ISSN 1064-5624, Doklady Mathematics, 2008, Vol. 77, No. 1, pp. 158–162. © Pleiades Publishing, Ltd., 2008.
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CONTROL THEORY

Filtering under Nonrandom Disturbances: The Method of Invariant Ellipsoids

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Presented by Academician A.B. Kurzhanskii July 23, 2007

Received October 8, 2007

DOI: 10.1134/S1064562408010390

1. The filtering problem (i.e., the state estimation of a dynamic system from measurements) under random disturbances can be solved nearly totally by applying the Kalman filter. However, in many situations, the randomness of disturbances is an unjustified assumption. Frequently, it is only known that all the disturbances are bounded and are arbitrary in the other respects. In this case, guaranteed (rather than probabilistic) state estimates can be constructed. This approach was suggested by American scientists H.S. Witsenhausen, D.P. Bertsekas, I.B. Rhodes, and F.C. Schweppe in the late 1960s and the early 1970s [1]. At about the same time, similar issues were analyzed by A.B. Kurzhanskii, A.I. Subbotin, Yu.S. Osipov, and others at the seminar led by N.N. Krasovskii (see [2]). A considerable contribution was made by Chernous'ko [3]. Specifically, the ellipsoidal filtering technique was developed in [1–3]. An overview of the results in this area can be found in [4–8].

This paper is also concerned with the problem of filtering under bounded nonrandom disturbances. We consider only stationary problems, i.e., all the parameters of the model are time-independent. Moreover, we search for a state estimate whose error is guaranteed to lie in a uniform ellipsoid (invariant ellipsoid) for all times; i.e., the estimate is uniform. The filter itself is also sought in the class of linear stationary filters. In this narrowed class of problems and estimates, the problem is totally solvable, i.e., an optimal filter and a state estimate can be constructed. It is this point that differs our setting from those mentioned above. The latter considered more general models, but the resulting solutions were only suboptimal, and no uniform estimates were obtained.

Technically, we follow the linear matrix inequality approach [9, 10], which proved to be an effective tool in the analysis and synthesis of control systems but was little used in filtering problems. An exception is [11], in contrast to which we (i) give simpler and more accurate estimates for the quality of filtering, (ii) extend them to the discrete case, and (iii) analyze the behavior of estimates for large initial deviations. An important new technical tool is the S-theorem for two constraints [12], while the same theorem for a single constraint was applied previously.

2. Consider a linear continuous dynamic system

$$\begin{aligned} \dot{x} &= Ax + D_1 w, \\ y &= Cx + D_2 w, \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$, $D_1 \in \mathbb{R}^{n \times m}$, $D_2 \in \mathbb{R}^{l \times m}$, $C \in \mathbb{R}^{l \times n}$, $x \in \mathbb{R}^n$ is the state of the system, $y \in \mathbb{R}^l$ is an observed output, and $w \in \mathbb{R}^m$ is an external disturbance (noise) that is bounded at any time: $\|w(t)\| \le 1$, $\forall t \ge 0$.[1] Thus, we consider L_∞-bounded external disturbances. Note that no other constraints are imposed on $w(t)$. Specifically, it is not assumed to be random or harmonic. Assume that the pair (A, D) is controllable and $D_1 D_2^T = 0$.

Suppose that x cannot be measured and information on the system is provided by its output y. We construct a filter governed by a linear differential equation with respect to a state estimator $\hat{x}$ that includes the mismatch between y and its prediction $C\hat{x}$:

$$\dot{\hat{x}} = A\hat{x} + F(y - C\hat{x}), \quad F \in \mathbb{R}^{n \times l}. \tag{2}$$

It should be stressed that the filter structure is specified beforehand (it is linear and stationary) and only the constant matrix F has to be chosen. This is the same structure as in the well-known Luenberger observer. Define the residual $e(t) = x(t) - \hat{x}(t)$, which characterizes the filtering error. According to (1) and (2), it satisfies the differential equation

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[1] Here and below, $\|\cdot\|$ denotes the Euclidean norm of a vector, I is the identity matrix of suitable size, and matrix inequalities are understood in the sense of the sign definiteness of the matrices involved.

