New challenges in nonlinear control: stabilization and synchronization of chaos

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To Lennart Ljung, with high respect.

Abstract: After the pioneering work Ott et al. (1990) the problem of chaos control attracted many researchers, mainly physicists and mathematicians. The problem differs dramatically from standard setup of control theory. First, it is essentially nonlinear. Second, goals of control are nonconventional - it may be stabilization of unstable periodic orbits or chaotization of a stable system. Third, the control itself should be small enough. A novel approach to chaos stabilization and synchronization, based on trajectory prediction, will be discussed. It is simple and effective (numerous simulation results demonstrate this); the theoretical validation will be also presented. On the other hand the approach has strong limitations because it is nonrobust with respect to system dynamics.

Introduction

Deterministic chaos is one of the fundamental concepts in the modern natural science. In fact, the existence of extremely intricate motions in simple deterministic systems prejudices the overall ideology of the deterministic Nature. First instances of such a complicated behavior exposed by systems having simple and transparent models have been observed quite a long time ago. Thus, as early as 1876, A. Cayley has discovered the irregular structure of the attraction basins of Newton's method when applied to solution of very simple equations (of the kind $z^3 = 1$) in the complex variable. The modern-language formulation of this phenomenon is that these basins are fractals. Later, in 1917–1920, two French scientists, G. Julia and P. Fatou performed a detailed analysis of the iterations of rational fractional mappings, with the Newton iterations being their special case. An important contribution to the theory of chaos was made in Sharkovskii (1964) on the co-existence of cycles of one-dimensional mappings. His studies demonstrated how complicated the behavior of a dynamic system may be even in one dimension. However, the true burst of interest to chaos arose after the publication of the works by E. Lorenz, D. Ruelle and F. Takens, and B. Mandelbrot at the middle of 1960th - beginning of 1970th. Since that time, the terms "*chaos*," "*strange attractor*," "*fractal*" have got common acceptance, and at present they are the subject of discussion in a vast literature.

The theory of chaos and its analysis techniques were being developed by joint efforts of mathematicians and physicists. In the last ten to fifteen years a new research direction had sprung in the area, and control theorists were involved in its exploration. This new topic is *control of chaos*; the problem was presumably first formulated in Ott et al. (1990); also, see survey works Chen and Yu (2003), Fradkov and Pogromsky (1998), Andrievskii and Fradkov (2003,2004), Arecchi et al. (1998), Bocaletti et al. (2000). Controlling chaos differs dramatically from the traditional statements of problems in control, both in the goals and methods. First, this theory is intrinsically nonlinear so that the standard linear theory techniques are not applicable. Second, the control objectives are pretty much different, i.e., these problems may have nothing in common with optimal control, but relate to the stabilization of chaos, synchronization of the chaotic motions, or, conversely, forcing trajectories to be chaotic. Third, we admit only small controls that still completely change the nature of system's behavior. In what follows, we devote our attention to the most important problem of *stabilizing* or *controlling* chaos. As stated in Arecchi et al. (1998), "controlling chaos consists in perturbing a chaotic system in order to stabilize a given unstable periodic orbit embedded in the chaotic attractor." Another problem under consideration will be synchronization of chaotic oscillators.

In this paper, we first propose a novel approach to stabilization of nonlinear discrete-time time-invariant systems, which, in the absence of control are given by

$$x_{k+1} = f(x_k). \tag{1}$$

The idea is to predict the trajectory of the system and apply an additive control action in the following form:

$$u(x) = \varepsilon\Big(f_{m+s}(x) - f_m(x)\Big), \tag{2}$$

where ε is the small step-size (a simple rule for its calculation will be given below), m is the prediction horizon, and s is the desired period of the cycle. Hereinafter, f_m denotes the m-th iteration of the mapping f, i.e.,

$$f_1(x) = f(x), \quad f_m(x) = f\Big(f_{m-1}(x)\Big).$$

Of course such control looks strange for an expert in control theory. On one hand it is a sort of feedback (control is in the form $u(x)$). But on the other hand to design the control we should know *precisely* the right-hand side $f(x)$. The prediction $f_m(x)$ exploits iterations of this function, which is in general unstable. In what follows we indeed assume that $f(x)$ is given analytically or by some algorithm, and the only source of uncertainties is round-off errors in calculations. Such problems do exist, for instance numerous examples of chaos-generating maps in mathematical research of chaos. There are are also physical systems (mechanical or electrical) where equations of motion are known and contain no uncertainty.

