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DETERMINATE SYSTEMS

Suppression of Bounded Exogenous Disturbances: Output Feedback

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Abstract—The paper was devoted to rejection of bounded exogenous disturbances and considered design of the static output feedback minimizing the invariant ellipsoids of the dynamic system. The problems of control analysis and design come to the equivalent conditions in the form of linear matrix inequalities and to the problem of semidefinite programming. The state estimate obtained using the Luenberger observer was used at that.

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1. INTRODUCTION

The present paper continues [1] devoted to rejection of the bounded exogenous disturbances and design of the static state feedback minimizing the invariant ellipsoids of the dynamic system. The problems of control analysis and design were reduced to the equivalent conditions in the form of linear matrix inequalities (LMI) and the problem of semidefinite programming (SDP). The present paper solves the same problem and applies the same approach to the output feedback. At that, the state estimate established using the Luenberger observer [2, 3] is used.

Studies of the difficult problem of rejection the bounded exogenous disturbances were reviewed in detail in [1], and we do not intend to repeat it. The same applies to the approach based on the notion of invariant ellipsoids and the LMI theory. We just note that in the works on LMI [4, 5] consideration was given to the methods of using observers for the output feedback and not to the problems of rejecting the L_∞-bounded disturbances. For example, [5, Ch. 8] used the LMI technique to reject the disturbances bounded in the L_2-norm, that is, decreasing on infinity.

It seems that [6] is the nearest paper in terms of the research area. Therefore, we focus on the differences in the results obtained. First, [6] studied only the continuous problem, whereas here consideraiton is given to the cases of both continuous and discrete times. Second, in contrast to [6] we consider various generalizations of the problem such as the nonzero initial conditions, bounded controls, and various optimality criteria. Third, the present paper aims at using methodically the technique of linear matrix inequalities and reduce the problems to the semidefinite programming format for which powerful computer facilities exist [7,8]. This enables numerical solutions of various problems such as that of the double oscillator used in the paper by way of example.

2. CONTINUOUS SYSTEM

Let us consider the linear continuous control system

$$\begin{cases} \dot{x} = Ax + B_1 u + D_1 w \\ y = C_1 x + D_2 w \\ z = C_2 x + B_2 u, \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times p}$, $B_2 \in \mathbb{R}^{r\times p}$, $D_1 \in \mathbb{R}^{n\times m}$, $D_2 \in \mathbb{R}^{l\times m}$, $C_1 \in \mathbb{R}^{l\times n}$, $C_2 \in \mathbb{R}^{r\times n}$, with the state $x \in \mathbb{R}^n$, observed output $y \in \mathbb{R}^l$, optimized output $z \in \mathbb{R}^r$, control $u \in \mathbb{R}^p$, and exogenous disturbance $w \in \mathbb{R}^m$ bounded at each time instant:[1]

$$\|w(t)\| \leqslant 1, \quad \forall t \geqslant 0.$$

Therefore, we consider the L_∞-bounded exogenous disturbances. We note that no other constraints are imposed on the disturbance $w(t)$ which is not assumed to be either random or harmonic.

The matrix A is not presumed to be stable, but the pair (A, B_1) is assumed to be controllable and also $B_2^{\mathrm{T}} C_2 = 0$. Additionally, we suppose that

$$D_1 D_2^{\mathrm{T}} = 0.$$

Let the system state x be unobservable and the information about the system be represented by its output y. We have to determine the minimum ellipsoid, in a sense, comprising the optimized output z. This philosophy of invariant ellipsoids is described in detail in [1].

Definition 1. The ellipsoid with a center at the origin

$$\mathcal{E}_x = \left\{ x \in \mathbb{R}^n : \quad x^{\mathrm{T}} P^{-1} x \leqslant 1 \right\}, \quad P > 0, \tag{2}$$

is called *state-invariant* in the variable x (in state) for the dynamic system (1) if

(1) $x(t) \in \mathcal{E}_x$ follows from the condition $x(0) \in \mathcal{E}_x$ for all time instants $t \geqslant 0$;

(2) $x(t) \to \mathcal{E}_x$, $t \to \infty$, for $x(0) \notin \mathcal{E}_x$, (at that, for some $T > 0$ it is possible that $x(t) \in \mathcal{E}_x$ for $t \geqslant T$).

Stated differently, any system trajectory outgoing from a point lying inside the ellipsoid $\mathcal{E}_x$ belongs to this ellipsoid at any time instant, and the system trajectory outgoing from a point lying outside the ellipsoid $\mathcal{E}_x$ tends with time to the ellipsoid $\mathcal{E}_x$.

If $\mathcal{E}_x$ is an invariant ellipsoid (2), then for $x(0) \in \mathcal{E}_x$ the output $y = Cx$ belongs to the ellipsoid

$$\mathcal{E}_y = \left\{ y \in \mathbb{R}^l : \quad y^{\mathrm{T}} (CPC^{\mathrm{T}})^{-1} y \leqslant 1 \right\}, \tag{3}$$

and for $x(0) \notin \mathcal{E}_x$ tends to it. It will be called the output-bounding ellipsoid.

The invariant ellipsoids may be regarded as the characterization of the impact of the exogenous disturbances on the trajectories of the dynamic system. In the case at hand, the problem lies in estimating the degree of effect of the exogenous disturbances on the system output vector $z(t)$. In this connection, we are interested in the minimal, in a sense, ellipsoids bounding the system output.

The trace criterion corresponding to the sum of squares of the semiaxes of the bounding ellipsoid in the output of the original system will be considered below as the objective function. Other functions such as the volume or greatest ellipsoid semiaxis may be considered as criteria as well, but the trace criterion is the simplest one in virtue of its linearity. Thus, the degree of influence of the L_∞-bounded exogenous disturbances on the system output is reduced to determination of the minimum-trace bounding ellipsoid.

We note that the notion of invariant ellipsoid is more useful and robust than the reachability set where it is assumed that the initial conditions are zero, but a minor deviation in the initial

[1] Here and below, $\|\cdot\|$ is the Euclidean norm of the vector, I is the identity matrix of the corresponding size, and the matrix inequalities are understood in the sense of matrix sign-definiteness.

condition may drive the trajectory outside the reachability set. For the invariant ellipsoid, it is possible to take into consideration the uncertainty in the initial state

$$x(0) \in \mathcal{E}_0 = \{x\colon\ x^{\mathrm{T}} P_0^{-1} x \leqslant 1\}, \quad P_0 > 0,$$

by requiring that $\mathcal{E}_0 \subset \mathcal{E}_x$, that is,

$$P \geqslant P_0. \tag{4}$$

If the initial condition $x(0) \neq 0$ is defined directly, then, instead of the constraint (4) on the matrix P, added is the condition

$$x^{\mathrm{T}}(0) P^{-1} x(0) \leqslant 1$$

which is obviously representable as the matrix linear inequality

$$\begin{pmatrix} I & x^{\mathrm{T}}(0) \\ x(0) & P \end{pmatrix} \geqslant 0.$$

Let us construct an observer obeying the linear differential equation including the deviation of the output y from the forecast $C_1\widehat{x}$:

$$\dot{\widehat{x}} = A\widehat{x} + B_1 u + F(y - C_1\widehat{x}), \quad F \in \mathbb{R}^{n\times l}. \tag{5}$$

We note that for filtration, that is, state estimation in the absence of control, the properties of observer (5) were studied in [9].

We consider the *residual* $e(t) = x(t) - \widehat{x}(t)$ which, according to (1) and (5), satisfies the differential equation

$$\dot{e} = (A - FC_1)e + (D_1 - FD_2)w.$$

Therefore, construction of the feedback by means of the dynamic controller $u = K\widehat{x}$ leads to the system

$$\begin{cases} \dot{\widehat{x}} = (A + B_1K)\widehat{x} + FC_1 e + FD_2 w \\ \dot{e} = (A - FC_1)e + (D_1 - FD_2)w. \end{cases}$$

We turn to the problem of minimization of the output z of system (1) and obtain

$$z = C_2 x + B_2 u = C_2(\widehat{x} + e) + B_2K\widehat{x} = (C_2 + B_2K)\widehat{x} + C_2 e = \mathcal{C}g,$$

where $\mathcal{C} = \begin{pmatrix} C_2 + B_2K & C_2 \end{pmatrix}$ and $g = \begin{pmatrix} \widehat{x} \\ e \end{pmatrix}$. At that, the vector g is the solution of the differential equation

$$\dot{g} = \underbrace{\begin{pmatrix} A + B_1K & FC_1 \\ 0 & A - FC_1 \end{pmatrix}}_{\widetilde{A}} g + \underbrace{\begin{pmatrix} FD_2 \\ D_1 - FD_2 \end{pmatrix}}_{\widetilde{D}} w. \tag{6}$$

We enclose g in the ellipsoid $\mathcal{E}_g$ generated by the matrix

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \in \mathbb{R}^{2n\times 2n}, \quad P > 0,$$

and use the trace criterion to minimize the ellipsoid bounding the output z generated by the matrix $\mathcal{C}P\mathcal{C}^{\mathrm{T}}$.