$$\dot{e} = (A - FC)e + (D_1 - FD_2)w. \tag{3}$$

The task is to find a minimal (in a certain sense) uniform ellipsoid that contains e. The method of invariant ellipsoids was applied to the analysis and synthesis of control systems in [9, 11, 13, 14]. We use it for filtering problems. Accordingly, we somewhat change the standard definition in order to include the case of large initial deviations. An ellipsoid centered at the origin

$$\mathcal{E} = \{e \in \mathbb{R}^n: e^T P^{-1} e \le 1\}, \quad P > 0 \tag{4}$$

is called invariant for dynamic system (3) if (i) $e(0) \in \mathcal{E}$ (small deviations) implies $e(t) \in \mathcal{E}$ for all times $t \ge 0$; and (ii) for $e(0) \notin \mathcal{E}$ (large deviations), $e(t) \to \mathcal{E}$ as $t \to \infty$ (possibly, $e(t) \in \mathcal{E}$ for $t \ge T$ at some $T > 0$). Thus, we estimate the asymptotic (and t-uniform for small deviations) filtering error.

First, we note that the controllability condition implies the existence of at least one invariant ellipsoid. Invariant ellipsoids are numerous, and our goal is (given fixed stabilizing F) to find the minimum one and, then, to minimize it over F. Minimality can be understood variously. For our purposes, it is convenient to define the minimal ellipsoid as one having the smallest sum of the squared semiaxes, i.e., one for which the trace of P is minimal. Other criteria will be mentioned below.

Theorem 1. *The solution $\hat{Q}$ and $\hat{Y}$ of the minimization problem*

$$\mathrm{tr} H \to \min$$

subject to the constraints

$$\begin{pmatrix} A^TQ + QA - YC - C^TY^T + \alpha Q & QD_1 - YD_2 \\ (QD_1 - YD_2)^T & -\alpha I \end{pmatrix} \le 0, \tag{5}$$

$$\begin{pmatrix} H & I \\ I & Q \end{pmatrix} \ge 0, \tag{6}$$

where the minimum is sought over the matrix variables $Q = Q^T \in \mathbb{R}^{n \times n}$, $Y \in \mathbb{R}^{n \times l}$, and $H = H^T \in \mathbb{R}^{n \times n}$ and the numerical parameter $\alpha > 0$, defines the matrix $\hat{P} = \hat{Q}^{-1}$ of the minimal invariant ellipsoid and the corresponding filter matrix

$$\hat{F} = \hat{Q}^{-1}\hat{Y}.$$

Proof. Consider the Lyapunov function $V(e) = e^TQe$, $Q \in \mathbb{R}^{n \times n}$, $Q > 0$, which is constructed for solutions to Eq. (3). Then $\dot{V}(e) = ((A - FC)e + (D_1 - FD_2)w)^TQe + e^TQ((A - FC)e + (D_1 - FD_2)w) = e^T((A - FC)^TQ + Q(A - FC))e + 2e^TQ(D_1 - FD_2)w$. For the trajectories of system (3) to remain within the ellipsoid $\mathcal{E}$ for $e(0) \in \mathcal{E}$, we require that $\dot{V}(e) \le 0$ whenever $V(e) \ge 1$ and $w^Tw \le 1$ (this condition also guarantees that $V(e)$ decreases monotonically for $V(e) \le 1$; i.e., the second property of invariant ellipsoids is satisfied). Thus,

$$e^T((A - FC)^TQ + Q(A - FC))e$$
$$+ 2w^T(D_1 - FD_2)^TQe \le 0, \tag{7}$$
$$\forall (e, w)\colon e^TQe \ge 1, \quad w^Tw \le 1.$$

Let

$$s = \begin{pmatrix} e \\ w \end{pmatrix},$$

$$M_0 = \begin{pmatrix} (A - FC)^TQ + Q(A - FC) & Q(D_1 - FD_2) \\ (D_1 - FD_2)^TQ & 0 \end{pmatrix},$$

$$M_1 = \begin{pmatrix} -Q & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix},$$

and $\tilde{f}_i(s) = s^TM_is$, $i = 0, 1, 2$. Then (7) can be rewritten as $\tilde{f}_0(s) \le 0$, $\forall s$: $\tilde{f}_1(s) \le -1$, $\tilde{f}_2(s) \le 1$. Applying the S-theorem for two quadratic forms [12], we find that (7) is equivalent to a linear matrix inequality $M_0 \le \alpha M_1 + \beta M_2$ for some α and β such that $\alpha \ge \beta \ge 0$, or

$$\begin{pmatrix} (A - FC)^TQ + Q(A - FC) + \alpha Q & Q(D_1 - FD_2) \\ (D_1 - FD_2)^TQ & -\beta I \end{pmatrix} \le 0.$$

