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TOPICAL ISSUE

The D-decomposition Technique for Linear Matrix Inequalities[1]

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Abstract—In the framework of the theory of linear matrix inequalities, a method is proposed for determining all the domains in the parameter space having the property that an affine family of symmetric matrices has the same fixed number of like-sign eigenvalues inside each of the domains. The approach leans on the ideas of D-decomposition; it is particularly efficient in the problems involving few parameters. Generalizations of the method are considered along with its modifications to the presence of uncertainty.

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1. INTRODUCTION AND STATEMENT OF THE PROBLEM

Considered is the following problem: In the space of parameters $x \in \mathbb{R}^{\ell}$, find the domains over which the matrix

$$A(x) \doteq A_0 + \sum_{i=1}^{\ell} x_i A_i, \qquad A_i \in \mathbb{S}^{n\times n}, \tag{1}$$

has a fixed number of negative eigenvalues. Here, $\mathbb{S}^{n\times n}$ denotes the space of real, symmetric $n \times n$ matrices.

A particular case of this problem is finding a point x such that the linear combination of symmetric matrices is negative definite; this is the well-known feasibility problem for the linear matrix inequality (LMI) $A(x) \le 0$. The first application of LMIs to control theory is due to A.M. Lyapunov; specifically, he established the equivalence between the stability of the system $\dot{x} = Ax$ and solvability of the LMI $A^{\mathrm{T}}P + PA < 0$, $P > 0$, where the matrix P defines the quadratic Lyapunov function $V(x) = x^{\mathrm{T}}Px$. The notation "matrix inequality" was put in use by V.A. Yakubovich [1] in 1962 as applied to the problems of absolute stability; later, this apparatus has been further developed by Yakubovich in a number of other works (e.g., see [2–4]). The role of Yakubovich in creating the LMI theory is widely recognized; thus, according to [5], "It is fair to say that Yakubovich is the father of the field, and Lyapunov the grandfather of the field."

After a while it has been realized that linear matrix inequalities represent quite a universal and fruitful tool in optimization, control, and system theory. Thus, it has been shown in [5] that many classical and modern problems in these areas of research can be formulated in terms of linear matrix inequalities and efficiently solved using convex programming methods.

An important step in the formation of the theory was understanding of the presently evident fact that linear matrix inequalities can be formulated as convex optimization problems [6], so that

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to solve them, it remains to make use of one or another numerical method. The first systematic studies of numerical implementations of the LMI algorithms were also conducted by E.S. Pyatnitski and co-authors; in particular, a special version of the method of ellipsoids was proposed.

Since mid-eighties, A.S. Nemirovski and Yu.V. Nesterov have published a number of papers, where numerically efficient procedures of convex optimization were developed based on interior point methods [7]; moreover, these procedures were proved to have reasonable computational complexity. These methods and their modifications had been successfully used for solving LMI problems in various formulations, and with time, a customary viewpoint became prevailing in system theory, according to which a problem is considered to be solved if it is represented in the LMI format. By mid-nineties, the theory and methods of linear matrix inequalities had shaped into a self-contained discipline. As far as the use of the LMI theory in control is concerned, the monograph [8] (the only one published in Russian) is worth mentioning, which presents the most recent results on applications of linear matrix inequalities technique to control system design.

At present, this area of research is also well developed from the application point of view—there exist powerful solvers and handy interfaces (front-ends, parsers); among the most popular ones are LMILab, SeDuMi, Yalmip (e.g., see [9]). At the same time, both the theoretical grounds and numerical procedures for finding solutions suffer certain limitations. Thus, the LMI theory is not aimed at describing the whole feasible set, not to mention other signature invariant domains; various problems involving model uncertainty often defy efficient solution, etc.

The primary goal of this work is not only to present a description of the *whole* feasible set $\{x \in \mathbb{R}^\ell \colon A(x) \le 0\}$, but also characterize *all* the domains in the parameter space over which the signature of the matrix $A(x)$ remains unaltered. This is accomplished via a technique known as D-decomposition [10], which was originally proposed as a tool for stability analysis of SISO systems, and its modification to the case of symmetric matrices. For a small number of parameters, $\ell = 2$, the domains are easy to depict graphically on the plane. A complete description of the feasible set allows for the new statements and design of solution methods for problems of semidefinite programming, and the determination of other domains with invariant number of negative eigenvalues is highly important in checking the applicability conditions of the generalized S-procedure (see [11], p. 312). Moreover, this potentially leads to relatively simple solution methods for systems of matrix inequalities in the presence of uncertainty in the matrix coefficients. In addition to other potential applications, the approach developed here is of a general theoretic interest.

We note that matrix versions of the D-decomposition technique were first proposed in [12] as applied to stability of affine combinations of real matrices of a general form. However, with this technique, such a problem can be analyzed in detail only for the case of one real or complex parameter.

2. CONSTRUCTION OF THE D-DECOMPOSITION DOMAINS

2.1. The Scalar Case

We first consider the one-parameter case, $\ell = 1$ in (1):

$$A(x) \;=\; A + xB, \qquad A, B \in \mathbb{S}^{n\times n}, \qquad x \in \mathbb{R}.$$

The goal is to determine the segments on the parameter axis x, over which the signature of the matrix $A(x)$ is constant:

$$D_m \;=\; \{x \in \mathbb{R}\colon\; A(x) \;\text{ has exactly } m \text{ negative eigenvalues}\,\}.$$

All basic ideas of the approach developed here become apparent in this simplest scalar case.

Assume that for a certain real x^0, the symmetric matrix $A(x) = A + x^0 B$ is nonsingular and has m negative and $n - m$ positive eigenvalues. As x varies, the number of like-sign eigenvalues can only alter if one of them (or several at once) crosses the origin, i.e., $\det(A + xB) = 0$ for some $x \in \mathbb{R}$. This means that there exists a nonzero vector e such that $(A + xB)e = 0$, i.e., $Ae = -xBe$. In other words, x is a generalized eigenvalue of the pair of matrices A and $-B$, and e is the associated generalized eigenvector. Therefore, the determination of the boundaries (points) of the D-decomposition domains in the scalar case reduces to finding all real generalized eigenvalues: $\texttt{eig}\,(A, -B) \in \mathbb{R}$.