More traditional is control based on the previous iterations (the so-called delayed feedback control, or DFC, which was devised by K. Pyragas in 1992 for continuous-time systems and extended later to the discrete-time case in Ushio (1996); see Morgul (2003) for recent developments). However there are many difficulties and limitations peculiar to the DFC-method; the main one is that stabilizing control can not be made small enough. A particular case of control in the form (2) with $m = 0$ was studied in Ushio and Yamamoto (1999) (for further results, see Hino et al. (2002)); however, again the quantity ε can not be made small. Some additional essential features that distinguish the method from many others including the original OGY-method elaborated in Ott et al. (1990) are worth noting. The control is applied at all time instants, not only in the vicinity of the desired cycle (in the latter case, the cycle must be known in advance). On top of that, the function f is not assumed to contain a parameter which is used to control the behavior of the system.

One of the prospective applications of our approach is not control itself but rather checking the existence of periodic orbits of nonlinear iterations. It is quite hard to reveal such an orbit if it is unstable, while with the method developed, the orbit is stabilized and, hence, becomes easily detectable. Moreover, one may attempt to detect all unstable periodic orbits; below, it will be shown that a system may possess many such orbits.

The second problem we consider in the paper is synchronization of chaotic oscillators. We assume that there are identical nonlinear systems, each of them performs chaotic behavior. The goal is to make these chaotic trajectories synchronous via small interactions. This can be done by using predictive control as well.

The first conference version of this paper, Polyak and Maslov (2005) was presented at the 16th World Congress of IFAC; originally, it was my student V.P. Maslov, who suggested to make use of predictive control in the form (2). Later, students S.V. Efremov, N.A. Meshcheryakov and E.N. Gryazina were involved in computer simulations and discussion. Some further results can be found in our publications Polyak (2005); Gryazina and Polyak (2006); Efremov and Polyak (2005).

The paper is organized as follows. In Section 1, method (2) is analyzed as applied to scalar systems; the results of numerical simulations are given for a number of classical examples of chaotic systems such as those specified by the logistic, tent, and cubic mappings. Section 2 is devoted to the analysis of the n-dimensional case, and the Hénon map is taken as an illustrating example. Various issues of computer implementation are discussed in Section 3. Stabilization problem is analyzed in Section 4. Final discussion and comments are provided in Section 5.

1 The Scalar Case

We consider the following open-loop nonlinear scalar discrete-time time-invariant system:

$$x_{k+1} = f(x_k), \quad x_k \in \mathbb{R}^1, \quad k = 1, \ldots, \tag{3}$$

which possesses an s-*cycle* (a *periodic orbit of length* s) $x_1^*, x_2^*, \ldots, x_s^*$, i.e., $x_{i+1}^* = f_i(x_1^*)$ for $i = 1, \ldots, s-1$, $x_1^* = f_s(x_1^*)$ and $x_1^* \neq f_m(x_1^*)$ for $m < s$. The case $s = 1$ corresponds to a fixed point of the function f. Throughout the paper, the knowledge of the cycle is not assumed; the only requirement is the existence of a cycle with period s. Such information about the system is often available in advance; for instance, the famous Sharkovskii's theorem on the ordering of cycles (see Sharkovskii (1964)) testifies to this possibility. The exposition to follow concentrates on *unstable cycles* (unstable periodic orbits), and the primary goal is to stabilize them using small controls.