Thus, the invariance condition for an ellipsoid with the matrix $P = Q^{-1} > 0$ is equivalent to the last linear matrix inequality with some $\alpha \ge \beta > 0$. Since we are interested in minimal ellipsoids, $\beta = \beta_{\max} = \alpha$. Introducing the matrix variable $Y = QF$ and eliminating F yields (5). To reduce the minimization of $\mathrm{tr}\,Q^{-1}$ to a linear problem, we introduce a matrix $H = H^T$ such that $Q^{-1} \le H$, which is equivalent to (6). As a result, we obtain the minimization of $\mathrm{tr} H \to \min$ under constraints (5) and (6).

Note that, for fixed α, this problem is reduced to the minimization of a linear function under constraints given by linear matrix inequalities, i.e., to a semi-definite programming (SDP) problem, which is a convex optimization problem. Numerous software packages are available for its numerical solution, for example, SeDuMi Toolbox, YALMIP Toolbox, and LMI Toolbox in Matlab [15]. One-dimensional minimization with respect to α is always convex, but strict substantiation of this fact remains an open problem.

3. Consider the linear discrete system

$$
\begin{aligned}
x_{k+1} &= Ax_k + D_1 w_k, \\
y_k &= Cx_k + D_2 w_k,
\end{aligned}
\tag{8}
$$

where $A \in \mathbb{R}^{n\times n}$, $D_1 \in \mathbb{R}^{n\times m}$, $D_2 \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{l\times n}$, $x_k \in \mathbb{R}^n$ is the state of the system, $y_k \in \mathbb{R}^l$ is an observed output, and $w_k \in \mathbb{R}^m$ is an external disturbance bounded for all times: $\|w_k\| \le 1$, $k = 0, 1, 2, \ldots$. Thus, we consider l_∞-bounded disturbances. Assume that the pair (A, D_1) is controllable and $D_1 D_2^T = 0$.

We construct a filter described by a linear equation with a constant matrix F with respect to a state estimator $\hat{x}_k$:

$$\hat{x}_{k+1} = A\hat{x}_k + F(y_k - C\hat{x}_k), \quad F \in \mathbb{R}^{n\times l}.$$

Consider the residual $e_k = x_k - \hat{x}_k$. It satisfies the difference equation

$$e_{k+1} = (A - FC)e_k + (D_1 - FD_2)w_k. \tag{9}$$

The task is to find a matrix F that ensures the minimality of an invariant ellipsoid containing e_k. In fact, the definition of an invariant ellipsoid remains the same as in the continuous case: an ellipsoid

$$\mathscr{E} = \{e_k \in \mathbb{R}^n: e_k^T P^{-1} e_k \le 1\}, \quad P > 0 \tag{10}$$

is called invariant for discrete system (9) if $e_0 \in \mathscr{E}$ (small deviations) implies $e_k \in \mathscr{E}$ for all times $k = 1, 2, \ldots$ and if $e_0 \notin \mathscr{E}$ (large deviations) implies $e_k \to \mathscr{E}$ as $k \to \infty$.

Theorem 2. *The solution $\hat{Q}$ and $\hat{Y}$ of the minimization problem*

$$\operatorname{tr} H \to \min$$

with the constraints

$$\begin{pmatrix} \Psi_1 & \Psi_2 \\ \Psi_2^T & \Psi_3 \end{pmatrix} \le 0, \quad \begin{pmatrix} Z & Y^T \\ Y & Q \end{pmatrix} \ge 0, \quad \begin{pmatrix} H & I \\ I & Q \end{pmatrix} \ge 0,$$