Theorem 1. *The solution $\widehat{P}_1$, $\widehat{Q}_2$, $\widehat{Y}_1$, and $\widehat{Y}_2$ of the minimization problem*

$$\operatorname{tr}\left[C_2(P_1+H)C_2^{\mathrm{T}}+B_2Z_1B_2^{\mathrm{T}}\right]\to\min$$

under the constraints

$$\begin{pmatrix} Z+\dfrac{1}{\alpha}R & Y_2C_1-\dfrac{1}{\alpha}R & 0\\ (Y_2C_1)^{\mathrm{T}}-\dfrac{1}{\alpha}R & A^{\mathrm{T}}Q_2+Q_2A-Y_2C_1-C_1^{\mathrm{T}}Y_2^{\mathrm{T}}+\alpha Q_2+\dfrac{1}{\alpha}R & Q_2D_1\\ 0 & D_1^{\mathrm{T}}Q_2 & -\alpha I\end{pmatrix}\leqslant 0,$$

$$\begin{pmatrix} 2Q_2+Z & I\\ I & -(AP_1+P_1A^{\mathrm{T}}+B_1Y_1+Y_1^{\mathrm{T}}B_1^{\mathrm{T}}+\alpha P_1)\end{pmatrix}\geqslant 0,$$

$$\begin{pmatrix} R & Y_2D_2\\ D_2^{\mathrm{T}}Y_2^{\mathrm{T}} & I\end{pmatrix}\geqslant 0,\quad \begin{pmatrix} Z_1 & Y_1\\ Y_1^{\mathrm{T}} & P_1\end{pmatrix}\geqslant 0,\quad \begin{pmatrix} H & I\\ I & Q_2\end{pmatrix}\geqslant 0,$$

where minimization is carried out with respect to the matrix variables $P_1=P_1^{\mathrm{T}}\in\mathbb{R}^{n\times n}$, $Q_2=Q_2^{\mathrm{T}}\in\mathbb{R}^{n\times n}$, $Y_1\in\mathbb{R}^{m\times n}$, $Y_2\in\mathbb{R}^{n\times l}$, $Z=Z^{\mathrm{T}}\in\mathbb{R}^{n\times n}$, $Z_1=Z_1^{\mathrm{T}}\in\mathbb{R}^{p\times p}$, $R=R^{\mathrm{T}}\in\mathbb{R}^{n\times n}$, $H=H^{\mathrm{T}}\in\mathbb{R}^{n\times n}$ and the numerical parameter $\alpha\in\mathbb{R}$ defines the matrix $\widehat{\mathcal{C}}\,\widehat{P}\,\widehat{\mathcal{C}}^{\mathrm{T}}$ of the bounding ellipsoid for the optimized output of system (1), *where*

$$\widehat{\mathcal{C}}=\begin{pmatrix} C_2+B_2\widehat{K} & C_2\end{pmatrix},\quad \widehat{P}=\begin{pmatrix}\widehat{P}_1 & 0\\ 0 & \widehat{Q}_2^{-1}\end{pmatrix},$$

as well as the dynamic controller

$$\widehat{K}=\widehat{Y}_1\widehat{P}_1^{-1}$$

and the observer matrix

$$\widehat{F}=\widehat{Q}_2^{-1}\widehat{Y}_2$$

corresponding to this invariant ellipsoid.

Theorem 1 is proved in the Appendix.

We note that under a fixed α the problem comes to minimizing the linear function under constraints representing linear matrix inequalities, that is, to semidefinite programming belonging to the class of convex optimization problems. There exist numerous packages to solve it such as SeDuMi Toolbox and YALMIP Toolbox for the MATLAB system [7,8].

As follows from the structure of the matrix $\widehat{P}$, the matrix $\widehat{P}_1$ defines the invariant ellipsoid for the observer $\widehat{x}$, and the matrix $\widehat{Q}_2^{-1}$ defines the invariant ellipsoid for the mismatch e.

Within the framework of this approach to rejection of the exogenous disturbances, it is only natural to require constraints on control. Let $u\in\mathbb{R}^p$, $\mu>0$, and

$$\|u\|\leqslant\mu. \tag{7}$$

The following lemma [4] reduces the constraint (7) to consideration of the equivalent linear matrix inequality.

Lemma 1. *Let P be the matrix of the state-invariant ellipsoid for a linear system with control like $u=Kx$. Let also $Y=KP$. Then, for the matrices P and Y, the constraint* (7) *is equivalent to satisfaction of the linear matrix inequality*

$$\begin{pmatrix} P & Y^{\mathrm{T}}\\ Y & \mu^2I\end{pmatrix}\geqslant 0.$$

Lemma 1 gives rise to the following corollary.

Corollary. *For system* (1), *constraint* (7) *amounts to satisfying the linear matrix inequality*

$$\begin{pmatrix} P_1 & Y_1^{\mathrm{T}} \\ Y_1 & \mu^2 I \end{pmatrix} \geqslant 0 \tag{8}$$

for the matrices P_1 *and* Y_1.

Therefore, in the presence of a control constraint like (7), LMI (8) is added to the LMI conditions of Theorem 1.

3. CONTROL OF THE DOUBLE OSCILLATOR: CONTINUOUS CASE

We illustrate the above approach to the ellipsoid-based invariant rejection of the exogenous disturbances by way of example of control of the double oscillator, that is, a system of two solid bodies of masses m_1 and m_2 connected by a spring with the elastic stiffness k and sliding without friction along a fixed horizontal rod (see Fig. 1).

The control action $u \in \mathbb{R}$ is applied to the left body with the aim of compensating the effect of exogenous disturbance

$$w = \begin{pmatrix} w_1 & w_2 \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^2$$

whose components act, respectively, on the left and right bodies. The disturbance is assumed to be arbitrary, but bounded at any time instant: $\|w(t)\| \leqslant 1$.

We denote by x_1, v_1 and x_2, v_2 the coordinates and velocities, respectively, of the left and right bodies. Then,

$$x = \begin{pmatrix} x_1 & x_2 & v_1 & v_2 \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^4$$

is the phase state vector of the considered dynamic system completely describing its behavior.

The vector

$$y = \begin{pmatrix} x_1 & x_2 + w_3 \end{pmatrix}^{\mathrm{T}}$$

is used as the observed system output, and the vector

$$z = \begin{pmatrix} u & x_2 \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^2$$

characterized by the value of control and the coordinates of the right body on which the control does not act directly is used as the minimized system output.

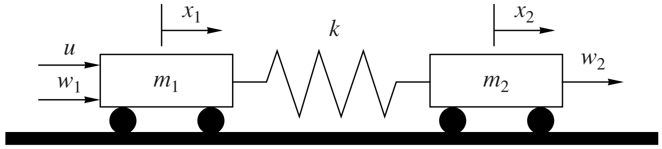

Fig. 1. Double oscillator.

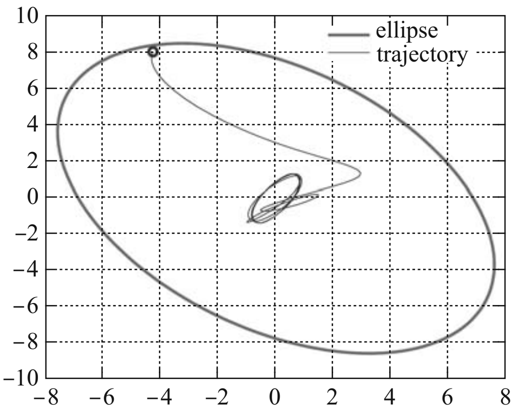

Fig. 2. Output-bounding ellipse.