It is seen that the maximum possible amount of domains (segments on the x-axis) is equal to $n+1$, since the equation $\det(A+xB) = 0$ has at most n real solutions. On the other hand, there might be no real solutions at all, in which case the D-decomposition is represented by a unique domain—the number of negative eigenvalues of the matrix $A+xB$ remains unaltered for all values of x.

In the general case of nonsymmetric matrices, the number of stable eigenvalues changes as they cross the imaginary axis, so that the additional parameter, frequency comes into play. As a result, graphical representation of the D-decomposition domains is limited to the case of one free real or complex parameter (see [12]); two-parameter families admit complete analysis only for matrices of special types.

The stability domain D_n. Of a particular interest is the stability domain D_n of the family; below, it will be sometimes referred to as the *feasibility domain* D_{feas}. The following lemma holds.

Lemma 1. *Let $A, B \in \mathbb{S}^{n\times n}$ and B be nonsingular. Let λ_i, e_i, $i = 1, \dots, n$, be the generalized eigenvalues and the associated generalized eigenvectors of the pair $(A, -B)$. Then*

(1) *If there exist complex-valued eigenvalues λ_i, then D_n is empty.*

(2) *If all λ_i are real, denote*

$$\underline{x} = \begin{cases} \max\limits_{i\in I_-} \lambda_i, & I_- \doteq \{i\colon\ (Be_i, e_i) < 0\} \neq \emptyset \\ -\infty, & I_- = \emptyset; \end{cases}$$

and

$$\overline{x} = \begin{cases} \min\limits_{i\in I_+} \lambda_i, & I_+ \doteq \{i\colon\ (Be_i, e_i) > 0\} \neq \emptyset \\ +\infty, & I_+ = \emptyset. \end{cases}$$

Then

$$D_n = \begin{cases} (\underline{x}, \quad \overline{x}) & \text{if} \quad \underline{x} < \overline{x} \\ \quad\emptyset & \text{if} \quad \underline{x} \geq \overline{x}. \end{cases}$$

If A is known to be negative definite, the formulas above take the following form:

$$\underline{x} = \begin{cases} \max\limits_{\lambda_i<0} \lambda_i \\ -\infty, & \text{provided all} \quad \lambda_i > 0; \end{cases} \tag{2}$$

and

$$\overline{x} = \begin{cases} \min\limits_{\lambda_i>0} \lambda_i \\ +\infty, & \text{provided all} \quad \lambda_i < 0. \end{cases} \tag{3}$$

2.2. Two Parameters

We now turn to the two-parameter case:

$$A(x) \;=\; A_0 + x_1 A_1 + x_2 A_2, \qquad A_i \in \mathbb{S}^{n\times n}, \qquad i = 0, 1, 2.$$

Let one of the matrices A_1, A_2, say, A_2 be nonsingular. We fix x_1 and denote $A = \overline{A}(x_1) \doteq A_0 + x_1 A_1$ and $B \doteq A_2$. Then we are in the conditions of the previous subsection, and the critical values of the parameter x_2 for the given x_1 are determined as the real generalized eigenvalues $x_2(x_1) = \mathtt{eig}\,(\overline{A}(x_1), -B)$. By varying x_1 we obtain the boundaries of the D-decomposition domains. For every value of x_1, the equation $\det(\overline{A}(x_1) + x_2 B) = 0$ has no more than n real roots, hence, the boundary of the D-decomposition consists of no more than n branches.

The problem is particularly simple to analyze in the situation where all the matrices are diagonal. In that case we have $A_0 = \operatorname{diag}(a_1, \ldots, a_n)$, $A_1 = \operatorname{diag}(b_1, \ldots, b_n)$, $A_2 = \operatorname{diag}(c_1, \ldots, c_n)$ and $A(x) = \operatorname{diag}(d_1, \ldots, d_n)$, where $d_i = a_i + x_1 b_i + x_2 c_i$. Therefore, the boundaries of the domains are given by the n straight lines $a_i + x_1 b_i + x_2 c_i = 0$. If the D_n domain is not empty, it is represented by a convex polygon (perhaps unbounded) defined by the system of inequalities $a_i + x_1 b_i + x_2 c_i \le 0$, $i = 1, \ldots, n$. Any n lines of the general position are known to partition the plane into $n(n+1)/2+1$ domains; i.e., the overall amount of the D-decomposition domains in that case is equal to this quantity. The number of the D_{n-1} domains does not exceed n; these are the domains adjacent to the edges of the polygon D_n having n vertices.

In the general situation, the stability domain

$$D_n \;=\; \{x \in \mathbb{R}^2 :\; A(x) < 0\}$$

can be described separately (it is obviously convex). Namely, for every x_1, determine the segment $(\underline{x}_2(x_1),\ \overline{x}_2(x_1))$ according to Lemma 1 (it may happen to be empty for some or all values of x_1); as x_1 varies, the endpoints of the segment sweep the boundary of the stability domain.

Note that D_n is knowingly unbounded if one of the matrices A_i, $i = 1, \ldots, \ell$, is sign-definite.

Marking the domains of D-decomposition. The marking can be performed in a standard way, i.e., upon constructing the D_m domains on the plane, pick a point inside each of them and compute the eigenvalues of the respective matrix.

Intersections of the boundaries of D-decomposition correspond to multiple zero eigenvalues of the $A(x)$ matrix. Let x^* be a point of double intersection, $\lambda_1(x^*) = \lambda_2(x^*) = 0$ and let the rest $n-2$ eigenvalues be negative. Then the D_n domain is nonempty (it is one of the four adjacent domains).