where

$$\Psi_1 = A^T QA - A^T YC - C^T Y^T A + C^T ZC - \alpha Q,$$

$$\Psi_2 = A^T QD_1 - C^T Y^T D_1 - A^T YD_2 + C^T ZD_2,$$

$$
\begin{aligned}
\Psi_3 = {} & D_1^T QD_1 - D_2^T Y^T D_1 - D_1^T YD_2 \\
& + D_2^T ZD_2 - (1 - \alpha)I,
\end{aligned}
$$

and the minimum is sought over the matrix variables $Q = Q^T \in \mathbb{R}^{n\times n}$, $Z = Z^T \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{n\times l}$, $H = H^T \in \mathbb{R}^{n\times n}$, and the numerical parameter $\alpha > 0$, *defines the matrix $\hat{P} = \hat{Q}^{-1}$ of the minimal invariant ellipsoid for the residual of system* (8) *and the corresponding filter*

$$\hat{F} = \hat{Q}^{-1}\hat{Y}.$$

The proof is essentially the same as in the continuous case. Note only some of the differences. For the Lyapunov function

$$V(e_k) = e_k^T Q e_k, \quad Q = P^{-1} \in \mathbb{R}^{n\times n}, \quad Q > 0$$

we have

$$
\begin{aligned}
V(e_{k+1}) &= ((A - FC)e_k + (D_1 - FD_2)w_k)^T \\
&\quad \times Q((A - FC)e_k + (D_1 - FD_2)w_k) \\
&= e_k^T (A - FC)^T Q(A - FC)e_k \\
&\quad + w_k^T (D_1 - FD_2)^T Q(D_1 - FD_2)w_k \\
&\quad + 2w_k^T (D_1 - FD_2)^T Q(A - FC)e_k.
\end{aligned}
$$

For the trajectories of system (9) to remain within the ellipsoid $\mathscr{E}$ for $e_0 \in \mathscr{E}$, we require that $V(e_k) \le 1$ whenever $V(e_{k+1}) \le 1$. After some rearrangements, applying the Schur lemma and the S-theorem, we see that this condition is equivalent to

$$
\begin{aligned}
(A - FC)^T Q(A - FC) - \alpha Q &\le (A - FC)^T Q(D_1 \\
&\quad - FD_2)((D_1 - FD_2)^T Q(D_1 - FD_2) - \beta I)^{-1} \\
&\quad \times (D_1 - FD_2)^T Q(A - FC).
\end{aligned}
$$

Note that the same inequality can be derived starting from the second property of invariant ellipsoids, which is written as $V(e_{k+1}) \le V(e_k)$ for $V(e_k) \ge 1$. The subsequent rearrangements are similar to those presented in the continuous case.

4. Possible generalizations. In some cases, there is a priori information on the system's initial state $x(0) \in E_0$, where $E_0 = \{x: x^T P_0^{-1} x \le 1\}$. Then, choosing $\hat{x}(0) = 0$, we guarantee that $e(0) \in E_0$. Assuming that $E_0 \subset \mathscr{E}$, we guarantee that $e(t) \in \mathscr{E}$ for all t. Thus, if the system of linear matrix inequalities in Theorem 1 is supplemented with one more inequality $Q \le P_0^{-1}$, then the filtering error estimate we obtain is not only asymptotic but also holds for all times.

Often, one needs to estimate the quality of filtering some of the coordinates of x rather than all of them. Given the output $y_1 = C_1 x$ (for example, one of the state coordinates), it is desirable to minimize the error of its estimator $e_1 = y_1 - \hat{y}_1 = C_1(x - \hat{x})$. Then the problem is reduced to the minimization of $\operatorname{tr} C_1 P C_1^T$ instead of $\operatorname{tr} P$,

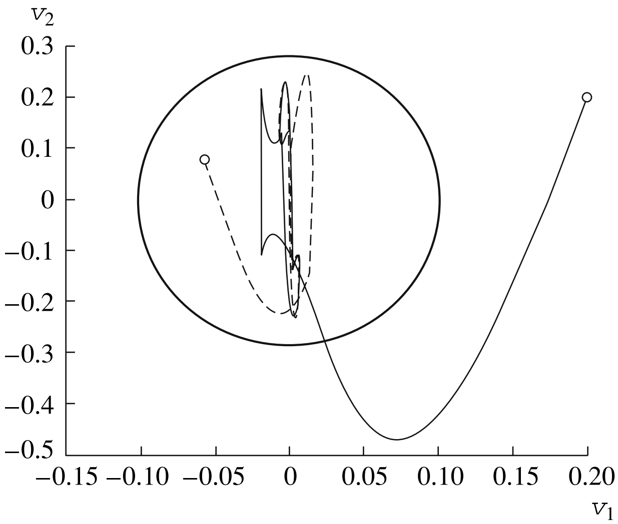

Fig. 1. Estimate (ellipse) and the trajectories of the residuals.

