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Probabilistic robust design with linear quadratic regulators

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Abstract

In this paper, we study robust design of uncertain systems in a probabilistic setting by means of linear quadratic regulators (LQR). We consider systems affected by random bounded nonlinear uncertainty so that classical optimization methods based on linear matrix inequalities cannot be used without conservatism. The approach followed here is a blend of *randomization* techniques for the uncertainty together with *convex optimization* for the controller parameters. In particular, we propose an iterative algorithm for designing a controller which is based upon subgradient iterations. At each step of the sequence, we first generate a random sample and then we perform a subgradient step for a convex constraint defined by the LQR problem. The main result of the paper is to prove that this iterative algorithm provides a controller which quadratically stabilizes the uncertain system with probability one in a finite number of steps. In addition, at a fixed step, we compute a lower bound of the probability that a quadratically stabilizing controller is found.

Keywords: Randomized algorithms; Probabilistic robust design; Linear quadratic regulators

1. Introduction

In recent years, we have seen a growing interest in combining robust and probabilistic approaches for analysis and design of control systems. The main idea is to use worst-case uncertainty bounds together with probabilistic information. That is, the uncertainty Δ entering into the system is bounded within a set $\boldsymbol{\Delta}$ and a certain probability distribution for Δ, for example uniform, is assumed. In this context, several problems have already been studied and solved, including the computation of bounds on the sample size [9,13,22], the connections between randomized algorithms and learning theory [14,24], randomized algorithms for robustness analysis [6,7,21] and other probabilistic-based problems [3,19,27]. In the control community, randomization is now accepted as a viable tool for performing analysis when a classical worst-case solution is not easily computable or when a larger robustness margin is of interest at the expense of a small risk expressed in probability. In addition, there is a widespread agreement that controller design should be carried on in a guaranteed fashion through convex optimization. In this paper, following this line of research, we first use randomization to handle parametric or nonparametric uncertainty, then we perform convex optimization to find the controller parameters.

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In particular, using the standard setting of linear quadratic regulators (LQR), we develop a sequential algorithm, based on subgradient iterations, for designing a controller which quadratically stabilizes the uncertain system with probability one in a finite number of steps. At each step of the sequence, the algorithm is based upon two stages:

- Generation of a random sample Δ^k of the uncertainty $\Delta \in \mathbf{\Delta}$.
- Subgradient step for a convex constraint defined by Δ^k.

The first stage depends on the specific uncertainty structure under attention and can be performed using for example the randomization techniques described in [6,7]. The outcome of this stage is the generation of a random vector or matrix sample $\Delta^k \in \mathbf{\Delta}$. The second stage requires the computation of an approximate solution of a convex inequality constraint $V(Q, \Delta^k) \leqslant 0$ defined by the LQR problem, where Q represents the controller parameters. This stage can be immediately performed by a subgradient step and projection on a cone of nonnegative definite matrices obtaining a set of controller parameters.

The main result of this paper is to demonstrate that this sequential algorithm provides a controller which quadratically stabilizes the uncertain system with probability one in a finite number of steps. Subsequently, at a fixed step k, we also compute a lower bound of the probability that a quadratically stabilizing controller is found. These results are based on two assumptions. Roughly speaking, the first assumption is that there exists a feasible solution, i.e. a quadratically stabilizing controller. Secondly, we assume that the probability to distinguish if an approximation Q is feasible or not is positive, see Section 5 for details on these assumptions and Section 7 for a discussion regarding the case when the first assumption is not satisfied.

Contrary to existing worst-case solutions, we remark that one of the advantages of the method proposed in this paper is that very general uncertainty structures and nonlinearities can be easily handled. For example, if we deal with an uncertain system affected by parametric uncertainty entering into the system in a nonlinear fashion, standard linear matrix inequalities (LMIs) methods cannot be used without introducing overbounding. In addition, even if this conservatism is acceptable, the LMI solution generally requires to simultaneously solve a number of convex inequalities which is exponential in the number of parameters. Unless the problem size is very small, this issue is computationally critical so that finding a feasible solution with standard interior point methods may be very difficult or not even tractable. On the other hand, the solution proposed here deals with only one constraint at each step of the sequence and therefore computation is not an issue. The drawback of this method is obviously the fact that probabilistic estimates are given instead of guaranteed solutions.

We notice that methods similar to those developed in this paper have already been used in a deterministic setting in the context of adaptive control, unknown-but-bounded identification and alternating projection techniques, see e.g. [2,4,11,20]. From the technical point of view, one of the novelties of this paper is to study fast iterative methods for solving linear and quadratic matrix inequalities (QMIs) and, more importantly, to formulate the problem in a probabilistic setting and to state probabilistic convergence results. Other recent papers following the same approach developed here are [8,10].

The paper is organized as follows: In Section 2, we introduce definitions and notation and in Section 3 we study linear quadratic regulators with random uncertainty. In particular, we study systems in state space form subject to bounded random uncertainty and we formulate a guaranteed cost problem in this context. In Section 4, this problem is shown to be equivalent to the solution of a set of quadratic matrix inequalities. In Section 5, the randomized algorithm is presented and its convergence properties are rigorously stated and proved. In Section 6, we study an application example of quadratic stabilization of an aircraft subject to parametric uncertainty entering multiaffinely into the system. In Section 7, we provide some extensions and, in particular, we discuss the probabilistic solution of infeasible sets of QMIs. Finally, in Section 8, we briefly present some conclusions.