2.3. The General Case. Boundary Oracle

For $\ell > 2$, graphical representation of the D-decomposition domains is intricate ($\ell = 3$) or impossible to implement ($\ell > 3$); however, the domains can be characterized in the following way. Consider a point $x \in \mathbb{R}^\ell$ and a direction $y \in \mathbb{R}^\ell$. For $\lambda \in \mathbb{R}$ we have

$$A(x + \lambda y) \;=\; A_0 + \sum_{i=1}^{\ell} x_i A_i + \lambda \sum_{i=1}^{\ell} y_i A_i,$$

and denoting

$$A \;\doteq\; A_0 + \sum_{i=1}^{\ell} x_i A_i, \qquad B \;\doteq\; \sum_{i=1}^{\ell} y_i A_i,$$

we arrive at the one-parameter case $A(\lambda) = A + \lambda B$. Hence, the determination of the boundary points in the given direction y reduces to finding real generalized eigenvalues λ_i for the pair of matrices $(A, -B)$. By generating the direction vectors uniformly on the surface of the unit ℓ-dimensional ball as $y = \eta/\|\eta\|$, where η has the standard ℓ-dimensional Gaussian distribution, we obtain the points $x + \lambda_i y$ of the boundaries of the D-decomposition. Such a procedure can be referred to as a *boundary oracle* by analogy with membership oracle and separation oracle, which are widely used in the modern convex optimization theory. For any $x, y \in \mathbb{R}^\ell$, the boundary oracle returns the points of intersection of the ray $x + \lambda y$ and the boundaries of implicitly specified domains D_m or reports about the absence of intersections.

This boundary oracle is particularly efficient when describing the stability domain D_n, which is convex. Assume that for some $x \in \mathbb{R}^\ell$, the matrix $A(x)$ (1) is negative definite, $x \in D_n$, and y is a direction; then the critical values $\underline{\lambda}$, $\overline{\lambda}$, retaining the sign-definiteness of the matrix $A(x + \lambda y)$ are obtained using (2), (3). Such an approach may be useful in generating random uniform distributions over D_n (cf. [13]) and design of the new methods of semidefinite programming. Moreover, it can be applied to approximate determination of the center of gravity of the stability domain. The authors plan to address this issue in a separate paper.

3. EXAMPLES

We first demonstrate by examples that even for simplest low-dimensional families, D-decomposition may have a complicated structure, and the variety of types of the boundaries is wide. We consider problems involving two parameters, in which case the domains D_m are easily visualizable on the plane (x_1, x_2). For ease of exposition, denote $A \doteq A_0$, $B \doteq A_1$, $C \doteq A_2$, $D(x) \doteq A + x_1 B + x_2 C$. In the simplest cases, the equations for the boundary can be obtained in closed form by solving the equation $\det D(x) = 0$; in the situations where solution is not straightforward, we exemplify use of the technique from Subsections 2.1 and 2.2.

<u>2 × 2 matrices.</u> This simplest case is easy to analyze analytically, since the boundaries are defined by a quadratic equation in the variables x_1, x_2; at the same time, it illustrates the diversity of possible types of the D-decomposition domains.

(1) *The boundaries are straight lines.* All types of such boundaries can be obtained with diagonal matrices A, B and C. The boundary consists of one or two straight lines which define two, three or four domains. For example, with $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, we have $D(x) = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$, i.e., the boundaries of the D-decomposition are represented by the coordinate axes $x_1 = 0$ and $x_2 = 0$. The four resulting domains are shown in Fig. 1a; hereinafter, the numbers in the figures correspond to the amount of negative eigenvalues, and the boundary of the stability domain is plotted in bold.

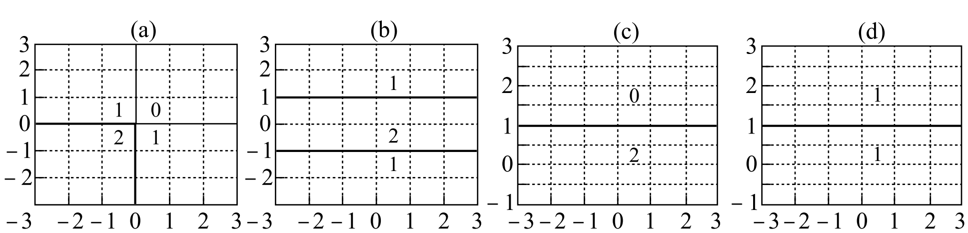

Fig. 1. Examples of D-decomposition for 2×2 matrices.

For $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, we have $D(x) = \begin{pmatrix} -x_2 - 1 & 0 \\ 0 & x_2 - 1 \end{pmatrix}$, and the boundary consists of the two parallel lines $x_2 = \pm 1$ yielding the three domains and in particular, the stability domain, which is represented by the unbounded (convex) strip $\{-1 < x_2 < 1\}$, see Fig. 1b. For $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the boundaries are the same, but the stability domain is empty. By interchanging B and C, we arrive at the vertical strip $\{-1 < x_1 < 1\}$; in this case, the equation $\det D(x) = 0$ is solvable only for two values of x_1.

For $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we obtain one line $x_2 = 1$, and the stability domain is represented by the whole half plane $\{x_2 < 1\}$ (Fig. 1c).

Of a certain interest is the case where $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.
Here, the boundary is also given by the single line $x_2 = 1$, however, both eigenvalues change their signs as the parameters cross this boundary; namely, one eigenvalue changes its sign from + to −, and another from − to +. In this example, the D-decomposition consists of the two D_1 domains separated by a "twofold" boundary (Fig. 1d).

For families of the form

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}, \tag{4}$$

the boundaries are defined by the equation

$$b_1 b_2 x_1^2 - c^2 x_2^2 + x_1(a_1 b_2 + a_2 b_1) + a_1 a_2 = 0, \tag{5}$$

and for different combinations of the coefficients we have the following cases.

(2) *No boundaries.* Equation (5) may possess no real solutions; for example, for $b_1 = b_2 = 0$, $c \neq 0$, $a_1 a_2 < 0$, we have $x_2^2 = a_1 a_2 / c^2 < 0$, and the D-decomposition consists of a single domain D_1.

(3) *The boundary consists of a point.* For $a_1 = a_2 = 0$, $b_1 b_2 = -d^2 < 0$, $c \neq 0$, the equation takes the form $d^2 x_1^2 + c^2 x_2^2 = 0$ with the unique solution $x_1 = x_2 = 0$. Another such possibility is realized with $a_1/b_1 = a_2/b_2$. For example, with $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = -A$, $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, Eq. (5) takes the form $x_1^2 + x_2^2 - 2x_1 + 1 = 0$ and the boundary of the D-decomposition also consists of the single point (1; 0).