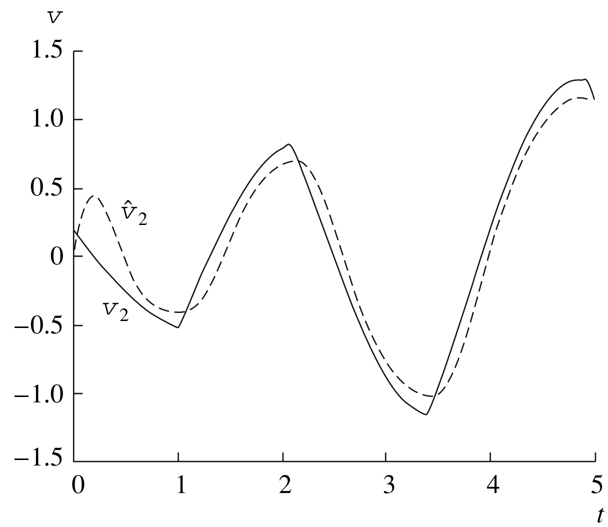

Fig. 2. Filtering the coordinate v_2.

which is easy to write in a form similar to Theorems 1 and 2.

Note that other optimality criteria can be used instead of the sum of the ellipsoid's squared semiaxes. For example, we can minimize the L_∞ norm of the residual (as was done in [11]), i.e., the radius of a ball containing $\mathscr{E}$. For this purpose, we require that $r \to \max$ under the additional constraint $Q \geq rI$.

Finally, we can consider robust versions of the problem, when the description of the system involves uncertainties (i.e., the matrices A and D involve bounded uncertainties). The task is to construct a filter and its guaranteed error estimates that hold for any admissible uncertainties. This problem can be solved using the same technique as before.

5. By way of illustration, the approach for filtering bounded external disturbances as based on invariant ellipsoids is applied to state estimation for a linearized double pendulum placed in a viscous medium. The state vector of the system is $x = (x_1^T \ x_2^T \ v_1^T \ v_2^T)^T$, where x_1 and x_2 are the coordinates of the upper and lower bobs and v_1 and v_2 are their velocities. Suppose that the observed output of the system is given by $y = (x_1^T \ x_2^T)^T$ and the output to be minimized is $y_1 = (v_1^T \ v_2^T)^T$. Assume that the lower bob's velocity is influenced by an external disturbance w. For unit parameters of the system and a medium drag of 0.2, we obtain linear equations of motion. Let $P_0 = 0.01I$. Solving the SDP problem with the help of YALMIP and SeDuMi and minimizing the result over α, we find the matrices of the filter $\hat{F}$ and the invariant ellipsoid $\hat{P}$ that ensures the minimum ellipse containing e_1. Figure 1 shows this ellipse and two trajectories $e_1(t)$ (for large and small deviations). The disturbance was specified as the locally worst one, i.e., a disturbance that maximizes $\dot{V}(e)$ for given e. Specifically, it was given by the formula $w^* = \dfrac{D^T\hat{P}^{-1}e}{\|D^T\hat{P}^{-1}e\|}$. Figure 2 displays the trajectories $v_2(t)$ (solid) and $\hat{v}_2(t)$ (dashed). It can be seen that the accuracy of filtering is fairly high (for $v_1(t)$, it is even higher).

6. A simple and universal approach has been proposed for filtering arbitrary bounded external disturbances with the use of an observer. The approach is based on the method of invariant ellipsoids. By applying this concept, the original problem is reformulated in terms of linear matrix inequalities and is reduced to semi-definite programming and one-dimensional minimization problems, which are easy to solve numerically. The efficiency of the method is demonstrated as applied to a double pendulum.

ACKNOWLEDGMENTS

The authors are grateful to P.S. Shcherbakov for helpful discussions.

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