2. Definitions and notation

In this paper, we use the following notation. For $x \in \mathbf{R}$, $x > 0$, we denote by $\lceil x \rceil$ the minimum integer greater or equal than x. Next, we recall the notion of projection of a symmetric matrix on a cone of nonnegative definite matrices. Let E_n be a Euclidean space of real symmetric matrices $A \in \mathbf{R}^{n,n}$ equipped with Frobenius norm $\|A\| = (\sum_{i,k=1}^{n} a_{ik}^2)^{1/2}$ and inner product $\langle A, B \rangle = \operatorname{tr} AB$. Let

$$\mathscr{C} \doteq \{A \in E_n\colon A \geqslant 0\},$$

be the cone of symmetric nonnegative definite matrices. The projection $A^+ \in \mathscr{C}$ of a matrix $A \in E_n$ is defined as

$$A^+ \doteq \arg\min_{X \in \mathscr{C}} \|A - X\|.$$

This projection can be found explicitly ([17], see also [20] for other methods based on the Schur decomposition) as follows: For a real symmetric matrix A, we write

$$A = T\Lambda T^{\mathrm{T}},$$

where T is an orthogonal matrix and $\Lambda \doteq \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, then

$$A^+ = T\Lambda^+ T^{\mathrm{T}}, \tag{1}$$

where $\Lambda^+ \doteq \operatorname{diag}(\lambda_1^+, \lambda_2^+, \ldots, \lambda_n^+)$ and $\lambda_i^+ \doteq \max\{0, \lambda_i\}$ for $i = 1, 2, \ldots, n$.

Next, we define a *random matrix* Δ with probability density function (p.d.f.) $f_\Delta(\Delta)$. A real random matrix $\Delta \in \mathbf{R}^{n,m}$ is a matrix of random variables Δ_{ik}, $i = 1, \ldots, n$; $k = 1, \ldots, m$. The probability density function $f_\Delta(\Delta)$ is defined as the joint probability density of the elements of Δ. We assume that the random matrix Δ is bounded in the set $\mathbf{\Delta} \subset \mathbf{R}^{n,m}$. A classical example of p.d.f. is when Δ is uniformly distributed within $\mathbf{\Delta}$. In this case, we have

$$f_\Delta(\Delta) = \mathscr{U}[\Delta] \doteq \begin{cases} \dfrac{1}{\operatorname{vol}(\mathbf{\Delta})} & \text{if } \Delta \in \mathbf{\Delta}, \\ 0 & \text{otherwise}, \end{cases} \tag{2}$$

where $\operatorname{vol}(\mathbf{\Delta})$ is the volume of the set $\mathbf{\Delta}$.

3. LQ regulators with random uncertainty

In this section, we formulate the LQ regulator problem in the presence of random bounded uncertainty. We consider the state space system

$$\dot{x}(t) = A(\Delta)x(t) + Bu(t), \quad x(0) = x_0,\ x(t) \in \mathbf{R}^n,\ u(t) \in \mathbf{R}^m \tag{3}$$

where the uncertainty $\Delta \in \mathbf{\Delta}$ enters into the state matrix. In this paper, we do not make any specific assumption on the dependence of $A(\Delta)$ on Δ, except for boundedness of $A(\Delta)$ for all $\Delta \in \mathbf{\Delta}$. For example, $A(\Delta)$ may be affected by parametric (possibly nonlinear) and nonparametric uncertainty. In the former case, the entries of $A(\Delta)$ are functions of uncertain parameters which are bounded within intervals. In the latter case, Δ is a norm-bounded matrix perturbing the *nominal* state matrix denoted by A_0. That is,

$$A_0 = A(0).$$

In the LQ regulator problem studied in this paper, the performance index is the standard quadratic cost function

$$J \doteq \int_0^\infty (x^{\mathrm{T}}(t)Sx(t) + u^{\mathrm{T}}(t)Ru(t))\,\mathrm{d}t, \tag{4}$$

where the weighting factor $S = S^{\mathrm{T}} > 0$ may depend on the uncertainty Δ while $R = R^{\mathrm{T}} > 0$ is fixed. Next, we take the control law

$$u(t) = -R^{-1}B^{\mathrm{T}}Q^{-1}x(t), \tag{5}$$

where $Q = Q^{\mathrm{T}} > 0$ is chosen so that quadratic stability is guaranteed. It is well-known [5,15] that this is equivalent to the solution of the LMI

$$A(\Delta)Q + QA^{\mathrm{T}}(\Delta) - 2BR^{-1}B^{\mathrm{T}} < 0, \quad Q = Q^{\mathrm{T}} > 0 \tag{6}$$

for all $\Delta \in \mathbf{\Delta}$. That is, if a common solution Q of (6) exists, then the control law (5) quadratically stabilizes the system. In many important cases, which include interval matrices, this problem can be reduced to the solution of a finite number of LMIs. Let $A(\Delta)$ denote an $n \times n$ interval matrix with 2^{ℓ}, $\ell = n^2$, vertex matrices