(4) *The boundary is a closed curve*; correspondingly, one of the D-decomposition domains is bounded (in other words, the equation $\det D(x) = 0$ is only solvable for the values of x_1 from a certain interval). This is the case for $b_1 b_2 < 0$, $a_1 a_2 > 0$, $a_1 b_2 + a_2 b_1 = 0$, i.e., the boundary is an ellipse centered at the origin. In particular, for $b_1 b_2 = -c^2$ we obtain a circle; for example, $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ lead to the boundary defined by $x_1^2 + x_2^2 = 1$.

(5) *Second-order open curves.* For $b_1 b_2 > 0$, $a_1 b_2 + a_2 b_1 = 0$, $a_1 a_2 \neq 0$, we obtain the two branches of a hyperbola of the form $d^2 x_1^2 - c^2 x_2^2 = a$ (for $a > 0$, the equation $\det D(x) = 0$ has no solutions for x_1 from a certain segment). Hyperbolas of the form $x_1 x_2 = 1$ can be obtained with $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

The boundary is parabolic of the form $x_1 = a x_2^2 + b$ for the case where $a_1 b_2 + a_2 b_1 = 0$ and either $b_1 = 0$, or $b_2 = 0$ (the equation $\det D(x) = 0$ is solvable for x_1 belonging to the semiaxis).

Finally, the situation opposite to that of case 2 above can be realized:

(6) *Every point on the plane is a boundary point.* For $\det A = 0$ and $B = \beta A$, $C = \gamma A$ with arbitrary β, γ, we have $\det D(x) = (1 + \beta x_1 + \gamma x_2)^2 \det A = 0$ for all x_1, x_2 (this is also true for $\ell > 2$ parameters).

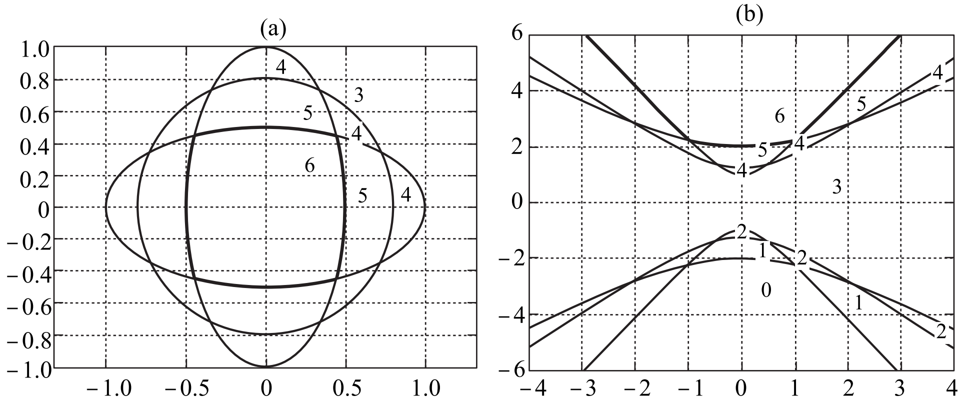

Fig. 2. Examples of D-decomposition for families (8) and (9) with $b_1 = 1$, $b_2 = 1.25$, $b_3 = 2$, $c_1 = 2$, $c_2 = 1.25$, $c_3 = 1$.

In general, all possible types and locations of the curves defining the boundaries for 2×2 matrices can be obtained by analyzing the coefficients of the equation $\det(A + x_1B + x_2C) = 0$, which has the form

$$s_{11}x_1^2 + s_{22}x_2^2 + s_{12}x_1x_2 + s_1x_1 + s_2x_2 + s_0 \;=\; 0, \tag{6}$$

where

$$\begin{gathered} s_{11} = |B|, \quad s_{22} = |C|, \quad s_{12} = |B + C| - |B| - |C|, \\ s_1 = |A + B| - |A| - |B|, \quad s_2 = |A + C| - |A| - |C|, \quad s_0 = |A|, \end{gathered} \tag{7}$$

and $|\cdot|$ denotes $\det(\cdot)$. Thus, using (6), (7), the boundaries in the form of a circle, ellipse, hyperbolas, parabola of the general position, are readily obtainable.

Some special cases, $n > 2$. For matrices of higher dimensions n and for larger values of ℓ, such a detailed analysis is hard to perform; however, for a number of special cases, the boundaries of the domains can also be described in closed form.

As a simple example of this sort, consider the following $2m \times 2m$ matrices:

$$A = -I; \quad B = \operatorname{diag}(b_1, \ldots, b_m, -b_m, \ldots, -b_1); \quad C = \overline{\operatorname{diag}}(c_1, \ldots, c_m, c_m, \ldots, c_1), \tag{8}$$

where $\overline{\operatorname{diag}}$ denotes an antidiagonal matrix. The corresponding determinant is straightforward to compute:

$$\det(A + x_1B + x_2C) \;=\; (x_1^2b_1^2 + x_2^2c_1^2 - 1) \cdot \ldots \cdot (x_1^2b_m + x_2^2c_m^2 - 1),$$

so that the boundaries of D-decomposition are represented by $m = n/2$ ellipses of the form $x_1^2b_i^2 + x_2c_i^2 = 1$, which decompose the plane into $n(n-2)/2 + 2$ domains. In Fig. 2a, an example of D-decomposition is depicted for some 6×6 matrices of this form.

Interchanging A and C in (8), i.e.,

$$A = \overline{\operatorname{diag}}(c_1, \ldots, c_m, c_m, \ldots, c_1); \quad B = \operatorname{diag}(b_1, \ldots, b_m, -b_m, \ldots, -b_1); \quad C = -I, \tag{9}$$

leads to the following expression for the determinant:

$$\det(A + x_1B + x_2C) \;=\; (x_1^2b_1^2 - x_2^2 + c_1^2) \cdot \ldots \cdot (x_1^2b_m - x_2^2 + c_m^2).$$

The boundaries of D-decomposition are now defined by the family of m pairs of hyperbolas $x_1^2b_i^2 - x_2^2 + c_i^2 = 0$, and the overall amount of domains does not exceed $n^2/2 + 1$. For the values of b_i, c_i from the previous example, the respective domains are depicted in Fig. 2b.

