization and Guaranteed Cost Control for Discrete-Time Linear Systems by Static Output Feedback, *Automatica*, 2003, vol. 39, pp. 1635–1641.

17. Kimura, H., Pole Assignment by Gain Output Feedback, *IEEE Trans. Automat. Control*, 1975, vol. 20, pp. 509–516.

18. Davison, E.J. and Wang, S.H., On Pole Assignment in Linear Multivariable Systems Using Output Feedback, *IEEE Trans. Automat. Control*, 1975, vol. 20, pp. 516–518.

19. Brockett, R. and Byrnes, C., Multivariable Nyquist Criteria, Root Loci and Pole Placement: A Geometric Viewpoint, *IEEE Trans. Automat. Control*, 1981, vol. 26, pp. 271–284.

20. Byrnes, C.I. and Anderson, B.D.O., Output Feedback and Generic Stabilizability, *SIAM J. Control Optimiz.*, 1984, vol. 22, pp. 362–380.

21. Byrnes, C., Pole Assignment by Output Feedback, in *Three Decades of Mathematical System Theory*, Lect. Notes Control Inf. Sci., 1989, vol. 135, pp. 31–78.

22. Eremenko, A. and Gabrielov, A., Pole Placement by Static Output Feedback for Generic Linear Systems, *SIAM J. Control Optimiz.*, 2002, vol. 41, no. 1, pp. 303–312.

23. Boyd, S.L., El Ghaoui, L., Feron, E., and Balakrishnan, V., *Linear Matrix Inequalities in Systems and Control Theory*, Philadelphia: SIAM, 1994.

24. Kharitonov, V.L., Asymptotic Stability of an Equilibrium Position of a Family of Systems of Linear Differential Equations, *Diff. Uravn.*, 1979, vol. 14, no. 11, pp. 1483–1485.

25. Tsypkin, Ya.Z. and Polyak, B.T., Frequency Domain Criteria for ℓ^p-robust Stability of Continuous Linear Systems, *IEEE Trans. Automat. Control*, 1991, vol. 36, no. 12, pp. 1464–1469.

26. Bartlett, A.C., Hollot, C. V., and Lin, H., Root Location of an Entire Polytope of Polynomials: It Suffices to Check the Edges, *Mat. Control Sig. Syst.*, 1988, vol. 1, pp. 61–71.

27. Barmish, B.R., *New Tools for Robustness of Linear Systems*, New York: Macmillan, 1994.

28. Ackermann, J., *Robust Control. The Parameter Space Approach*, London: Springer, 2002.

29. Bhattacharyya, S.P., Chapellat, H., and Keel, L., *Robust Control: The Parametric Approach*, Upper Saddle River: Prentice Hall, 1995.

30. Hinrichsen, D. and Pritchard A., Stability Radius for Structured Perturbations and the Algebraic Riccati Equation, *Syst. Control Lett.*, 1986, no. 8, pp. 105–113.

31. Qiu, L., Bernhardsson, B., Rantzer A., *et al.*, A Formula for Computation of the Real Stability Radius, *Automatica*, 1995, vol. 31, no. 6, pp. 879–890.

32. Zadeh, L.A. and Desoer, C.A., *Linear System Theory. The State Space Approach*, New York: McGraw-Hill, 1963. Translated under the title *Teoriya lineinykh sistem*, Moscow: Nauka, 1970.

33. Kats, A.M., Finding the Parameters of a Regulator from the Desired Characteristic Equation of a Regulation System, *Avtom. Telemekh.*, 1955, no. 3, pp. 269–272.

34. Volgin, L.N., *Elementy teorii upravlyayushchikh mashin* (Elemens of the Theory of Governors), Moscow: Sovetskoe Radio, 1962.

35. Larin, V.M., Naumenko, K.I., and Suntsev, V.N., *Spektral'nye metody sinteza lineinykh sistem s obratnoi svyaz'yu* (Spectral Methods for Linear Feedback System Design), Kiev: Naukova Dumka, 1971.

36. Bongiorno, J.J. and Youla, D.C., On the Design of Single-Loop Single-Input-Output Feedback Control Systems in the Complex Frequency Domain, *IEEE Trans. Automat. Control*, 1977, vol. 22, no. 3, pp. 416–423.