$$A^1, A^2, \ldots, A^{2^{\ell}}.$$

In this case, if

$$A^k Q + Q(A^k)^{\mathrm{T}} - 2BR^{-1}B^{\mathrm{T}} < 0, \quad Q = Q^{\mathrm{T}} > 0$$

for all $k = 1, 2, \ldots, 2^{\ell}$, then (6) holds for all $\Delta \in \mathbf{\Delta}$, see e.g. [12] for details. A major computational issue, however, is that the number of LMIs which should be simultaneously solved may be large. For example, if a 5×5 interval matrix is considered, we need to solve 2^{25} LMIs and this is beyond the capabilities of LMIs solvers. To overcome this drawback, in this paper we follow a nondeterministic approach which "approximately solves" the system of LMIs in a probabilistic sense.

4. Guaranteed cost with random uncertainty

In this section, we generalize the problem formulation of the previous section and we study the so-called guaranteed cost problem for uncertain systems, see e.g. [16]. That is, given a parameter $\gamma > 0$ and initial conditions x_0, the objective is to guarantee that

$$J \leqslant \gamma^{-1} x_0^{\mathrm{T}} Q^{-1} x_0 \tag{7}$$

for all $\Delta \in \mathbf{\Delta}$. It is well known that this problem can be formulated in terms of block structured LMIs, see e.g. [5]. However, we use an alternative formulation which contains no block structure and requires the solution of QMIs. This is formally stated in the next lemma, see e.g. [16] for similar results.

Lemma 1. *Let Q be a solution of the QMIs*

$$A(\Delta)Q + QA^{\mathrm{T}}(\Delta) - 2BR^{-1}B^{\mathrm{T}} + \gamma(BR^{-1}B^{\mathrm{T}} + QSQ) \leqslant 0, \quad Q = Q^{\mathrm{T}} > 0 \tag{8}$$

for all $\Delta \in \mathbf{\Delta}$. Then, the control

$$u(t) = -R^{-1}B^{\mathrm{T}}Q^{-1}x(t)$$

quadratically stabilizes system (3) *and the cost*

$$J \leqslant \gamma^{-1} x_0^{\mathrm{T}} Q^{-1} x_0,$$

is guaranteed for all $\Delta \in \mathbf{\Delta}$.

Proof. The proof of this result is standard and it easily follows from a well-known result given in [26] which is now recalled. Consider the system

$$\dot{x}(t) = A_c x(t), \quad x(0) = x_0$$

and a cost function

$$J = \int_0^{\infty} x^{\mathrm{T}}(t) W x(t)\,\mathrm{d}t$$

for $W = W^{\mathrm{T}} > 0$, then $J = x_0^{\mathrm{T}} X x_0$ where $X = X^{\mathrm{T}} > 0$ is a solution of the Lyapunov equation

$$A_c^{\mathrm{T}} X + X A_c = -W \tag{9}$$

and

$$A_c \doteq A(\Delta) - BR^{-1}B^{\mathrm{T}}Q^{-1},$$

$$W \doteq S + Q^{-1}BR^{-1}B^{\mathrm{T}}Q^{-1}.$$

We now pre- and post-multiply inequality (8) by Q^{-1} obtaining

$$\gamma^{-1}A_c^{\mathrm{T}}Q^{-1} + \gamma^{-1}Q^{-1}A_c \leqslant -W.$$

Subtracting this inequality and (9), we get

$$A_c^{\mathrm{T}}(\gamma^{-1}Q^{-1} - X) + (\gamma^{-1}Q^{-1} - X)A_c \leqslant 0. \tag{10}$$

Then, we observe that A_c is stable since it satisfies the Lyapunov inequality with $W > 0$ and $\gamma^{-1}Q^{-1} > 0$. Hence, from (10) we obtain $\gamma^{-1}Q^{-1} - X \geqslant 0$ and $J = x_0^{\mathrm{T}}Xx_0 \leqslant \gamma^{-1}x_0^{\mathrm{T}}Q^{-1}x_0$. □

We remark that guaranteed cost is an extension of the quadratic stability problem studied in the previous section. That is, taking sufficiently small γ we immediately recover equation (6). Therefore, the special case of quadratic stability can be easily handled in this more general setting by a suitable selection of the parameter γ. Thus, γ may be viewed as a design parameter which can be adequately chosen.

5. Probabilistic design with guaranteed cost

In this section, we present the main assumptions and results of the paper.

5.1. Uncertainty randomization and assumptions

First, we generate a random matrix Δ^k in the set $\mathbf{\Delta}$ according to the p.d.f. $f_\Delta(\Delta)$. Thus, we immediately obtain the QMI

$$A(\Delta^k)Q + QA^{\mathrm{T}}(\Delta^k) - 2BR^{-1}B^{\mathrm{T}} + \gamma(BR^{-1}B^{\mathrm{T}} + QSQ) \leqslant 0, \quad Q = Q^{\mathrm{T}} > 0. \tag{11}$$

In this inequality, if the weighting factor S depends on the uncertainty Δ, we need to replace it with $S(\Delta^k)$.