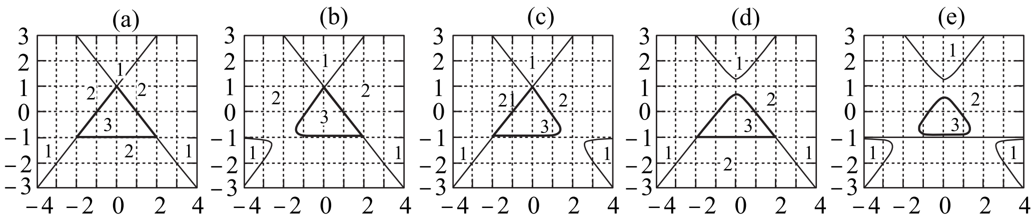

Fig. 3. Examples of D-decomposition for various perturbations in the A matrix.

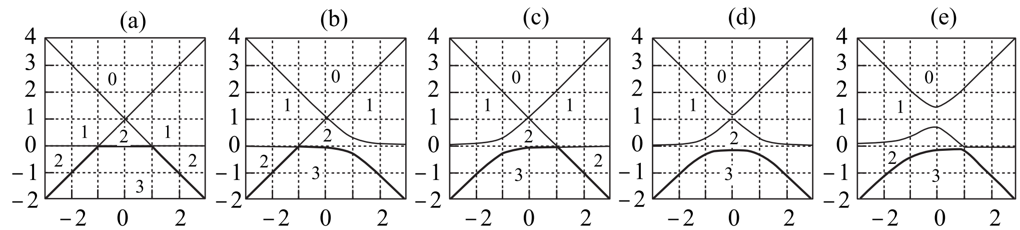

Fig. 4. Examples of D-decomposition for family (10) under various perturbations in A: (a) $\Delta = 0$; (b) $\Delta = \Delta_2$; (c) $\Delta = \Delta_3$; (d) $\Delta = \Delta_2 + \Delta_3$; (e) $\Delta = \Delta_1 + \Delta_3$. In all cases, $\varepsilon = 0.3$.

In general, for $\ell > 2$, the domains are impossible to visualize directly, however sometimes they can be characterized explicitly. The simplest situation is when the matrices $A_i \in \mathbb{S}^{n\times n}$ commute. In that case, they are simultaneously diagonalizable and the domains $D_m \subset \mathbb{R}^\ell$ are represented by polyhedral sets defined by the intersections of n hyperplanes of the form $a_i^0 + v_i^{\mathrm{T}} x = 0$, where $a_i^0 = (A_0)_{ii}$ and the vector v is composed of the (i,i)th entries of the matrices $A_1, \ldots, A_\ell$ (here, the matrices are already diagonalized).

Evolution of the domains under perturbations of the coefficients. It is interesting to analyze how the D_m domains change when uncertainty is incorporated in the matrices. Analytic solution of the problem is bulky, while the technique developed in Subsections 2.1 and 2.2 is simple and descriptive; we illustrate it by examples of 3×3 matrices.

Consider the family

$$D(x) = A + \Delta + x_1 B + x_2 C, \quad A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\Delta \in \mathbb{S}^{3\times 3}$ is a fixed symmetric perturbation. In the unperturbed problem ($\Delta = 0$), the boundaries of D-decomposition are given by the three lines in Fig. 3a.

Introducing uncertainty changes the picture. Figures 3b–3e show the domains of D-decomposition after the "elementary" perturbations $\Delta_1 = \begin{pmatrix} 0 & \varepsilon & 0 \\ \varepsilon & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\Delta_2 = \begin{pmatrix} 0 & 0 & \varepsilon \\ 0 & 0 & 0 \\ \varepsilon & 0 & 0 \end{pmatrix}$, and $\Delta_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \varepsilon \\ 0 & \varepsilon & 0 \end{pmatrix}$ with $\varepsilon = 0.3$, and their sum, respectively, have been incorporated into the A matrix.

For the unperturbed family with the matrices

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{10}$$

the boundaries are also given by the three crossing lines in Fig. 4a. However, introducing the same uncertainties leads to a totally different picture, see Figs. 4b–4e.

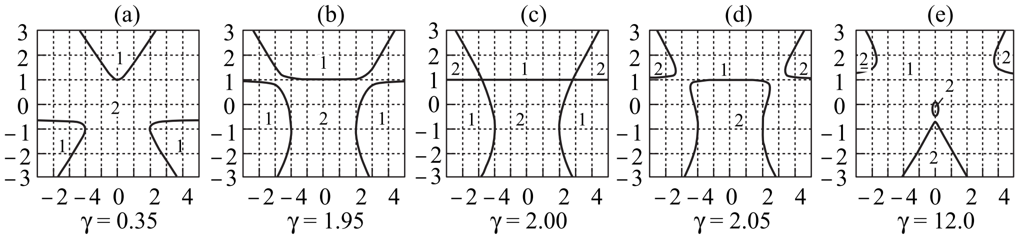

Fig. 5. Evolution of the boundaries of D-decomposition with growth of perturbation.

We further illustrate how the D-decomposition domains evolve as the level of perturbations increases. For the matrices from the first example, consider the perturbation matrix $\Delta = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and perform D-decomposition for the family $A + \gamma\Delta + x_1 B + x_2 C$ with gradually increasing γ. For small $\gamma \lesssim 0.3$, the shape of the boundaries is close to that given in Fig. 3e, and the stability domain shrinks as γ grows. Further increase in the level of perturbation leads to dramatic changes in the geometry of the domains. Figures 5a–5e depict the results of D-decomposition for γ taking the values 0.35, 1.95, 2.0, 2.05, and 12.0, respectively.

4. ROBUSTNESS

One of the valuable generalizations of the proposed approach is its modification to the presence of uncertainty. There are a number of various formulation of LMI problems and assumptions on the uncertainty available in the literature; e.g., see [5, 14, 15]. Here we concentrate on the situation where uncertainty is bounded in the spectral norm.