37. Vidyasagar, M., *Control System Synthesis: A Factorization Approach*, Boston: MIT Press, 1985.

38. Blondel, V., *Simultaneous Stabilization of Linear Systems*, London: Springer, 1995.

39. *Osnovy avtomaticheskogo regulirovaniya* (Foundations of Automatic Regulation Theory), Solodovnikov, V.V., Ed., Moscow: Mashgiz, 1954.

40. Dahleh, M. and Pearson, J., l_1-optimal Controllers for MIMO Discrete-Time Systems, *IEEE Trans. Automat. Control*, 1987, vol. 3, pp. 314–322.

41. Dahleh, M. and Diaz-Bobillo, I., *Control of Uncertain Systems: A Linear Programming Approach*, Englewood Cliffs: Prentice Hall, 1995.

42. Barabanov, A.E., *Sintez minimaksnykh regulyatorov* (Design of Minimax Regulators), St. Petersburg: S.-Peterburg. Gos. Univ., 1996.

43. Evans, W.R., *Control System Dynamics*, New York: McGraw-Hill, 1954.

44. Uderman, E.G., *Metod kornevogo godografa v teorii avtomaticheskogo upravleniya* (The Root Locus Method in Automatic Control Theory), Leningrad: Gostekhizdat, 1963.

45. Maxwell, J.C., Vyshnegradskii, I.A., and Stodola, A., *Teoriya avtomaticheskogo regulirovaniya* (Automatic Regulation Theory), Moscow: Akad. Nauk SSSR, 1949.

46. Neimark, Yu.I., *Ustoichivost' linearizovannykh sistem* (Stability of Linearized Systems), Leningrad: LKVVIA, 1949.

47. Šiljak, D.D., *Nonlinear Systems*, New York: Wiley. 1969.

48. Datta, A., Ho, M.T., and Bhattacharyya, S.P., *Structure and Synthesis of PID Controllers*, London: Springer-Verlag, 2000.

49. Nikolaev, Yu.P., The Set of Stable Polynomials of Linear Discrete Systems: Its Geometry, *Avtom. Telemekh.*, 2002, no. 7, pp. 44–53.

50. Gryazina, E.N., The D-Decomposition Theory, *Avtom. Telemekh.*, 2004, no. 12, pp. 15–28.

51. Petrov, N.P. and Polyak, B.T., Robust D-Decomposition, *Avtom. Telemekh.*, 1991, no. 11, pp. 41–53.

52. Frazer, R.A. and Duncan, W.J., On the Criteria for the Stability of Small Motion, *Proc. Roy. Soc., Ser. A*, 1929, vol. 124, pp. 642–654.

53. Polyak, B.T. and Tsypkin, Ya.Z., Frequency-Domain Criteria for Robust Stability and Aperiodicity of Linear Systems, *Avtom. Telemekh.*, 1990, no. 9, pp. 45–54.

54. Barmish, B.R., Corless, M., and Leitmann, G., A New Class of Stabilizing Controllers for Uncertain Dynamical Systems, *SIAM J. Control Optimiz.*, 1983, vol. 21, no. 2, pp. 246–255.

55. Calafiore, G. and Polyak, B.T., Stochastic Algorithms for Exact and Approximate Feasibility of Robust LMIs, *IEEE Trans. Automat. Control*, 2001, vol. 46, no. 11, pp. 1755–1759.

56. Liberzon, D. and Tempo, R., Common Lyapunov Functions and Gradient Algorithms, *IEEE Trans. Automat. Control*, 2004, vol. 49, no. 6, pp. 990–994.

57. Meilakhs, A.M., On the Stabilization of Linear Control Systems in the Presence of Uncertainty, *Avtom. Telemekh.*, 1975, no. 2, pp. 182–184.

58. Petersen, I.R. and Hollot, C.V., A Riccati Equation Approach to the Stabilization of Uncertain Linear Systems, *Automatica*, 1986, vol. 22, pp. 397–411.

59. Petersen, I.R. and McFarlane, D.C., Optimal Guaranteed Cost Control and Filtering for Uncertain Linear Systems, *IEEE Trans. Automat. Control*, 1994, vol. 39, pp. 1971–1977.