Then, we define a matrix-valued function

$$V(Q, \Delta^k) \doteq A(\Delta^k)Q + QA^{\mathrm{T}}(\Delta^k) - 2BR^{-1}B^{\mathrm{T}} + \gamma(BR^{-1}B^{\mathrm{T}} + QSQ) \tag{12}$$

and introduce the scalar function

$$v(Q, \Delta^k) \doteq \|[V(Q, \Delta^k)]^+\|, \tag{13}$$

where $[\,\cdot\,]^+$ is the projection defined in (1) and $\|\cdot\|$ denotes Frobenius norm.

We now formally state a feasibility assumption regarding the set of solutions Q.

Assumption 1. *The solution set*

$$\mathscr{S}_{\mathbf{\Delta}} \doteq \{Q = Q^{\mathrm{T}} > 0\colon V(Q, \Delta) \leqslant 0 \ \textit{for all}\ \Delta \in \mathbf{\Delta}\}$$

has a nonempty interior.

Next, we introduce a probabilistic assumption on the set $\mathbf{\Delta}$.

Assumption 2. *For any $Q \notin \mathscr{S}_{\mathbf{\Delta}}$, we assume that*

$$\mathrm{Prob}\{v(Q, \Delta) > 0\} > 0.$$

This is a mild assumption which can be explained as follows: For any $Q \notin \mathscr{S}_{\mathbf{\Delta}}$, the probability of generating $\Delta^k \in \mathbf{\Delta}$ so that Q is infeasible, i.e. $v(Q, \Delta^k) > 0$, is nonzero. In addition, we observe that if the set $\mathbf{\Delta}$ is finite and the probability to generate any element in this set is positive, then Assumption 2 is satisfied.

5.2. Iterative algorithm and subgradient computation

With given initial conditions $Q^0 > 0$, we consider iterations of the form

$$Q^{k+1} = \begin{cases} [Q^k - \mu^k \partial_Q\{v(Q^k, \Delta^k)\}]^+ & \text{if } v(Q^k, \Delta^k) > 0, \\ Q^k & \text{otherwise,} \end{cases} \tag{14}$$

where ∂_Q denotes the subgradient.

The stepsize μ^k is given by

$$\mu^k = \frac{v(Q^k, \Delta^k) + r\|\partial_Q\{v(Q^k, \Delta^k)\}\|}{\|\partial_Q\{v(Q^k, \Delta^k)\}\|^2}, \tag{15}$$

where $r > 0$ is the radius of a ball $\mathscr{B}_r \subseteq \mathscr{S}_{\mathbf{\Delta}}$ with center Q^*. Then, given initial conditions $Q^0 > 0$ and $Q^* \in \mathscr{B}_r$, we define

$$M \doteq \left\lceil \frac{\|Q^0 - Q^*\|^2}{r^2} \right\rceil.$$

In the next lemma, we explicitly compute the subgradient of $v(Q^k, \Delta^k)$.

Lemma 2. *The function $v(Q, \Delta^k)$ is convex in Q and its subgradient $\partial_Q\{v(Q, \Delta^k)\}$ is given by*

$$\partial_Q\{v(Q, \Delta^k)\} = \frac{[V(Q, \Delta^k)]^+(A(\Delta^k) + \gamma QS) + (A(\Delta^k) + \gamma QS)^{\mathrm{T}}[V(Q, \Delta^k)]^+}{v(Q, \Delta^k)} \tag{16}$$

if $v(Q, \Delta^k) > 0$ or

$$\partial_Q\{v(Q, \Delta^k)\} = 0,$$

otherwise.

Proof. The convexity of $v(Q, \Delta^k)$ can be checked by direct calculation observing that the quadratic term in $V(Q, \Delta^k)$, which is γQSQ, is nonnegative definite. To show that the subgradient is given by (16), we observe that $v(Q, \Delta^k)$ is differentiable if $v(Q, \Delta^k) > 0$. Indeed, for $v(Q, \Delta^k) > 0$ and $Q + Q_\Delta > 0$, we write

$$\begin{aligned} v((Q + Q_\Delta), \Delta^k) = \|[&A(\Delta^k)Q + QA^{\mathrm{T}}(\Delta^k) + (\gamma - 2)BR^{-1}B^{\mathrm{T}} \\ &+ \gamma QSQ + A(\Delta^k)Q_\Delta + Q_\Delta A^{\mathrm{T}}(\Delta^k) + \gamma Q_\Delta SQ + \gamma QSQ_\Delta + \mathrm{o}(\|Q_\Delta\|)]^+\|. \end{aligned}$$