Let

$$A_i(\Delta_i) \;=\; A_i + \Delta_i, \qquad A_i, \Delta_i \in \mathbb{S}^{n\times n}, \qquad \|\Delta_i\| \le \varepsilon_i, \qquad i = 0, \ldots, \ell, \tag{11}$$

where $\|\cdot\|$ is the *spectral* norm and $\varepsilon_i \ge 0$ are given numbers. Note that in order to retain the LMI structure of the problem, only symmetric perturbations Δ_i are considered. We thus arrive at the following uncertain linear function:

$$A(x,\Delta) \;=\; A_0(\Delta_0) + \sum_{i=1}^{\ell} x_i A_i(\Delta_i), \qquad \Delta \in \mathcal{D} \doteq \Big\{ \{\Delta_i = \Delta_i^{\mathrm{T}}\}_0^{\ell} : \; \|\Delta_i\| \le \varepsilon_i \Big\}. \tag{12}$$

The domains D_m of robust D-decomposition are now defined as follows:

$$D_m^{\mathrm{rob}} \;=\; \Big\{ x \in \mathbb{R}^{\ell} : \; A(x,\Delta) \;\text{ has exactly } m \text{ negative eigenvalues } \; \forall\, \Delta \in \mathcal{D} \Big\}; \tag{13}$$

in particular, the robustly feasible domain is defined by

$$D_n^{\mathrm{rob}} \;=\; D_{\mathrm{feas}}^{\mathrm{rob}} \;=\; \Big\{ x \in \mathbb{R}^{\ell} : \; A(x,\Delta) \le 0 \quad \forall\, \Delta \in \mathcal{D} \Big\}.$$

The problem is to describe the boundaries of the domains of robust D-decomposition; these boundaries are now defined as the values of the parameters x such that the matrix $A(x,\Delta)$ becomes singular for some $\Delta \in \mathcal{D}$. In contrast to the uncertainty-free formulation, the boundaries of robust D-decomposition smear into "strips" inside which the matrix $A(x,\Delta)$ may have different values of signature depending on one or another admissible value of Δ.

4.1. The One-parameter Case

Similarly to the analysis of the problem without uncertainty, we first consider one-parameter families in the simplest case where only the A matrix is subjected to uncertainty:

$$A(x,\Delta) \;=\; (A+\Delta)+xB; \qquad \Delta \in \mathcal{D} \doteq \{\Delta \in \mathbb{S}^{n\times n}\colon\ \|\Delta\| \le \varepsilon\}.$$

The boundaries of the domains (segments) of robust D-decomposition are defined from the condition that the matrix $A+xB+\Delta$ be singular for some $\Delta \in \mathcal{D}$, i.e., the problem is to determine the radius of nonsingularity for the matrix $A+xB$. The lemma below is the main tool for the subsequent analysis.

Lemma 2. *For a nonsingular matrix $M \in \mathbb{S}^{n\times n}$, its symmetric radius of nonsingularity*

$$\rho(M) \;\doteq\; \inf\{\|P\|\colon\ P \in \mathbb{S}^{n\times n},\ \ M+P \text{ is singular}\}$$

is equal to

$$\rho(M) \;=\; 1/\|M^{-1}\| \;=\; \min_i |\lambda_i(M)|.$$

The critical value of P is given by $P = -\lambda ee^{\mathrm{T}}$, where λ is the minimal (in absolute value) eigenvalue of M, and e is the associated eigenvector.

Lemma 2 is a counterpart of Theorem 3 in [16] (also, see [17]) for the symmetric case; for non-symmetric matrices, the radius of nonsingularity is given by the general formula in Theorem 3 [16]. Note that the lemma explicitly indicates the minimum-norm perturbation which makes the matrix M singular; specifically, it is the symmetric matrix $P = -\lambda ee^{\mathrm{T}}$ of rank one.

By the lemma, the matrix $(A+xB)+\Delta$ with perturbations $\|\Delta\| \le \varepsilon$ remains robustly nonsingular for the values of x satisfying

$$\|(A+xB)^{-1}\| \;<\; \frac{1}{\varepsilon}; \tag{14}$$

hence, introducing the function

$$\varphi(x) \;\doteq\; \|(A+xB)^{-1}\|,$$

the segments of robust nonsingularity are found numerically as $\{x\colon\ \varphi(x) < 1/\varepsilon\}$.

Example 1. Consider $A(x,\Delta) = A+\Delta_A+xB$ with $A = \mathrm{diag}\,(-4\ \ -6\ 20\ 27)$, $B = \mathrm{diag}\,(7\ 1\ 3\ 6)$, and $\|\Delta_A\| \le \varepsilon = 0.03\|A\|$. In Fig. 6, the boundaries of robust D-decomposition are represented by the bold segments obtained by smearing the points $x^i = \mathtt{eig}_i(A,-B)$, which are the critical values of the parameter for the unperturbed problem. These segments separate the robustness domains D_i^{rob}, inside which the matrix $A+xB+\Delta_A$ has a fixed number of negative eigenvalues for all admissible perturbations $\|\Delta_A\| \le \varepsilon$.

A more general problem where the uncertainty enters both matrices,

$$A(x,\Delta) \;=\; (A+\Delta_A)+x(B+\Delta_B); \quad \|\Delta_A\| \le \varepsilon_A;\ \|\Delta_B\| \le \varepsilon_B, \tag{15}$$

is analyzed similarly. Represent $A(x,\Delta) = (A+xB)+(\Delta_A+x\Delta_B)$. For the perturbation $\Delta_A+x\Delta_B$ of the matrix $A+xB$ we have the estimate $\|\Delta_A+x\Delta_B\| \le \varepsilon_A+|x|\varepsilon_B$, which is sharp (the equality is attainable), since Δ_A and Δ_B are chosen independently. Introduce the function $\varphi(x) = \|(A+xB)^{-1}\|$; then by Lemma 2, the segments of robust nonsingularity or equivalently, the domains D_m^{rob} are defined by the condition

$$\varphi(x) \;<\; \frac{1}{\varepsilon_A+|x|\varepsilon_B}.$$

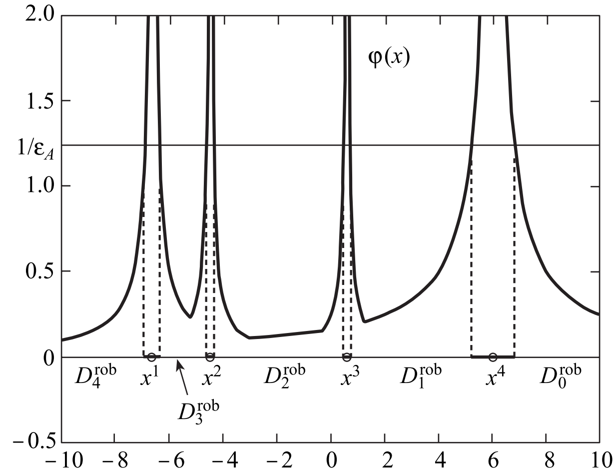

Fig. 6. Example of robust D-decomposition with respect to a single parameter, with uncertainty in A.