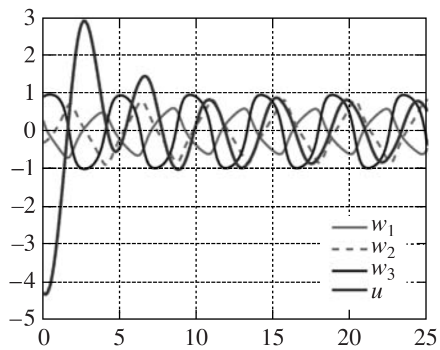

Fig. 3. Disturbances $w^*(t)$ and control $u(t)$.

The continuous model of the disturbed system oscillations obeys Eqs. (1) where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k}{m_1} & \dfrac{k}{m_1} & 0 & 0 \\ \dfrac{k}{m_2} & -\dfrac{k}{m_2} & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \dfrac{1}{m_1} \\ 0 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \dfrac{1}{m_1} & 0 & 0 \\ 0 & \dfrac{1}{m_2} & 0 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The optimal controller $\widehat{K}$ and the observer matrix $\widehat{F}$ minimizing (for the trace criterion) the bounding output ellipse were established using Theorem 1 for the unit values of the parameters. For numerical solution of the problem of semidefinite programming, SeDuMi Toolbox and YALMIP Toolbox based on the Matlab system were used. As the result, for the system at hand we obtained:

$$\widehat{K} \approx \begin{pmatrix} -1.5925 & 0.2679 & -1.9023 & -1.3278 \end{pmatrix},$$

$$\widehat{F} \approx \begin{pmatrix} 1.3747 & 0.0822 \\ 0.2346 & 1.1966 \\ 1.1805 & 0.1155 \\ 0.3140 & 0.5991 \end{pmatrix},$$

$$\widehat{P}_1 \approx \begin{pmatrix} 45.2779 & 31.0798 & -9.4031 & -19.0728 \\ 31.0798 & 66.1396 & 6.4236 & -12.3619 \\ -9.4031 & 6.4236 & 30.6355 & -13.5222 \\ -19.0728 & -12.3619 & -13.5222 & 32.4699 \end{pmatrix},$$

$$\widehat{Q}_2 \approx \begin{pmatrix} 0.2555 & -0.0894 & -0.0609 & -0.0174 \\ -0.0894 & 0.2434 & -0.0214 & -0.0892 \\ -0.0609 & -0.0214 & 0.1410 & 0.0240 \\ -0.0174 & -0.0892 & 0.0240 & 0.1760 \end{pmatrix}.$$

Figure 2 depicts the minimal output bounding ellipse and the trajectory $z(t)$ for some choice of the initial position inside this ellipse and some exogenous disturbances $w(t)$. Figure 3 shows the graphs of exogenous disturbances $w_1(t)$, $w_2(t)$, $w_3(t)$ the control $u(t)$.

4. DISCRETE SYSTEM

Let us consider the discrete linear control system

$$\begin{cases} x_{k+1} = Ax_k + B_1 u_k + D_1 w_k \\ y_k = C_1 x_k + D_2 w_k \\ z_k = C_2 x_k + B_2 u_k, \end{cases} \tag{9}$$

where $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times p}$, $B_2 \in \mathbb{R}^{l\times p}$, $D_1 \in \mathbb{R}^{n\times m}$, $D_2 \in \mathbb{R}^{l\times m}$, $C_1 \in \mathbb{R}^{l\times n}$, $C_2 \in \mathbb{R}^{n\times r}$ with the state $x_k \in \mathbb{R}^n$, observed output $y_k \in \mathbb{R}^l$, optimized output $z_k \in \mathbb{R}^r$, control $u_k \in \mathbb{R}^p$, and exogenous disturbance $w_k \in \mathbb{R}^m$ bounded at all time instants:

$$\|w_k\| \leqslant 1, \quad k = 0, 1, 2, \ldots .$$

Thus, consideration is given to the l_∞-bounded exogenous disturbances.

The matrix A is not assumed to be stable. However, we assume that the pair (A, B_1) is controllable and also $B_2^{\mathrm{T}} C_2 = 0$. Additionally, we assume that[2] $D_1 D_2^{\mathrm{T}} = 0$.

Let the system state x_k be unobservable and the information about the system represented by its output y_k. The problem lies in determining the minimal, in a sense, ellipsoid comprising the optimized output z_k.

As in the continuous case, we define the invariant ellipsoid.

Definition 2. The ellipsoid with a center at the origin

$$\mathcal{E}_x = \left\{ x_k \in \mathbb{R}^n : \ x_k^{\mathrm{T}} P^{-1} x_k \leqslant 1 \right\}, \quad P > 0, \tag{10}$$

is called *invariant* in the variable x_k (in state) for the dynamic system (9) if

(1) $x_k \in \mathcal{E}_x$ follows from the condition $x_0 \in \mathcal{E}_x$ for all time instants $k = 1, 2, \ldots;$

(2) $x_k \to \mathcal{E}_x$, $k \to \infty$, for $x_0 \notin \mathcal{E}_x$ (at that, for $k \geqslant K$, possibly, $x_k \in \mathcal{E}_x$ for some $K \in \mathbb{N}$).

Let us construct an observer following the linear difference equation including the deviation of the output y_k from its forecast $C_1 \widehat{x}_k$:

$$\widehat{x}_{k+1} = A\widehat{x}_k + B_1 u_k + F(y_k - C_1 \widehat{x}_k), \quad F \in \mathbb{R}^{n\times l}.$$

We consider the residual $e_k = x_k - \widehat{x}_k$ satisfying the difference equation

$$e_{k+1} = (A - FC_1)e_k + (D_1 - FD_2)w_k.$$

Construction of the feedback with the use of the dynamic controller $u_k = K\widehat{x}_k$ provides the system

$$\begin{cases} \widehat{x}_{k+1} = (A + B_1 K)\widehat{x}_k + FC_1 e_k + FD_2 w_k \\ e_{k+1} = (A - FC_1)e_k + (D_1 - FD_2)w_k. \end{cases}$$

Now, we turn to the problem of minimization of the output z of system (9) and obtain

$$z_k = C_2 x_k + B_2 u_k = C_2(\widehat{x}_k + e_k) + B_2 K\widehat{x}_k = (C_2 + B_2 K)\widehat{x}_k + C_2 e_k = \mathcal{C} g_k,$$

[2] In contrast to the continuous case, the following Theorem 2 is valid even if this condition is not met.

where $\mathcal{C} = \begin{pmatrix} C_2 + B_2 K & C_2 \end{pmatrix}$ and $g_k = \begin{pmatrix} \widehat{x}_k \\ e_k \end{pmatrix}$. At that, the vector g_k is the solution of the difference equation

$$g_{k+1} = \underbrace{\begin{pmatrix} A + B_1 K & FC_1 \\ 0 & A - FC_1 \end{pmatrix}}_{\widetilde{A}} g_k + \underbrace{\begin{pmatrix} FD_2 \\ D_1 - FD_2 \end{pmatrix}}_{\widetilde{D}} w_k. \tag{11}$$

We include g_k in the ellipsoid $\mathcal{E}_g$ generated by the matrix

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \in \mathbb{R}^{2n\times 2n}, \quad P > 0,$$

and minimize (for the trace criterion) the output bounding ellipsoid z_k generated by the matrix $\mathcal{C}P\mathcal{C}^{\mathrm{T}}$.