Then, we obtain

$$\begin{aligned} v((Q + Q_\Delta), \Delta^k) &= v(Q, \Delta^k) \\ &\quad + \left\langle \frac{[V(Q, \Delta^k)]^+}{v(Q, \Delta^k)}, (A(\Delta^k)Q_\Delta + Q_\Delta A^{\mathrm{T}}(\Delta^k) + \gamma Q_\Delta SQ + \gamma QSQ_\Delta) \right\rangle + \mathrm{o}(\|Q_\Delta\|) \end{aligned}$$

and, using the fact that $\langle A, B\rangle = \langle B, A\rangle$, we finally get

$$\begin{aligned} v((Q + Q_\Delta), \Delta^k) &= v(Q, \Delta^k) + \mathrm{tr}\left[\left(\frac{[V(Q, \Delta^k)]^+}{v(Q, \Delta^k)}(A(\Delta^k) + \gamma QS)^{\mathrm{T}} + (A(\Delta^k) + \gamma QS)\frac{[V(Q, \Delta^k)]^+}{v(Q, \Delta^k)}\right) Q_\Delta\right] \\ &\quad + \mathrm{o}(\|Q_\Delta\|). \end{aligned}$$

The proof of the lemma is completed observing that the above relation coincides with the definition of gradient. Finally, the proof for the case $v(Q, \Delta^k) = 0$ is immediate. □

5.3. Probabilistic convergence results

We are now ready to state the main result of the paper which holds when μ^k is given by (15) and r is known.

Theorem 1. *Let Assumptions* 1 *and* 2 *be satisfied. Then, algorithm* (14) *with stepsize* (15) *converges with probability one in a finite number of iterations. That is,*

$$\Pr\{\text{There exists } k_0 < \infty\text{: } Q^k \in \mathscr{S}_\Delta \text{ for all } k \geqslant k_0\} = 1. \tag{17}$$

Moreover, for any $Q \notin S_\Delta$ *if*

$$\Pr\{v(Q,\Delta) > 0\} \geqslant p > 0 \tag{18}$$

holds, then for $k > M/p$

$$\Pr\{Q^k \in \mathscr{S}_\Delta\} \geqslant 1 - \exp(-2(pk - M)^2/k). \tag{19}$$

Proof. Define

$$\bar{Q} = Q^* + \frac{r}{\|\partial_Q\{v(Q^k,\Delta^k)\}\|}\partial_Q\{v(Q^k,\Delta^k)\},$$

where $Q^* \in \mathscr{B}_r$. Then, due to Assumption 1, $\bar{Q}$ is a feasible solution of (8) and, in particular, $v(\bar{Q},\Delta^k) \leqslant 0$. Now, if $v(Q^k,\Delta^k) > 0$, it follows from the properties of a projection that

$$\begin{aligned}\|Q^{k+1} - Q^*\|^2 &\leqslant \|Q^k - Q^* - \mu^k\partial_Q\{v(Q^k,\Delta^k)\}\|^2\\ &= \|Q^k - Q^*\|^2 + (\mu^k)^2\|\partial_Q\{v(Q^k,\Delta^k)\}\|^2 - 2\mu^k\langle\partial_Q\{v(Q^k,\Delta^k)\},(Q^k - \bar{Q})\rangle\\ &\quad - 2\mu^k\langle\partial_Q\{v(Q^k,\Delta^k)\},(\bar{Q} - Q^*)\rangle.\end{aligned}$$

We now consider the last two terms in the inequality above. Due to the convexity of $v(Q,\Delta)$ and the feasibility of $\bar{Q}$, we obtain

$$\langle\partial_Q\{v(Q^k,\Delta^k)\},(Q^k - \bar{Q})\rangle \geqslant v(Q^k,\Delta^k)$$

while, due to the definition of $\bar{Q}$

$$\langle\partial_Q\{v(Q^k,\Delta^k)\},(\bar{Q} - Q^*)\rangle = r\|\partial_Q\{v(Q^k,\Delta^k)\}\|.$$

Thus, we write

$$\|Q^{k+1} - Q^*\|^2 \leqslant \|Q^k - Q^*\|^2 + (\mu^k)^2\|\partial_Q\{v(Q^k,\Delta^k)\}\|^2 - 2\mu^k(v(Q^k,\Delta^k) + r\|\partial_Q\{v(Q^k,\Delta^k)\}\|).$$

Now, substituting the value of μ^k (15), we get

$$\|Q^{k+1} - Q^*\|^2 \leqslant \|Q^k - Q^*\|^2 - \frac{(v(Q^k,\Delta^k) + r\|\partial_Q\{v(Q^k,\Delta^k)\}\|)^2}{\|\partial_Q\{v(Q^k,\Delta^k)\}\|^2} \leqslant \|Q^k - Q^*\|^2 - r^2.$$

Therefore, if $v(Q^k,\Delta^k) > 0$, we obtain

$$\|Q^{k+1} - Q^*\|^2 \leqslant \|Q^k - Q^*\|^2 - r^2.$$

From this formula, we conclude that no more than $\lceil\|Q^0 - Q^*\|^2/r^2\rceil$ correction steps can be executed. On the other hand, if Q^k is infeasible, then, due to Assumption 2, there is a nonzero probability of making a correction step. Thus, with probability one, the method cannot terminate at an infeasible point. We therefore conclude that the algorithm must terminate after a finite number of iterations at a feasible solution.