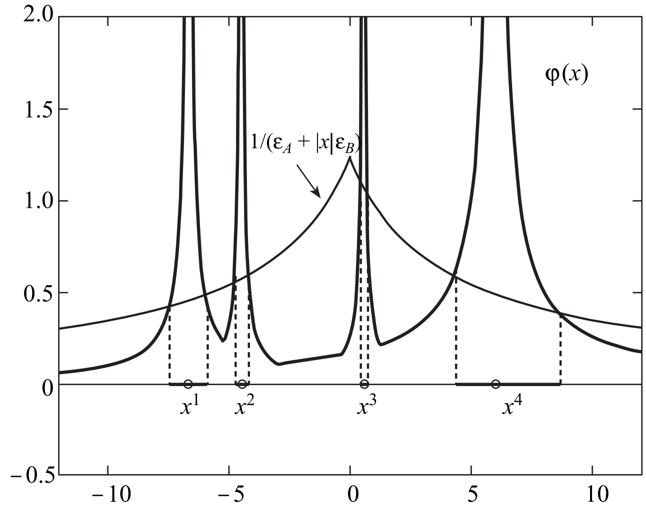

Fig. 7. Robust D-decomposition with respect to a single parameter, with uncertainty entering A and B.

Example 2. Consider the same matrices as in the previous example but with the additional uncertainty (of the same level) in the B matrix: $\|\Delta_B\| \le \varepsilon_B = 0.03\|B\|$. The constructions described above are depicted in Fig. 7; the robustness segments shrink due to the presence of additional uncertainty (respectively, the boundaries get wider).

4.2. The Two-parameter Case

Slightly changing the notation, we now address the case of two-parameter families:

$$A(x, \Delta) \;=\; (A_0 + \Delta_0) + x_1(A_1 + \Delta_1) + x_2(A_2 + \Delta_2); \quad \|\Delta_i\| \le \varepsilon_i,\ i = 0, 1, 2.$$

We fix x_1 and denote $A = A_0 + x_1 A_1$ and $B = A_2$ and also $\varepsilon_A \doteq \varepsilon_0 + |x_1|\varepsilon_1$ and $\varepsilon_B = \varepsilon_2$. This leads us to the setup of problem (15). Indeed, to determine the boundaries, we compose the functions $\varphi(x_2) = \|(A + x_2 B)^{-1}\|$ and $\varepsilon(x_2) = 1/(\varepsilon_A + |x_2|\varepsilon_B)$ and check the condition $\varphi(x_2) < \varepsilon(x_2)$, which defines the segment of robustness in x_2 for a given value of x_1. As x_1 varies, these segments fill

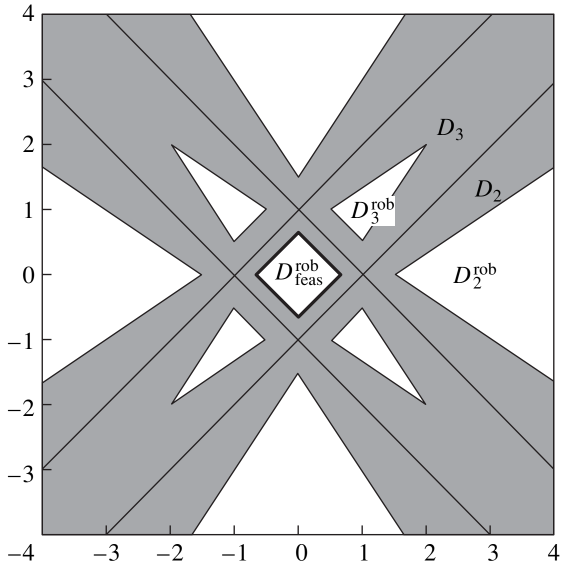

Fig. 8. Example of robust D-decomposition with respect to two parameters.

up two-dimensional domains of robust D-decomposition. On the other hand, the values of x_2 that violate the condition $\varphi(x_2) < \varepsilon(x_2)$ correspond to the absence of robustness; as x_1 varies, these segments of violation sweep certain two-dimensional regions which are considered the boundaries of robust D-decomposition.

We also note that similarly to the uncertainty-free case, the domain $D_n^{\rm rob}$ can be described separately from the rest of $D_m^{\rm rob}$ by combining the results of Lemmas 1 and 2.

Example 3. For the two-parameter family in $\mathbb{S}^{4\times 4}$ with matrices $A_0 = -I$, $A_1 = \mathrm{diag}\,(-1\ 1\ 1\ -1)$ and $A_2 = \mathrm{diag}\,(1\ -1\ 1\ -1)$, the boundaries of D-decomposition in the absence of uncertainty are given by the two pairs of parallel lines in Fig. 8. Introducing uncertainty smears the boundaries into stripes and the D_m domains shrink. In the example considered, for equal relative levels of uncertainty $\|\Delta_i\| \le \varepsilon_i = 0.2\|A_i\|$, $i = 0, 1, 2$, the boundaries of robust D-decomposition are given by the shaded region and the uncolored regions represent the domains of robustness.

4.3. The General Case. Robust Boundary Oracle

Let us now turn to the general case and describe the boundaries of the domains $D_m^{\rm rob}$ (13) for family (11), (12) when $\ell > 2$. We follow the same approach as in Subsection 2.3; i.e., we seek the intersection points of one-dimensional rays and the boundaries of robust D-decomposition. For simplicity, the exposition below is restricted to the description of the robust stability domain.