Theorem 2. *The solution $\widehat{P}_1$, $\widehat{Q}_2$, $\widehat{Y}_1$, and $\widehat{Y}_2$ of the minimization problem*

$$\operatorname{tr}\left[C_2(P_1 + H)C_2^{\mathrm{T}} + B_2 Z_1 B_2^{\mathrm{T}}\right] \to \min$$

under the constraints

$$\begin{pmatrix} Z & Y_2 C_1 & Y_2 D_2 \\ C_1^{\mathrm{T}} Y_2^{\mathrm{T}} & \Lambda_1 + C_1^{\mathrm{T}} Z_2 C_1 & \Lambda_2 + C_1^{\mathrm{T}} Z_2 D_2 \\ D_2^{\mathrm{T}} Y_2^{\mathrm{T}} & \Lambda_2^{\mathrm{T}} + D_2^{\mathrm{T}} Z_2 C_1 & \Lambda_3 + D_2^{\mathrm{T}} Z_2 D_2 \end{pmatrix} \leqslant 0,$$

$$\begin{pmatrix} 2Q_2 + Z & I \\ I & -\dfrac{1}{\alpha}\left(AP_1A^{\mathrm{T}} + B_1Y_1A^{\mathrm{T}} + AY_1^{\mathrm{T}}B_1^{\mathrm{T}} + B_1Z_1B_1^{\mathrm{T}}\right) + P_1 \end{pmatrix} \geqslant 0,$$

$$\begin{pmatrix} Z_1 & Y_1 \\ Y_1^{\mathrm{T}} & P_1 \end{pmatrix} \geqslant 0, \quad \begin{pmatrix} Z_2 & Y_2^{\mathrm{T}} \\ Y_2 & Q_2 \end{pmatrix} \geqslant 0, \quad \begin{pmatrix} H & I \\ I & Q_2 \end{pmatrix} \geqslant 0,$$

where

$$\Lambda_1 = A^{\mathrm{T}} Q_2 A - A^{\mathrm{T}} Y_2 C_1 - C_1^{\mathrm{T}} Y_2^{\mathrm{T}} A - \alpha Q_2,$$
$$\Lambda_2 = A^{\mathrm{T}} Q_2 D_1 - C_1^{\mathrm{T}} Y_2^{\mathrm{T}} D_1 - A^{\mathrm{T}} Y_2 D_2,$$
$$\Lambda_3 = D_1^{\mathrm{T}} Q_2 D_1 - D_2^{\mathrm{T}} Y_2^{\mathrm{T}} D_1 - D_1^{\mathrm{T}} Y_2 D_2 - (1-\alpha) I,$$

and the minimization is carried out with respect to the matrix variables $P_1 = P_1^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, $Q_2 = Q_2^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, $Y_1 \in \mathbb{R}^{m\times n}$, $Y_2 \in \mathbb{R}^{n\times l}$, $Z = Z^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, $Z_1 = Z_1^{\mathrm{T}} \in \mathbb{R}^{p\times p}$, $Z_2 = Z_2^{\mathrm{T}} \in \mathbb{R}^{l\times l}$, $H = H^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ and the numerical parameter $\alpha \in \mathbb{R}$, defines the matrix $\widehat{\mathcal{C}}\widehat{P}\widehat{\mathcal{C}}^{\mathrm{T}}$ of the invariant ellipsoid for the optimized output of the system (9), *where*

$$\widehat{\mathcal{C}} = \begin{pmatrix} C_2 + B_2\widehat{K} & C_2 \end{pmatrix}, \quad \widehat{P} = \begin{pmatrix} \widehat{P}_1 & 0 \\ 0 & \widehat{Q}_2^{-1} \end{pmatrix},$$

as well as the dynamic controller

$$\widehat{K} = \widehat{Y}_1 \widehat{P}_1^{-1}$$

and the observer matrix

$$\widehat{F} = \widehat{Q}_2^{-1} \widehat{Y}_2$$

corresponding to this invariant ellipsoid.

Theorem 2 is proved in the Appendix.

We note that for a fixed α we again encounter the semidefinite programming problem as in the continuous case.

5. CONTROL OF THE DOUBLE OSCILLATOR: DISCRETE CASE

We demonstrate the proposed approach to rejection of the exogenous disturbances using the invariant ellipsoids by way of example of the discrete control problem for the double oscillator obtained by discretizing the above case of the continuous system.

The discrete model of the disturbed system oscillations obeys the equations

$$\begin{cases} x_{k+1} = A_d x_k + B_{1d} u_k + D_{1d} w_k \\ y_k = C_1 x_k + D_2 w_k \\ z_k = C_2 x_k + B_2 u_k, \end{cases} \tag{12}$$

where $A_d = e^{A\Delta}$, $B_{1d} = \int\limits_0^\Delta e^{sA} B_1\, ds$, $D_{1d} = \int\limits_0^\Delta e^{sA} D_1\, ds$.

For the unit parameters of the original continuous system and $\Delta = 0.1000$, the optimal controller $\widehat{K}$ and the observer matrix $\widehat{F}$ minimizing (for the trace criterion) the output bounding ellipse were determined using Theorem 2. For the numerical solution of the semidefinite programming problem, the Matlab-based SeDuMi Toolbox and YALMIP Toolbox were used at that. As the result, we obtained for the system under study:

$$\widehat{K} \approx \begin{pmatrix} -1.7906 & 0.4197 & -2.0575 & -1.3572 \end{pmatrix},$$

$$\widehat{F} \approx \begin{pmatrix} 0.4263 & 0.0006 \\ 0.3753 & 0.1042 \\ 0.7814 & 0.0356 \\ 0.0465 & 0.0467 \end{pmatrix}, \quad \widehat{P}_1 \approx \begin{pmatrix} 84.0712 & 51.6668 & -15.3474 & -51.6557 \\ 51.6668 & 116.0064 & 31.6849 & -29.1613 \\ -15.3474 & 31.6849 & 64.0205 & -15.5231 \\ -51.6557 & -29.1613 & -15.5231 & 64.2882 \end{pmatrix},$$

$$\widehat{Q}_2 \approx \begin{pmatrix} 4.8560 & -0.0356 & -1.0556 & 0.0637 \\ -0.0356 & 0.4737 & -0.2407 & -0.1773 \\ -1.0556 & -0.2407 & 0.7502 & 0.0496 \\ 0.0637 & -0.1773 & 0.0496 & 0.2895 \end{pmatrix}.$$

Figure 4 depicts the minimal output bounding ellipse and the trajectory $z(t)$ for some choice of the initial position within the ellipse and the exogenous disturbances

$$w_k = \frac{1}{\sqrt{3}} \begin{pmatrix} \operatorname{sgn}(x_{1k} + 1.6x_{2k}) \\ \operatorname{sgn}(x_{3k} + 1.6x_{4k}) \\ \operatorname{sgn} \sin(k/10) \end{pmatrix}.$$

The graphs of exogenous disturbances w_{1k}, w_{2k}, and w_{3k} and the control u_k are plotted in Fig. 5.

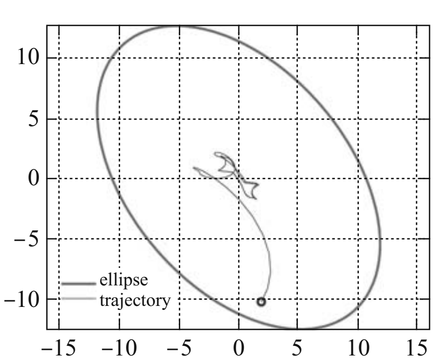

Fig. 4. Output-bounding ellipse.

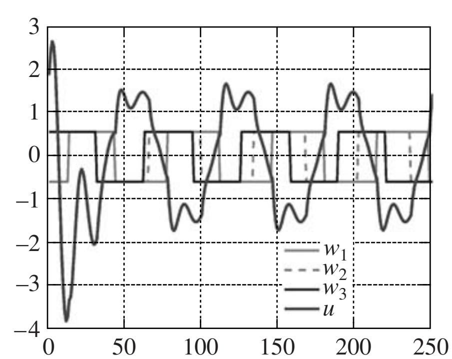

Fig. 5. Disturbances w_k and control u_k.

6. CONCLUSIONS

The paper proposed a simple and universal approach to rejection of arbitrary bounded exogenous disturbances relying on the observer-based linear output feedback. The approach is founded on the method of invariant ellipsoid enabling one to reduce the design of optimal controller and observer to the search of the least invariant ellipsoid of the closed dynamic system. The concept of invariant ellipsoids enables one to restate the original problem in terms of the linear matrix inequalities and reduce the controller design itself immediately to the problems of semidefinite programming and one-dimensional minimization which readily yield to numerical solution. Efficiency of this method was demonstrated by the example of control of the double oscillator. Consideration was given equally to the continuous and discrete cases.