To prove relation (19), for fixed k we define the indicator function

$$I^k \doteq \begin{cases} 1 & \text{if } v(Q^k,\Delta^k) > 0,\\ 0 & \text{otherwise}\end{cases}$$

and we let

$$L^k \doteq \sum_{i=1}^{k} I^i.$$

Then, $L^k \geqslant M$ implies that the algorithm terminates after a number of steps which is no greater than k. From (18), the probability of the event

$$\{v(Q^k, \Delta^k) > 0\}$$

is at least p. Next, recall that the one-sided Chernoff bound [25] gives

$$\Pr\{\hat{p} - p \geqslant -\varepsilon\} \geqslant 1 - \delta$$

where $\hat{p} = L^k/k$ and

$$\delta \leqslant \exp(-2\varepsilon^2 k). \tag{20}$$

Equivalently, we write

$$\Pr\{L^k \geqslant (p - \varepsilon)k\} \geqslant 1 - \delta.$$

Taking $\varepsilon = p - M/k > 0$, we get

$$\Pr\{L^k \geqslant M\} \geqslant 1 - \delta.$$

The desired results can be finally obtained using bound (20). □

The iterative algorithm proposed in Section 5.2 can be traced back to the Kaczmarz projection method for solving linear equations which was developed in 1937 and to the Agmon–Motzkin–Shoenberg method for solving linear inequalities which has been studied in 1954. A finite-convergent version of the latter method was proposed by Yakubovich in the middle of the sixties, see [4] and references therein. A similar approach for solving convex inequalities can also be found in [11,18]. However, in contrast with this literature, the method proposed here is random. Therefore, the probabilistic results given in Theorem 1 are new not only for the QMI case but also for the case of linear constraints. Finally, if compared with the method of alternating projections, see [20], we observe that iteration (14) given in this section does not require to find projections on QMIs. From the computational point of view, this is a crucial advantage since this operation may be hard to perform.

We remark that the lower bound of the probability given in (19) is based on the Chernoff bound (see e.g. [22]) and therefore may be very conservative. We also note that this bound is based on the a priori knowledge of the probability p and on the quantities $\|Q^* - Q^0\|$ and r, which may be difficult to acquire. Therefore, the contribution of the second part of Theorem 1 is to show that at step $k > M/p$, a bound on the probability of finding a feasible solution can be obtained, provided that some a priori information is available. If this information is not available, the first part of Theorem 1 shows that a feasible solution can be found with probability one in a finite number of steps.

6. Numerical example

In this section, we consider a multivariable example given in [1] (see also the original paper [23] for a slightly different model and set of data) which studies the design of a controller for the lateral motion of an aircraft. The model consists of four states and two inputs. The state space equation is given by

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & L_p & L_\beta & L_r \\ g/V & 0 & Y_\beta & -1 \\ N_{\dot{\beta}}(g/V) & N_p & N_\beta + N_{\dot{\beta}} Y_\beta & N_r - N_{\dot{\beta}} \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & -3.91 \\ 0.035 & 0 \\ -2.53 & 0.31 \end{bmatrix} u(t),$$

where x_1 is the bank angle, x_2 the derivative of the bank angle, x_3 is the sideslip angle, x_4 the yaw rate, u_1 the rudder deflection and u_2 the aileron deflection.

We consider the following nominal values for the aircraft parameters entering into the state matrix: $L_p = -2.93$, $L_\beta = -4.75$, $L_r = 0.78$, $g/V = 0.086$, $Y_\beta = -0.11$, $N_{\dot{\beta}} = 0.1$, $N_p = -0.042$, $N_\beta = 2.601$ and $N_r = -0.29$.

These values coincide with those given in [23] with the difference that here we take $N_{\dot{\beta}} = 0.1$ while in [23] the value of this parameter is set to zero. The perturbed matrix $A(\Delta)$ is constructed as follows: Each nominal aircraft parameter is perturbed by a relative uncertainty which is taken to be equal to 15%; e.g., the parameter L_β is bounded in the interval $[-5.4625, -4.0375]$. Then, we assume that the p.d.f. is uniform in these intervals. Finally, taking $\gamma = 0$ we study quadratic stability and we also select weights $R = I$ and $S = 0.01I$.

The uncertain parameters enter into the state space equation in a multiaffine fashion. Therefore, classical quadratic stability can be ascertained solving simultaneously $2^9 = 512$ LMIs corresponding to the vertex set, see e.g. [12]. From the computational point of view, we remark that this may be not an easy task.

We now describe the results of the simulations. The initial condition $Q^0 > 0$ was randomly generated obtaining

$$Q^0 = \begin{bmatrix} 0.6995 & 0.7106 & 0.6822 & 0.5428 \\ 0.7106 & 0.8338 & 0.9575 & 0.6448 \\ 0.6822 & 0.9575 & 1.3380 & 0.8838 \\ 0.5428 & 0.6448 & 0.8838 & 0.8799 \end{bmatrix}.$$