60. Polyak, B.T. and Tempo, R., Probabilistic Robust Design with Linear Quadratic Regulators, *Syst. Control Lett.*, 2001, vol. 43, pp. 343–353.

61. Polyak, B. and Halpern, M., Optimal Design for Discrete-Time Linear Systems via New Performance Index, *Int. J. Adaptive Control Sig. Proc.*, 2001, vol. 15, no. 2, pp. 129–152.

62. Lozinskii, S.M., Estimating the Errors of the Approximate Solution of Systems of Ordinary Differential Equations, *Dokl. Akad. Nauk SSSR*, 1953, vol. 92, no. 2, pp. 225–228.

63. Polyak, B.T. and Shcherbakov, P.S., Superstable Linear Control Systems. I. Analysis, *Avtom. Telemekh.*, 2002, no. 8, pp. 37-53.

64. Rosenbrock, H.H., *Computer-Aided Control System Design*, London: Academic, 1974.

65. Voronov, A.A., *Osnovy teorii avtomaticheskogo upravleniya. Avtomaticheskoe regulirovanie nepreryvnykh lineinykh sistem* (Foundations of Automatic Control Theory. Automatic Regulation of Continuous-Time Linear Systems), Moscow: Energiya, 1980.

66. Ogata, K., *Modern Control Engineering*, Englewood Cliffs: Prentice Hall, 1990.

67. Kaszkurevich, E. and Bhaya, A., *Matrix Diagonal Stability in Systems and Computation*, Boston: Birkhäuser, 2000.

68. Blanchini, F. and Sznaier, M., A Convex Optimization Approach for Fixed-Order Controller Design for Disturbance Rejection in SISO Systems, *IEEE Trans. Automat. Control*, 2000, vol. 45, pp. 784–789.

69. Polyak, B.T. and Shcherbakov, P.S., Superstable Linear Control Systems. II. Design, *Avtom. Telemekh.*, 2002, no. 11, pp. 56–75.

70. Polyak, B.T., Extended Superstability in Control Theory, *Avtom. Telemekh.*, 2004, no. 4, pp. 70–80.

71. Polyak, B.T. and Shcherbakov P.S., A New Approach to Robustness and Stabilization of Control Systems via Perturbation Theory, *Proc. 14th World Congress of IFAC*, Beijing, 1999, vol. C, pp. 13–18.

72. Polyak, B.T. and Shcherbakov, P.S., Numerical Search of Stable or Unstable Element in Matrix or Polynomial Families: A Unified Approach to Robustness Analysis and Stabilization, in *Robustness in Identification and Control*, Lect. Notes Control Inf. Sci., 1999, vol. 245, pp. 344–358.

73. Horn, R. and Johnson, C., *Matrix Analysis*, New York: Cambridge Univ. Press, 1985. Translated under the title *Matrichnyi analiz*, Moscow: Mir, 1989.

74. Anderson, B.D.O., Bose, N.K., and Jury, E.I., Output Feedback Stabilization and Related Problems—Solution via Decision Methods, *IEEE Trans. Automat. Control*, 1975, vol. 20, no. 1, pp. 53–66.

75. Polak, E. and Wardi, Y., Nondifferentiable Optimization Algorithm for Designing Control Systems Having Singular Value Inequalities, *IEEE Trans. Automat. Control*, 1982, vol. 18, no. 3, pp. 267–283.

76. Malan, S., Milanese, M., and Taragna, M., Robust Analysis and Design of Control Systems Using Interval Arithmetic, *Automatica*, 1997, vol. 33, no. 7, pp. 1363–1372.

77. Bobylev, N.A., Emelyanov, S.V., and Korovin, S.K., Estimates of Perturbations of Stable Matrices, *Avtom. Telemekh.*, 1998, no. 4, pp. 15–24.

78. Kiselev, O.N. and Polyak, B.T., Low-Order Controller Design via the H^∞ Criterion and the Criterion of Maximal Robustness, *Avtom. Telemekh.*, 1999, no. 3, pp. 119–130.

79. Francis, B., *A Course in H^∞ Control Theory*, Berlin: Springer, 1987.

80. Keel, L.H. and Bhattacharyya, S.P., Robust Stability and Performance with Fixed-Order Controllers, *Automatica*, 1999, vol. 35, no. 10, pp. 1717–1724.

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