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APPENDIX

Proposition A.1 (S-procedure). *Let the uniform quadratic forms* $f_i(x) = x^{\mathrm{T}} A_i x$, $i = 0, 1, \ldots, m$, *in* $\mathbb{R}^n$ *and the numbers* $\alpha_0, \alpha_1, \ldots, \alpha_m \in \mathbb{R}$ *be given. If there exist real numbers* $\tau_i \geqslant 0$, $i = 1, \ldots, m$, *such that*

$$A_0 \leqslant \sum_{i=1}^{m} \tau_i A_i, \quad \alpha_0 \geqslant \sum_{i=1}^{m} \tau_i \alpha_i, \tag{A.1}$$

then it follows from

$$f_i(x) \leqslant \alpha_i, \quad i = 1, \ldots, m, \tag{A.2}$$

that

$$f_0(x) \leqslant \alpha_0. \tag{A.3}$$

Inversely, if (A.3) *follows from* (A.2) *and any of the conditions*

(a) $m = 1$;

(b) $m = 2$, $n \geqslant 3$

is satisfied, and there exist the numbers $\mu_1, \mu_2 \in \mathbb{R}$ *and vector* $x^0 \in \mathbb{R}^n$ *such that*

$$\mu_1 A_1 + \mu_2 A_2 > 0, \quad f_1(x^0) < \alpha_1, \quad f_2(x^0) < \alpha_2,$$

then there exist numbers $\tau_i \geqslant 0$, $i = 1, \ldots, m$, *for which inequalities* (A.1) *are valid.*

The full proof of this statement can be found in [10].

Lemma A.1. (1) *If* $\begin{pmatrix} A & B \\ B^{\mathrm{T}} & C \end{pmatrix} \leqslant 0$ *and* $D = D^{\mathrm{T}} \leqslant A$, *then* $\begin{pmatrix} D & B \\ B^{\mathrm{T}} & C \end{pmatrix} \leqslant 0$.

(2) *If* $\begin{pmatrix} A + E^{\mathrm{T}} DE & B + E^{\mathrm{T}} DF \\ B^{\mathrm{T}} + F^{\mathrm{T}} DE & C + F^{\mathrm{T}} DF \end{pmatrix} \leqslant 0$ *and* $G = G^{\mathrm{T}} \leqslant D = D^{\mathrm{T}}$, *then*

$$\begin{pmatrix} A + E^{\mathrm{T}} GE & B + E^{\mathrm{T}} GF \\ B^{\mathrm{T}} + F^{\mathrm{T}} GE & C + F^{\mathrm{T}} GF \end{pmatrix} \leqslant 0.$$

(3) *If* $\begin{pmatrix} A+E^{\mathrm T}DE & B-E^{\mathrm T}DF \\ B^{\mathrm T}-F^{\mathrm T}DE & C+F^{\mathrm T}DF \end{pmatrix} \leqslant 0$ *and* $G=G^{\mathrm T}\leqslant D=D^{\mathrm T}$, *then*

$$\begin{pmatrix} A+E^{\mathrm T}GE & B-E^{\mathrm T}GF \\ B^{\mathrm T}-F^{\mathrm T}GE & C+F^{\mathrm T}GF \end{pmatrix} \leqslant 0.$$

(4)
$$Y^{\mathrm T}X^{-1}Y \geqslant Y+Y^{\mathrm T}-X, \quad X>0, \tag{A.4}$$

where X *is the quadratic invertible matrix and* Y *is the quadratic matrix of the corresponding size.*

Proof. (1) Since $D\leqslant A$,

$$\begin{pmatrix} D & B \\ B^{\mathrm T} & C \end{pmatrix} - \begin{pmatrix} A & B \\ B^{\mathrm T} & C \end{pmatrix} = \begin{pmatrix} D-A & 0 \\ 0 & 0 \end{pmatrix} \leqslant 0.$$

(2) Since $G-D\leqslant 0$,

$$\begin{pmatrix} A+E^{\mathrm T}GE & B+E^{\mathrm T}GF \\ B^{\mathrm T}+F^{\mathrm T}GE & C+F^{\mathrm T}GF \end{pmatrix} - \begin{pmatrix} A+E^{\mathrm T}DE & B+E^{\mathrm T}DF \\ B^{\mathrm T}+F^{\mathrm T}DE & C+F^{\mathrm T}DF \end{pmatrix}$$
$$= \begin{pmatrix} E^{\mathrm T} \\ F^{\mathrm T} \end{pmatrix} (G-D) \begin{pmatrix} E & F \end{pmatrix} \leqslant 0.$$

(3) Since $G-D\leqslant 0$,

$$\begin{pmatrix} A+E^{\mathrm T}GE & B-E^{\mathrm T}GF \\ B^{\mathrm T}-F^{\mathrm T}GE & C+F^{\mathrm T}GF \end{pmatrix} - \begin{pmatrix} A+E^{\mathrm T}DE & B-E^{\mathrm T}DF \\ B^{\mathrm T}-F^{\mathrm T}DE & C+F^{\mathrm T}DF \end{pmatrix}$$
$$= \begin{pmatrix} E^{\mathrm T} \\ -F^{\mathrm T} \end{pmatrix} (G-D) \begin{pmatrix} E & -F \end{pmatrix} \leqslant 0.$$

(4) Since $X>0$,

$$(Y-X)^{\mathrm T}X^{-1}(Y-X) = Y^{\mathrm T}X^{-1}Y - Y - Y^{\mathrm T} + X \geqslant 0.$$

The lemma is proved. □

Lemma A.2. (matrix inversion lemma) [11].

$$\left(X-YZY^{\mathrm T}\right)^{-1} = X^{-1} + X^{-1}Y\left(Z^{-1}-Y^{\mathrm T}X^{-1}Y\right)^{-1}Y^{\mathrm T}X^{-1}, \tag{A.5}$$

where X *and* Z *are invertible quadratic matrices,* Y *is a matrix of the corresponding size, and the matrix* $X-YZY^{\mathrm T}$ *is invertible.*

Proof of Theorem 1. Let us consider the quadratic function

$$V(g) = g^{\mathrm T}Qg, \quad Q=P^{-1} = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} \in \mathbb{R}^{2n\times 2n}$$

constructed on the solutions of Eq. (6). Then,

$$\dot V(g) = \left(\widetilde A g + \widetilde D w\right)^{\mathrm T} Qg + g^{\mathrm T}Q\left(\widetilde A g + \widetilde D w\right) = g^{\mathrm T}\left(\widetilde A^{\mathrm T}Q + Q\widetilde A\right)g + 2g^{\mathrm T}Q\widetilde D w.$$

For the trajectories of system (6) to remain within the boundaries of the ellipsoid $\mathcal{E}_g$, we require that $\dot{V}(g) \leqslant 0$ be satisfied for $V(g) \geqslant 1$ and $w^{\mathrm{T}} w \leqslant 1$, that is,

$$g^{\mathrm{T}}\left(\widetilde{A}^{\mathrm{T}} Q+Q \widetilde{A}\right) g+2 w^{\mathrm{T}} \widetilde{D}^{\mathrm{T}} Q g \leqslant 0, \quad \forall(g, w): \; g^{\mathrm{T}} Q g \geqslant 1, \quad w^{\mathrm{T}} w \leqslant 1. \tag{A.6}$$

Let $s=\begin{pmatrix} g & w \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^{2n+m}$ and

$$M_0=\begin{pmatrix} \widetilde{A}^{\mathrm{T}} Q+Q \widetilde{A} & Q \widetilde{D} \\ \widetilde{D}^{\mathrm{T}} Q & 0 \end{pmatrix}, \quad M_1=\begin{pmatrix} -Q & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2=\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix},$$

and also $\widetilde{f}_i(s)=s^{\mathrm{T}} M_i s$, $i=0,1,2$. Then, (A.6) is rearranged in the form

$$\widetilde{f}_0(s) \leqslant 0, \quad \forall s: \; \widetilde{f}_1(s) \leqslant-1, \quad \widetilde{f}_2(s) \leqslant 1.$$

Since conditions (b) of Proposition A.1 are satisfied, under some values of α, β such that $\alpha \geqslant \beta \geqslant 0$ or

$$\begin{pmatrix} \widetilde{A}^{\mathrm{T}} Q+Q \widetilde{A}+\alpha Q & Q \widetilde{D} \\ \widetilde{D}^{\mathrm{T}} Q & -\beta I \end{pmatrix} \leqslant 0$$