We assume that the differentiable function f is defined on a bounded interval $[a, b]$ and maps it onto itself: $f\colon [a, b] \to [a, b]$, $f \in C^1$. The number $\mu = f'(x_s^*) \cdot \ldots \cdot f'(x_1^*)$ is referred to as the *multiplicator* of the cycle. The condition $|\mu| < 1$ is sufficient for the cycle to be stable, and we also say that the cycle is an *attractor* in this case., while the condition $|\mu| > 1$ is sufficient for instability of the cycle, in which case it is referred to as a *repeller*. Hereinafter, by stability we mean local convergence of trajectories to the cycle; i.e., there exists $\varepsilon > 0$ such that $\rho(x_k) < \varepsilon$ implies $\lim\limits_{m\to\infty} \rho(x_m) = 0$, where $\rho(x) = \min\limits_i |x - x_i^*|$. Let the cycle under consideration be unstable and $|\mu| > 1$. To stabilize it, we add a control term of the form (2) to the function f in the right-hand side of (3). The resulting closed-loop system takes the form

$$x_{k+1} = F(x_k), \quad F(x) = f(x) - \varepsilon\Big(f_{(p+1)s+1}(x) - f_{ps+1}(x)\Big), \tag{4}$$

$$\frac{|\varepsilon - \varepsilon^*|}{|\varepsilon^*|} < \frac{1}{|\mu|^{1/s}}, \quad \varepsilon^* = \frac{1}{\mu^p(\mu - 1)}, \tag{5}$$

where p is a nonnegative integer. It is important to note that the quantity ε^* becomes arbitrarily small for sufficiently large values of p; respectively, the control also becomes small, since f_m are bounded for all m, and ε decreases together with ε^*.

Theorem 1
Assume that $f \in C^1$ and system (3) possesses an unstable s-cycle with multiplicator μ, $|\mu| > 1$. Then this same cycle is stable for system (4) with arbitrary $p \geq 1$ and any ε satisfying (5).

Proof: The cycle $x_1^*, x_2^*, \ldots, x_s^*$ of the mapping f remains a cycle for f_m for all m; therefore, we have $F(x_i^*) = f(x_i^*) - \varepsilon(f_{p(s+1)+1}(x_i^*) - f_{ps+1}(x_i^*)) = x_{i+1}^*$, i.e., it is also a cycle of F. We next find the multiplicator of (4): $\nu = F'(x_s^*) \cdot \ldots \cdot F'(x_1^*)$. Since $f_s'(x_i^*) = \mu$, $f_{ps}'(x_i^*) = \mu^p$, and $f_{ps+1}'(x_i^*) = \mu^p f'(x_{i+1}^*)$, we obtain $F'(x_i^*) = (1 - \varepsilon\mu^p(\mu - 1))f'(x_i^*)$. Multiplying these equalities for $i = 1, \ldots, s$, we arrive at the expression for the multiplicator of F:

$$\nu = (1 - \varepsilon\mu^p(\mu - 1))^s \mu.$$

To make certain that the cycle is stable, it is sufficient to show that $|\nu| < 1$. Indeed, we have

$$|\nu| = |(1 - \varepsilon\mu^p(\mu - 1))|^s|\mu| < |(1 - (\varepsilon^*(1 \pm (1/|\mu|^{1/s}))\mu^p(\mu - 1))|^s|\mu| = 1,$$

since the function $|1 - c\varepsilon|^s$ attains its maximum at the extreme values of ε. □

An important distinguishing feature of the proposed control setup is its global behavior. Note that Theorem 1 ensures only local convergence of the method; however, if it is applied to the stabilization of chaotic motion having the so-called *mixing property*, then the system with the added control is expected to retain this property (since F is close to f), i.e., after a certain number of iterations, the trajectory enters the domain of attraction of the stabilized orbit.

We illustrate the considerations above by several well-known examples of scalar chaotic systems.