Then, we sequentially generated 800 random matrices Δ^k obtaining a sequence of solutions Q^k according to the iteration given in (14). The final solution Q and the corresponding controller $K = B^T Q^{-1}$ are given by

$$Q = \begin{bmatrix} 0.7560 & -0.0843 & 0.1645 & 0.7338 \\ -0.0843 & 1.0927 & 0.7020 & 0.4452 \\ 0.1645 & 0.7020 & 0.7798 & 0.7382 \\ 0.7338 & 0.4452 & 0.7382 & 1.2162 \end{bmatrix}$$

and

$$K = \begin{bmatrix} 38.6191 & -4.3731 & 43.1284 & -49.9587 \\ -2.8814 & -10.1758 & 10.2370 & -0.4954 \end{bmatrix}.$$

The controller parameter Q satisfies all 512 LMIs given by the vertex set and therefore K is a quadratic stabilizing controller in a classical worst-case sense. Subsequently, we performed a few experiments for different values of relative uncertainty. The conclusion was that 15% is approximately the largest value of uncertainty for which there exists a probabilistic robust quadratic stabilizing controller.

For the sake of comparison, we checked the performance of the standard optimal controller $K_0 = B^T P_0$, where P_0 is obtained solving the Riccati equation for the nominal system

$$A_0^T P_0 + P_0 A_0 - P_0 B B^T P_0 + 0.01I = 0.$$

In this case, the optimal controller K_0 quadratically stabilizes only a subset of 240 vertex systems.

Finally, we performed a few experiments regarding the selection of the value r in the stepsize (15). We observed that its choice was not critical regarding the convergence rate of the algorithm. This convergence rate, however, can be improved by means of a suitable selection of the distribution used to generate the samples Δ^k. Since in this example the parameters enter into the state matrix in a multiaffine fashion, it suffices to randomly generate samples in the vertex set. In turn, from the randomization point of view, this is equivalent to selecting impulse distributions centered at the vertex systems. This observation suggests the use of impulse distributions instead of uniform or other distributions for systems affected by multiaffine uncertainty.

7. Further extensions

In this section, we discuss extensions and ramifications of the approach introduced in this paper. To this end, we observe that the algorithm proposed here enjoys some interesting convergence properties whenever a quadratic stabilizer exists or, in other words, the problem is strictly feasible. This assumption is standard in interior point methods but it may be restrictive in some cases. Therefore, for problems for which a solution of the system of QMIs does not necessarily exist, we define a *measure of violation*. This concept is not new and it has been introduced in [3]. The novelty here is to discuss a different and more effective measure of violation. In particular, as a violation function, we consider the function defined in (13)

$$v(Q,\Delta^k) = \|[V(Q,\Delta^k)]^+\|.$$

An interpretation in terms of violation can be immediately given when $V(Q,\Delta^k)$ is a scalar function. In this case $[V(Q,\Delta^k)]^+$ simply means setting $v(Q,\Delta^k)$ to zero if (11) is satisfied or, otherwise, to $V(Q,\Delta^k)$ if (11) is violated. For the case when $V(Q,\Delta^k)$ is a matrix, this threshold function is replaced with a projection.

Subsequently, if

$$\Delta^1, \Delta^2, \ldots, \Delta^N,$$

are N randomly generated matrices, we study the empirical mean of the violation function $v(Q,\Delta^k)$

$$\bar{v}(Q) \doteq \frac{1}{N}\sum_{k=1}^{N} v(Q,\Delta^k).$$

The final objective is then to minimize the empirical mean

$$v_{inf} = \inf_{Q>0} \bar{v}(Q), \tag{21}$$

which is equivalent to minimize the average violation. It is straightforward to show that this problem is convex. In addition, the set of QMIs (11) is feasible and a common solution exists if and only if

$$v_{inf} = 0.$$

Contrary to the solution proposed for the feasible case, in this section the randomization process is not sequential and the number N of QMIs is fixed a priori. Convergence properties of a sequential algorithm of form (14), with a different stepsize, for minimization of the empirical mean are addressed in [8].

In this paper, we studied the case when the uncertainty is affecting only the state matrix $A = A(\Delta)$ while the input matrix B is fixed. This approach can also be used when the uncertainty enters into $B = B(\Delta)$ in the system

$$\dot{x}(t) = A(\Delta)x(t) + B(\Delta)u(t), \quad x(0) = x_0,\ x(t) \in \mathbf{R}^n,\ u(t) \in \mathbf{R}^m,$$

where $\Delta \in \mathbf{\Delta}$. In this case, we consider the feedback law

$$u(t) = -GQ^{-1}x(t),$$

where both G and Q are design variables but G is not symmetric. Hence, the problem of quadratic stabilization is equivalent to the solution of the LMI

$$A(\Delta)Q + QA^{\mathrm{T}}(\Delta) + B(\Delta)G + G^{\mathrm{T}}B^{\mathrm{T}}(\Delta) < 0$$

for all $\Delta \in \mathbf{\Delta}$. The algorithm introduced in Section 5 can be applied to the pair of matrix variables $\{G, Q\}$ with suitable modifications in the subgradient computations.

We finally remark that many other control problems, including H_∞ norm minimization via state feedback and quadratic stability of linear time varying systems, can be recast in a suitable way, see e.g. [5], to be handled by the stochastic gradient algorithm proposed in this paper.