(A.6) is equivalent to the linear matrix inequality

$$M_0 \leqslant \alpha M_1+\beta M_2.$$

By pre- and post-multiplying the matrix $\begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix}$, we obtain

$$\begin{pmatrix} P \widetilde{A}^{\mathrm{T}}+\widetilde{A} P+\alpha P & \widetilde{D} \\ \widetilde{D}^{\mathrm{T}} & -\beta I \end{pmatrix} \leqslant 0,$$

or, by the Schur lemma,

$$P \widetilde{A}^{\mathrm{T}}+\widetilde{A} P+\alpha P+\frac{1}{\beta} \widetilde{D} \widetilde{D}^{\mathrm{T}} \leqslant 0. \tag{A.7}$$

Therefore, the condition for invariance of the ellipsoid with the matrix $P>0$ is equivalent to satisfying the last matrix inequality for some $\alpha \geqslant \beta>0$. Since it is the minimal ellipsoids that are sought,

$$\beta=\beta_{\max}=\alpha.$$

Inequality (A.7) takes the form

$$\begin{pmatrix} \Psi_1 & F C_1 P_2-\dfrac{1}{\alpha} F D_2 D_2^{\mathrm{T}} F^{\mathrm{T}} \\ (F C_1 P_2)^{\mathrm{T}}-\dfrac{1}{\alpha} F D_2 D_2^{\mathrm{T}} F^{\mathrm{T}} & \Psi_2 \end{pmatrix} \leqslant 0,$$

where

$$\Psi_1=P_1(A+B_1 K)^{\mathrm{T}}+(A+B_1 K) P_1+\alpha P_1+\frac{1}{\alpha} F D_2 D_2^{\mathrm{T}} F^{\mathrm{T}},$$

$$\Psi_2=P_2(A-F C_1)^{\mathrm{T}}+(A-F C_1) P_2+\alpha P_2+\frac{1}{\alpha}\left(D_1 D_1^{\mathrm{T}}+F D_2 D_2^{\mathrm{T}} F^{\mathrm{T}}\right).$$

By pre- and post-multiplying it by the matrix $\begin{pmatrix} Q_2 & 0 \\ 0 & Q_2 \end{pmatrix}$ and introducing the matrix variables

$$Y_1 = KP_1, \quad Y_2 = Q_2F,$$

we eliminate K and F:

$$\begin{pmatrix} Q_2\Psi_3Q_2 + \frac{1}{\alpha}Y_2D_2D_2^{\mathrm{T}}Y_2^{\mathrm{T}} & Y_2C_1 - \frac{1}{\alpha}Y_2D_2D_2^{\mathrm{T}}Y_2^{\mathrm{T}} \\ (Y_2C_1)^{\mathrm{T}} - \frac{1}{\alpha}Y_2D_2D_2^{\mathrm{T}}Y_2^{\mathrm{T}} & \Psi_4 + \frac{1}{\alpha}\left(Q_2D_1D_1^{\mathrm{T}}Q_2 + Y_2D_2D_2^{\mathrm{T}}Y_2^{\mathrm{T}}\right) \end{pmatrix} \leqslant 0$$

or

$$\begin{pmatrix} Q_2\Psi_3Q_2 + \frac{1}{\alpha}Y_2D_2D_2^{\mathrm{T}}Y_2^{\mathrm{T}} & Y_2C_1 - \frac{1}{\alpha}Y_2D_2D_2^{\mathrm{T}}Y_2^{\mathrm{T}} & 0 \\ (Y_2C_1)^{\mathrm{T}} - \frac{1}{\alpha}Y_2D_2D_2^{\mathrm{T}}Y_2^{\mathrm{T}} & \Psi_4 + \frac{1}{\alpha}Y_2D_2D_2^{\mathrm{T}}Y_2^{\mathrm{T}} & Q_2D_1 \\ 0 & D_1^{\mathrm{T}}Q_2 & -\alpha I \end{pmatrix} \leqslant 0, \tag{A.8}$$

where

$$\Psi_3 = AP_1 + P_1A^{\mathrm{T}} + B_1Y_1 + Y_1^{\mathrm{T}}B_1^{\mathrm{T}} + \alpha P_1,$$
$$\Psi_4 = A^{\mathrm{T}}Q_2 + Q_2A - Y_2C_1 - C_1^{\mathrm{T}}Y_2^{\mathrm{T}} + \alpha Q_2.$$

We introduce the matrix variable $R = R^{\mathrm{T}}$. According to Lemma A.1, if

$$Y_2D_2D_2^{\mathrm{T}}Y_2^{\mathrm{T}} \leqslant R \tag{A.9}$$

and

$$\begin{pmatrix} Q_2\Psi_3Q_2 + \frac{1}{\alpha}R & Y_2C_1 - \frac{1}{\alpha}R & 0 \\ (Y_2C_1)^{\mathrm{T}} - \frac{1}{\alpha}R & \Psi_4 + \frac{1}{\alpha}R & Q_2D_1 \\ 0 & D_1^{\mathrm{T}}Q_2 & -\alpha I \end{pmatrix} \leqslant 0, \tag{A.10}$$

then inequality (A.8) is valid as well. We note that condition (A.9) is equivalent to the linear matrix inequality

$$\begin{pmatrix} R & Y_2D_2 \\ D_2^{\mathrm{T}}Y_2^{\mathrm{T}} & I \end{pmatrix} \geqslant 0. \tag{A.11}$$

We introduce the matrix variable $Z = Z^{\mathrm{T}}$. By Lemma A.1, if

$$\begin{pmatrix} Z + \frac{1}{\alpha}R & Y_2C_1 - \frac{1}{\alpha}R & 0 \\ (Y_2C_1)^{\mathrm{T}} - \frac{1}{\alpha}R & A^{\mathrm{T}}Q_2 + Q_2A - Y_2C_1 - C_1^{\mathrm{T}}Y_2^{\mathrm{T}} + \alpha Q_2 + \frac{1}{\alpha}R & Q_2D_1 \\ 0 & D_1^{\mathrm{T}}Q_2 & -\alpha I \end{pmatrix} \leqslant 0 \tag{A.12}$$

and

$$Q_2\Psi_3Q_2 \leqslant Z, \tag{A.13}$$

then the matrix inequality (A.10) is satisfied as well.

By assuming in inequality (A.4) that

$$X = -\Psi_3^{-1}, \quad Y = Q_2,$$

we obtain

$$Q_2 \Psi_3 Q_2 \leqslant -2Q_2 - \Psi_3^{-1}.$$

Therefore, the matrix inequality (A.13) is satisfied if valid is the inequality

$$-2Q_2 - \Psi_3^{-1} \leqslant Z$$

which is equivalent to the linear matrix inequality

$$\begin{pmatrix} 2Q_2 + Z & I \\ I & -\left(AP_1 + P_1A^{\mathrm{T}} + B_1Y_1 + Y_1^{\mathrm{T}}B_1^{\mathrm{T}} + \alpha P_1\right) \end{pmatrix} \geqslant 0. \tag{A.14}$$

At that,

$$\begin{aligned} \mathcal{C}P\mathcal{C}^{\mathrm{T}} &= \begin{pmatrix} C_2 + B_2K & C_2 \end{pmatrix} \begin{pmatrix} P_1 & 0 \\ 0 & Q_2^{-1} \end{pmatrix} \begin{pmatrix} (C_2 + B_2K)^{\mathrm{T}} \\ C_2^{\mathrm{T}} \end{pmatrix} \\ &= (C_2 + B_2K)P_1(C_2 + B_2K)^{\mathrm{T}} + C_2Q_2^{-1}C_2^{\mathrm{T}} \\ &= C_2(P_1 + Q_2^{-1})C_2^{\mathrm{T}} + B_2Y_1P_1^{-1}Y_1^{\mathrm{T}}B_2^{\mathrm{T}}. \end{aligned}$$