Further manipulations are essentially a minor generalization of the considerations in Subsection 4.1. Indeed, let $x \in D_{\rm feas}^{\rm rob}$ be a robustly feasible point and $y \in \mathbb{R}^\ell$ be a certain direction. Consider the straight line $x + \lambda y$ and compute $\underline{\lambda}^{\rm rob}$ and $\overline{\lambda}^{\rm rob}$, the minimal and maximal values of λ which guarantee the negative definiteness of the matrix $A(x + \lambda y, \Delta)$ for all $\Delta \in \mathcal{D}$. We have

$$A(x + \lambda y, \Delta) \;=\; \widehat{A}(\lambda) + \Delta(\lambda),$$

where it is denoted

$$\widehat{A}(\lambda) \;=\; A_0 + \sum_{i=1}^{\ell}(x_i + \lambda y_i)A_i, \qquad \Delta(\lambda) \;=\; \Delta_0 + \sum_{i=1}^{\ell}(x_i + \lambda y_i)\Delta_i,$$

and by Lemma 2, the matrix $\widehat{A}(\lambda) + \Delta(\lambda)$ remains nonsingular (hence, negative definite) for all $\Delta \in \mathcal{D}$ satisfying

$$\|(\widehat{A}(\lambda))^{-1}\| \;<\; \frac{1}{\|\Delta(\lambda)\|}\,.$$

Since the perturbations independently sweep the respective domains of uncertainty, the estimate

$$\|\Delta(\lambda)\| \;\le\; \|\Delta_0\| + \sum_{i=1}^{\ell} |x_i + \lambda y_i|\,\|\Delta_i\| \;=\; \varepsilon_0 + \sum_{i=1}^{\ell} |x_i + \lambda y_i|\,\varepsilon_i$$

is sharp. Hence, by considering the two scalar functions

$$\varphi(\lambda) \;=\; \Big\|\Big(A_0 + \sum_{i=1}^{\ell}(x_i + \lambda y_i)A_i\Big)^{-1}\Big\|, \qquad \varepsilon(\lambda) \;=\; \frac{1}{\varepsilon_0 + \sum_{i=1}^{\ell} |x_i + \lambda y_i|\,\varepsilon_i}\,, \tag{16}$$

the segment $[\underline{\lambda}^{\rm rob},\ \overline{\lambda}^{\rm rob}]$ of robust negative definiteness of the family $A(x + \lambda y, \Delta)$ can be found numerically as $\{\lambda\colon\ \varphi(\lambda) \le \varepsilon(\lambda)\}$.

Clearly, the inclusion $[\underline{\lambda}^{\rm rob},\ \overline{\lambda}^{\rm rob}] \subset [\underline{\lambda},\ \overline{\lambda}]$ holds, where $\underline{\lambda}$, $\overline{\lambda}$ are the critical values of the parameter λ in the problem without uncertainty (the minimal and the maximal values of λ retaining the negative definiteness of the matrix $A(x + \lambda y, 0)$). Therefore, the test $\varphi(\lambda) \le \varepsilon(\lambda)$ should only be performed over the segment $[\underline{\lambda},\ \overline{\lambda}]$. We thus arrive at the following *robust boundary oracle.*

Lemma 3. *Let $A(x, 0) < 0$. For any $y \in \mathbb{R}^\ell$, the maximum and minimum values of λ retaining the negative definiteness of the matrix $A(x+\lambda y, \Delta)$ for all admissible perturbations Δ are given by the two solutions of the equation $\varphi(\lambda) = \varepsilon(\lambda)$ (16) over the segment $[\underline{\lambda},\ \overline{\lambda}]$ (2), (3).*

Similarly, to solve a simpler problem of checking if $x \in D^{\rm rob}_{\rm feas}$ for some $x \in \mathbb{R}^\ell$, it suffices to consider the unperturbed matrix at the point x

$$A(x, 0) \;=\; A_0 + \sum_{i=1}^{\ell} x_i A_i$$

and check if the inequality

$$\varepsilon_0 + \sum_{i=1}^{\ell} |x_i|\varepsilon_i \;<\; 1/\|(A(x, 0))^{-1}\|$$

holds.

5. CONCLUSIONS

In this paper, a technique is developed for constructing the domains in the parameter space having the property that an affine family of symmetric matrices has a fixed number of like-sign eigenvalues inside each of the domains; a generalization is proposed for the case of uncertain matrix coefficients. For a few parameters, the results are well visualized on the plane or axis. For the general case of $\ell > 2$ parameters, the boundary oracle is developed, which allows for a simple and efficient description of the feasibility domain; robust formulations of the problem are also covered.

Among the potential applications of the results obtained here is ellipsoidal estimation of the states of dynamical systems, quadratic optimization, finding the center of gravity of a convex set, to name just a few. A promising direction for further elaboration of the proposed technique is its use in solving semidefinite programs, particularly in the robust formulations, which lack satisfactory numerical solution methods.

APPENDIX

Proof of Lemma 1. Essentially, this result can be found in [18] (see Assertion 3.1); for completeness of exposition, the sketch of the proof is given below. Assume that all λ_i are real and distinct, then $(Be_i, e_j) = 0$ for $i \neq j$, so that any $v \in \mathbb{R}^n$ is representable as $v = \sum_i \alpha_i e_i$ for some $\alpha_i \in \mathbb{R}$. The following expression is immediate: $((A + xB)v, v) = \sum_i \alpha_i^2 (Be_i, e_i)(x - \lambda_i)$; hence, the function $f(x) = \sum_i \alpha_i^2 (Be_i, e_i)(x - \lambda_i)$ is negative for all α_i only for the values of x satisfying $\max\limits_{i:(Be_i,e_i)<0} \lambda_i < x < \min\limits_{i:(Be_i,e_i)>0} \lambda_i$.

Proof of Lemma 2. Assuming that M is nonsingular, λ_i are its eigenvalues, and λ is the one which attains $\min_i |\lambda_i|$, we have $\rho(M) = |\lambda|$. Then for any matrix with $\|P\|_2 < \rho(M)$ and any $\|x\|_2 = 1$, the following holds:

$$\big|((M + P)x, x)\big| \;=\; |(Mx, x) + (Px, x)| \;\geq\; |(Mx, x)| - |(Px, x)| \;\geq\; \min_i |\lambda_i| - \|P\|_2 \;>\; 0,$$

i.e., $M + P$ is nonsingular.

On the other hand, consider $P = -\lambda e e^{\mathrm{T}}$, where e, $\|e\|_2 = 1$, is the eigenvector of M associated with the eigenvalue λ: $Me = \lambda e$. We then have $\|P\|_2 = |\lambda|$ and $(M + P)e = \lambda e - \lambda e e^{\mathrm{T}} e = 0$.

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