8. Conclusion

In this paper, we studied randomized algorithms for probabilistic robust design of linear quadratic regulators via convex optimization and gradient-based methods. Current research directions include the extension of the results given here to the case of output feedback in the context of linear parameter varying systems (LPV) and gain scheduling controllers.

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References

[1] B.D.O. Anderson, J.B. Moore, Optimal Control, Linear Quadratic Methods, Prentice-Hall, Englewood Cliffs, NJ, 1989.
[2] E.W. Bai, K.M. Nagpal, R. Tempo, Bounded error parameter estimation: noise models, recursive algorithms and H_∞ optimality, Automatica 32 (1996) 985–999.
[3] B.R. Barmish, P.S. Shcherbakov, On avoiding vertexization of robustness problems: the approximate feasibility concept, Proceedings of the IEEE Conference on Decision and Control, Sydney, Australia, 2000.
[4] V.A. Bondarko, V.A. Yakubovich, The method of recursive aim inequalities in adaptive control theory, Internat. J. Adaptive Control Signal Process. 6 (1992) 141–160.
[5] S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, SIAM, Philadelphia, PA, 1994.
[6] G. Calafiore, F. Dabbene, R. Tempo, Radial and uniform distributions in vector and matrix spaces for probabilistic robustness, in: D.E. Miller, L. Qiu (Eds.), Topics in Control and its Applications, Springer, London, 1999, pp. 17–31 (see also Proceedings of the IEEE Conference on Decision and Control, Tampa, 1998).
[7] G. Calafiore, F. Dabbene, R. Tempo, Randomized algorithms for probabilistic robustness with real and complex structured uncertainty, IEEE Trans. Automat. Control AC-45 (2000) 2218–2235.
[8] G. Calafiore, B. Polyak, Fast algorithms for exact and approximate feasibility of robust LMIs, Proceedings of the IEEE Conference on Decision and Control, Sydney, Australia, 2000; IEEE Trans. Automat. Control, to appear.
[9] X. Chen, K. Zhou, Order statistics and probabilistic robust control, Systems Control Lett. 35 (1998) 175–182.
[10] F. Dabbene, P. Gay, B.T. Polyak, Inner ellipsoidal approximation of membership set: a fast recursive algorithm, Proceedings of the IEEE Conference on Decision and Control, Sydney, Australia, 2000.
[11] V.N. Fomin, Mathematical Theory of Learning Recognizing Systems, LGU, Leningrad, 1976 (in Russian).
[12] H.P. Horisberger, P.R. Belanger, Regulators for linear time invariant plants, IEEE Trans. Automat. Control AC-21 (1976) 705–708.
[13] P.P. Khargonekar, A. Tikku, Randomized algorithms for robust control analysis have polynomial time complexity, Proceedings of the Conference on Decision and Control, Kobe, Japan, 1996.
[14] V. Koltchinskii, C.T. Abdallah, M. Ariola, P. Dorato, D. Panchenko, Improved sample complexity estimates for statistical learning control of uncertain systems, IEEE Trans. Automat. Control AC-45 (2000) 2383–2388.
[15] A.M. Meilakhs, On stabilization of linear control systems under uncertainty, Automat. Remote Control 2 (1975) 182–184.
[16] I.R. Petersen, D.C. McFarlane, Optimal guaranteed cost control and filtering for uncertain linear systems, IEEE Trans. Automat. Control AC-39 (1994) 1971–1977.
[17] B.T. Polyak, Gradient methods for solving equations and inequalities, USSR Comput. Math. and Math. Phys. 4 (6) (1964) 17–32.
[18] B.T. Polyak, Minimization of nonsmooth functionals, USSR Comput. Math. and Math. Phys. 9 (3) (1969) 14–29.
[19] B.T. Polyak, P. Shcherbakov, Random spherical uncertainty in estimation and robustness, IEEE Trans. Automat. Control AC-45 (2000) 2145–2150.
[20] R.E. Skelton, T. Iwasaki, K. Grigoriadis, A Unified Algebraic Approach to Linear Control Design, Taylor & Francis, London, 1997.
[21] R.F. Stengel, L.R. Ray, Stochastic robustness of linear time-invariant control systems, IEEE Trans. Automat. Control AC-36 (1991) 82–87.
[22] R. Tempo, E.-W. Bai, F. Dabbene, Probabilistic robustness analysis: explicit bounds for the minimum number of samples, Systems Control Lett. 30 (1997) 237–242.
[23] J.S. Tyler, F.B. Tuteur, The use of a quadratic performance index to design multivariable invariant plants, IEEE Trans. Automat. Control AC-11 (1966) 84–92.
[24] M. Vidyasagar, Randomized algorithms for robust controller synthesis using statistical learning theory, Automatica, to appear.
[25] M. Vidyasagar, A Theory of Learning and Generalization with Applications to Neural Networks and Control Systems, Springer, London, 1997.
[26] K. Zhou, J.C. Doyle, K. Glover, Robust and Optimal Control, Prentice-Hall, Upper Saddle River, NJ, 1996.
[27] X. Zhu, Y. Huang, J. Doyle, Soft vs. hard bounds in probabilistic robustness analysis, Proceedings of the IEEE Conference on Decision and Control, Kobe, Japan, 1996.