To reduce the problem to minimization of the linear function, we introduce new matrix variables $Z_1 = Z_1^{\mathrm{T}}$ such that $Z_1 \geqslant Y_1P_1^{-1}Y_1^{\mathrm{T}}$, which is equivalent to the linear matrix inequality

$$\begin{pmatrix} Z_1 & Y_1 \\ Y_1^{\mathrm{T}} & P_1 \end{pmatrix} \geqslant 0, \tag{A.15}$$

and the matrix variable $H = H^{\mathrm{T}}$ such that $Q_2^{-1} \leqslant H$, which is equivalent to the linear matrix inequality

$$\begin{pmatrix} H & I \\ I & Q_2 \end{pmatrix} \geqslant 0. \tag{A.16}$$

As the result, we come to the problem of minimization

$$\mathrm{tr}\left[C_2(P_1 + H)C_2^{\mathrm{T}} + B_2Z_1B_2^{\mathrm{T}}\right] \to \min$$

under the constraints (A.11), (A.12), and (A.14)–(A.16), which proves the theorem. □

Proof of Theorem 2. Let us consider the quadratic function

$$V(g_k) = g_k^{\mathrm{T}}Qg_k, \quad Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} \in \mathbb{R}^{2n\times 2n}, \quad Q > 0,$$

constructed on the solutions of system (11). Then,

$$V(g_{k+1}) = \left(\widetilde{A}g_k + \widetilde{D}w_k\right)^{\mathrm{T}} Q \left(\widetilde{A}g_k + \widetilde{D}w_k\right) = g_k^{\mathrm{T}}\widetilde{A}^{\mathrm{T}}Q\widetilde{A}g_k + w_k^{\mathrm{T}}\widetilde{D}^{\mathrm{T}}Q\widetilde{D}w_k + 2w_k^{\mathrm{T}}\widetilde{D}^{\mathrm{T}}Q\widetilde{A}g_k.$$

For the trajectories of system (9) to remain within the boundaries of the ellipsoid $\mathcal{E}_g$, we require that $V(g_{k+1}) \leqslant 1$ be satisfied for $V(g_k) \leqslant 1$, that is,

$$g^{\mathrm{T}}\widetilde{A}^{\mathrm{T}}Q\widetilde{A}g + 2w^{\mathrm{T}}\widetilde{D}^{\mathrm{T}}Q\widetilde{A}g + w^{\mathrm{T}}\widetilde{D}^{\mathrm{T}}Q\widetilde{D}w \leqslant 1,\ \forall (g,w) : g^{\mathrm{T}}Qg \leqslant 1,\ w^{\mathrm{T}}w \leqslant 1. \tag{A.17}$$

Let $s = \begin{pmatrix} g & w \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^{2n+m}$,

$$M_0 = \begin{pmatrix} \widetilde{A}^{\mathrm{T}}Q\widetilde{A} & \widetilde{A}^{\mathrm{T}}Q\widetilde{D} \\ \widetilde{D}^{\mathrm{T}}Q\widetilde{A} & \widetilde{D}^{\mathrm{T}}Q\widetilde{D} \end{pmatrix}, \quad M_1 = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

and $\widetilde{f}_i(s) = s^{\mathrm{T}}M_is$, $i = 0,1,2$. Then, (A.17) is rearranged in the form

$$\widetilde{f}_0(s) \leqslant 1, \quad \forall s:\ \widetilde{f}_1(s) \leqslant 1, \quad \widetilde{f}_2(s) \leqslant 1.$$

According to Assertion A.1, for some values of $\alpha, \beta \geqslant 0$ such that $\alpha + \beta \leqslant 1$, condition (A.17) is equivalent to the linear matrix inequality

$$\begin{pmatrix} \widetilde{A}^{\mathrm{T}}Q\widetilde{A} - \alpha Q & \widetilde{A}^{\mathrm{T}}Q\widetilde{D} \\ \widetilde{D}^{\mathrm{T}}Q\widetilde{A} & \widetilde{D}^{\mathrm{T}}Q\widetilde{D} - \beta I \end{pmatrix} \leqslant 0. \tag{A.18}$$

Using the Schur lemma, we rearrange inequality (A.18) in

$$\widetilde{A}^{\mathrm{T}}Q\widetilde{A} - \alpha Q \leqslant \widetilde{A}^{\mathrm{T}}Q\widetilde{D}\left(\widetilde{D}^{\mathrm{T}}Q\widetilde{D} - \beta I\right)^{-1}\widetilde{D}^{\mathrm{T}}Q\widetilde{A}.$$

Since sought are the *minimal* ellipsoids, that is, those with the greatest matrix Q, and on the other hand, $\widetilde{D}^{\mathrm{T}}Q\widetilde{D} - \beta I \leqslant 0$ must be satisfied, we get

$$\beta = \beta_{\max} = 1 - \alpha.$$

With regard for this, we set down inequality (A.18) as

$$\begin{pmatrix} (A+B_1K)^{\mathrm{T}}Q_1(A+B_1K) - \alpha Q_1 & (A+B_1K)^{\mathrm{T}}Q_1FC_1 & (A+B_1K)^{\mathrm{T}}Q_1FD_2 \\ (FC_1)^{\mathrm{T}}Q_1(A+B_1K) & \Psi_1 & \Psi_2 \\ (FD_2)^{\mathrm{T}}Q_1(A+B_1K) & \Psi_2^{\mathrm{T}} & \Psi_3 \end{pmatrix} \leqslant 0,$$

where

$$\Psi_1 = (FC_1)^{\mathrm{T}}Q_1FC_1 + (A - FC_1)^{\mathrm{T}}Q_2(A - FC_1) - \alpha Q_2,$$
$$\Psi_2 = (FC_1)^{\mathrm{T}}Q_1FD_2 + (A - FC_1)^{\mathrm{T}}Q_2(D_1 - FD_2),$$
$$\Psi_3 = (FD_2)^{\mathrm{T}}Q_1FD_2 + (D_1 - FD_2)^{\mathrm{T}}Q_2(D_1 - FD_2) - (1-\alpha)I.$$

By the Schur lemma,

$$\begin{pmatrix} \Psi_1 & \Psi_2 \\ \Psi_3 & \Psi_4 \end{pmatrix} - \begin{pmatrix} (FC_1)^{\mathrm{T}}Q_1(A+B_1K) \\ (FD_2)^{\mathrm{T}}Q_1(A+B_1K) \end{pmatrix}$$
$$\times \left((A+B_1K)^{\mathrm{T}}Q_1(A+B_1K) - \alpha Q_1\right)^{-1}$$
$$\times \begin{pmatrix} (A+B_1K)^{\mathrm{T}}Q_1FC_1 & (A+B_1K)^{\mathrm{T}}Q_1FD_2 \end{pmatrix} \leqslant 0$$

or

$$\begin{pmatrix} (A-FC_1)^{\mathrm{T}}Q_2(A-FC_1)-\alpha Q_2 & (A-FC_1)^{\mathrm{T}}Q_2(D_1-FD_2) \\ (D_1-FD_2)^{\mathrm{T}}Q_2(A-FC_1) & (D_1-FD_2)^{\mathrm{T}}Q_2(D_1-FD_2)-(1-\alpha)I \end{pmatrix}$$
$$-\begin{pmatrix} (FC_1)^{\mathrm{T}} \\ (FD_2)^{\mathrm{T}} \end{pmatrix} W^{-1} \begin{pmatrix} FC & FD_2 \end{pmatrix} \leqslant 0,$$

where

$$W^{-1} = -Q_1 + Q_1(A+B_1K)\left((A+B_1K)^{\mathrm{T}}Q_1(A+B_1K)-\alpha Q_1\right)^{-1}(A+B_1K)^{\mathrm{T}}Q_1.$$

Using the matrix inversion Lemma A.2 for

$$X = Q_1^{-1} = P_1, \quad Y = A + B_1K, \quad Z = (\alpha Q_1)^{-1} = \frac{1}{\alpha}P_1,$$

we obtain

$$W^{-1} = \left(\frac{1}{\alpha}(A+B_1K)P_1(A+B_1K)^{\mathrm{T}} - P_1\right)^{-1}.$$

Now, by the Schur lemma,

$$\begin{pmatrix} W & FC_1 & FD_2 \\ (FC_1)^{\mathrm{T}} & (A-FC_1)^{\mathrm{T}}Q_2(A-FC_1)-\alpha Q_2 & (A-FC_1)^{\mathrm{T}}Q_2(D_1-FD_2) \\ (FD_2)^{\mathrm{T}} & (D_1-FD_2)^{\mathrm{T}}Q_2(A-FC_1) & (D_1-FD_2)^{\mathrm{T}}Q_2(D_1-FD_2)-(1-\alpha)I \end{pmatrix} \leqslant 0$$

or

$$\begin{pmatrix} Q_2WQ_2 & Q_2FC_1 & Q_2FD_2 \\ (FC_1)^{\mathrm{T}}Q_2 & (A-FC_1)^{\mathrm{T}}Q_2(A-FC_1)-\alpha Q_2 & (A-FC_1)^{\mathrm{T}}Q_2(D_1-FD_2) \\ (FD_2)^{\mathrm{T}}Q_2 & (D_1-FD_2)^{\mathrm{T}}Q_2(A-FC_1) & (D_1-FD_2)^{\mathrm{T}}Q_2(D_1-FD_2)-(1-\alpha)I \end{pmatrix} \leqslant 0.$$

By introducing the matrix variables

$$Y_1 = KP_1, \quad Y_2 = Q_2F,$$

we eliminate K and F. At that, $KQ_2K^{\mathrm{T}} = Y_1P_1^{-1}Y_1^{\mathrm{T}}$ is estimated as follows. We introduce the matrix variable $Z_1 = Z_1^{\mathrm{T}}$; since $P_1 > 0$, we get that $Z_1 \geqslant Y_1P_1^{-1}Y_1^{\mathrm{T}}$ if and only if

$$\begin{pmatrix} Z_1 & Y_1 \\ Y_1^{\mathrm{T}} & P_1 \end{pmatrix} \geqslant 0. \tag{A.19}$$

As the result, we come to the matrix inequality

$$\begin{pmatrix} Q_2\Omega Q_2 & Y_2C_1 & Y_2D_2 \\ C_1^{\mathrm{T}}Y_2^{\mathrm{T}} & \Lambda_1 + C_1^{\mathrm{T}}Y_2^{\mathrm{T}}Q_2^{-1}Y_2C_1 & \Lambda_2 + C_1^{\mathrm{T}}Y_2^{\mathrm{T}}Q_2^{-1}Y_2D_2 \\ D_2^{\mathrm{T}}Y_2^{\mathrm{T}} & \Lambda_2^{\mathrm{T}} + D_2^{\mathrm{T}}Y_2^{\mathrm{T}}Q_2^{-1}Y_2C_1 & \Lambda_3 + D_2^{\mathrm{T}}Y_2^{\mathrm{T}}Q_2^{-1}Y_2D_2 \end{pmatrix} \leqslant 0, \tag{A.20}$$

where

$$\Omega = \frac{1}{\alpha}\left(AP_1A^{\mathrm{T}} + B_1Y_1A^{\mathrm{T}} + AY_1^{\mathrm{T}}B_1^{\mathrm{T}} + B_1Z_1B_1^{\mathrm{T}}\right) - P_1,$$
$$\Lambda_1 = A^{\mathrm{T}}Q_2A - A^{\mathrm{T}}Y_2C_1 - C_1^{\mathrm{T}}Y_2^{\mathrm{T}}A - \alpha Q_2,$$
$$\Lambda_2 = A^{\mathrm{T}}Q_2D_1 - C_1^{\mathrm{T}}Y_2^{\mathrm{T}}D_1 - A^{\mathrm{T}}Y_2D_2,$$
$$\Lambda_3 = D_1^{\mathrm{T}}Q_2D_1 - D_2^{\mathrm{T}}Y_2^{\mathrm{T}}D_1 - D_1^{\mathrm{T}}Y_2D_2 - (1-\alpha)I.$$

By Lemma A.1, if

$$Y_2^{\mathrm{T}} Q_2^{-1} Y_2 \leqslant Z_2, \quad Z_2 = Z_2^{\mathrm{T}} \tag{A.21}$$

and

$$\begin{pmatrix} Q_2 \Omega Q_2 & Y_2 C_1 & Y_2 D_2 \\ C_1^{\mathrm{T}} Y_2^{\mathrm{T}} & \Lambda_1 + C_1^{\mathrm{T}} Z_2 C_1 & \Lambda_2 + C_1^{\mathrm{T}} Z_2 D_2 \\ D_2^{\mathrm{T}} Y_2^{\mathrm{T}} & \Lambda_2^{\mathrm{T}} + D_2^{\mathrm{T}} Z_2 C_1 & \Lambda_3 + D_2^{\mathrm{T}} Z_2 D_2 \end{pmatrix} \leqslant 0, \tag{A.22}$$

then inequality (A.20) is valid as well. We note that since $Q_2 > 0$, inequality (A.21) is equivalent to the linear matrix inequality

$$\begin{pmatrix} Z_2 & Y_2^{\mathrm{T}} \\ Y_2 & Q_2 \end{pmatrix} \geqslant 0. \tag{A.23}$$

We introduce the matrix variable $Z = Z^{\mathrm{T}}$; by Lemma A.1, if

$$\begin{pmatrix} Z & Y_2 C_1 & Y_2 D_2 \\ C_1^{\mathrm{T}} Y_2^{\mathrm{T}} & \Lambda_1 + C_1^{\mathrm{T}} Z_2 C_1 & \Lambda_2 + C_1^{\mathrm{T}} Z_2 D_2 \\ D_2^{\mathrm{T}} Y_2^{\mathrm{T}} & \Lambda_2^{\mathrm{T}} + D_2^{\mathrm{T}} Z_2 C_1 & \Lambda_3 + D_2^{\mathrm{T}} Z_2 D_2 \end{pmatrix} \leqslant 0 \tag{A.24}$$

and

$$Q_2 \Omega Q_2 \leqslant Z, \tag{A.25}$$

the matrix inequality (A.22) is satisfied as well.

By assuming in inequality (A.4) that

$$X = -\Omega^{-1}, \quad Y = Q_2,$$

we obtain

$$Q_2 \Omega Q_2 \leqslant -2Q_2 - \Omega^{-1}.$$

Therefore, the matrix inequality (A.25) is satisfied if valid is the inequality

$$-2Q_2 - \Omega^{-1} \leqslant Z$$

which is equivalent to the linear matrix inequality

$$\begin{pmatrix} 2Q_2 + Z & I \\ I & -\dfrac{1}{\alpha}\left(AP_1A^{\mathrm{T}} + B_1Y_1A^{\mathrm{T}} + AY_1^{\mathrm{T}}B_1^{\mathrm{T}} + B_1Z_1B_1^{\mathrm{T}}\right) + P_1 \end{pmatrix} \geqslant 0. \tag{A.26}$$

At that,

$$\begin{aligned} \mathcal{C}P\mathcal{C}^{\mathrm{T}} &= \begin{pmatrix} C_2 + B_2K & C_2 \end{pmatrix} \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} (C_2 + B_2K)^{\mathrm{T}} \\ C_2^{\mathrm{T}} \end{pmatrix} \\ &= (C_2 + B_2K)P_1(C_2 + B_2K)^{\mathrm{T}} + C_2P_2C_2^{\mathrm{T}} \\ &= C_2P_1C_2^{\mathrm{T}} + C_2P_2C_2^{\mathrm{T}} + B_2KP_1K^{\mathrm{T}}B_2^{\mathrm{T}} = C_2(P_1 + P_2)C_2^{\mathrm{T}} + B_2Y_1P_1^{-1}Y_1^{\mathrm{T}}B_2^{\mathrm{T}}. \end{aligned}$$

We introduce the matrix variable $H = H^{\mathrm{T}}$ such that $P_2 = Q_2^{-1} \leqslant H$ in order to reduce the problem to minimization of a linear function. The last matrix inequality is equivalent to the linear matrix inequality

$$\begin{pmatrix} H & I \\ I & Q_2 \end{pmatrix} \geqslant 0. \tag{A.27}$$

As the result, we get the minimization problem

$$\mathrm{tr}\left[C_2(P_1 + H)C_2^{\mathrm{T}} + B_2 Z_1 B_2^{\mathrm{T}}\right] \to \min$$

under the constraints (A.19), (A.23), (A.26), (A.24), and (A.27), which proves the theorem. □

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