Example 1: logistic map

We consider the function

$$f(x) = \lambda x(1 - x), \quad 0 \le \lambda \le 4, \tag{6}$$

with $f : [0, 1] \to [0, 1]$. The behavior of iterations (3) of this mapping is well studied, being the subject of discussion in many books on chaos. For $\lambda < 1$, there exists a fixed point $x^* = 0$, which is stable; for $1 < \lambda < 3$, there appears another stable fixed point $x^* = 1 - 1/\lambda$; with further increase of λ we observe bifurcation, and the system acquires a stable 2-cycle, etc. Importantly, for $\lambda > 3.84$, mapping (6) possesses s-cycles for all s, and all of them are unstable so that the system exhibits a completely chaotic behavior. Therefore, stabilization of periodic orbits of (6) for λ close to 4 is of the most interest, so that the value $\lambda = 3.9$ was taken in the simulation. The experiments were organized as follows. For each of the 100 initial points x_0 picked from the uniform grid over $[0, 1]$, we performed K iterations of method (4), (5) with various values of s, p and μ, and plot the points x_K. We set $\varepsilon = \varepsilon^* = 1/\mu^p(1 - \mu)$ in formula (5), and the value of μ for the desired s-cycle was calculated according to the rules given in Section 4 below.

Let us discuss the typical results of simulation. For $s = 1$ (stabilization of a fixed point), the value of $\mu = 2 - \lambda = -1.9$ for the fixed point $x^* = 1 - 1/\lambda = 0.7436$ can be obtained in closed form; moreover, quite quickly the method succeeds to stabilize the desired fixed point *globally*; see Fig. 1, where $K = 150$ and $p = 10$. The 2-cycle also stabilizes very quickly; we can manage

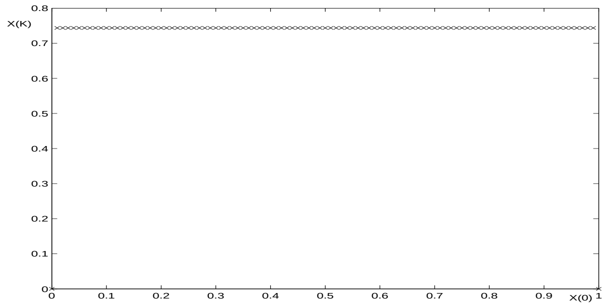

Figure 1.1: *Stabilization of the fixed point of the logistic map.*

with $p \simeq 15$ leading to $\varepsilon \simeq 10^{-10}$. For $s = 3$, the system possesses two 3-cycles, and the estimate $\mu = -5.17$ was obtained for one of them; global stabilization was observed for $K = 2000$ and $p = 5$; see Fig. 2. For $s = 7$

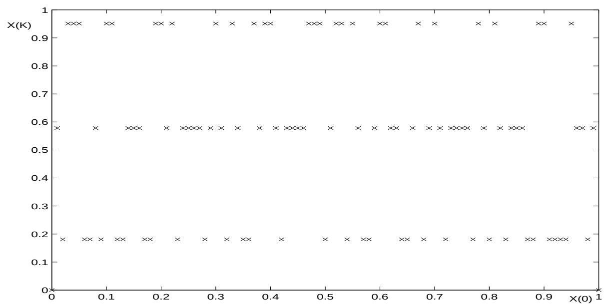

Figure 1.2: *Stabilization of a* 3*-cycle of the logistic map.*

we were able to stabilize two cycles of length 7 (with $\mu = -90$ and $\mu = 95$). However, to secure global convergence, more iterations were required, namely, $K = 10,000$. At the same time, for certain initial approximations, the fixed point $x^* = 1 - 1/\lambda = 0.7436$ was stabilized along with the 7-cycle. Cycles of higher periods can also be detected; for example, eight cycles of length 11 were revealed. As far as the "record" cycle length is concerned, a systematic study of cycles with period 31 was performed; we managed to detect and stabilize 133 such cycles by choosing $p = 0$, and the value of μ achieved the orders of 10^8, i.e., $\varepsilon \sim 10^{-8}$.

Example 2: tent map

Let us consider

$$f(x) = \lambda(1 - |2x - 1|), \quad 0 \leq \lambda \leq 1, \tag{7}$$

where $f : [0, 1] \to [0, 1]$. The iterations of this mapping have much in common with those of the logistic mapping; e.g., the chaotic behavior is observed for the values of λ close to unity. However, there is a substantial difference: all cycles of (7) are unstable for any $\lambda > 0.5$. Indeed, we have $|f'(x)| = 2\lambda > 1$ for any point $x \neq 0.5$ so that $|\mu| = (2\lambda)^s > 1$ for any s-cycle. Nevertheless, these cycles can be stabilized by control of the form (4), (5); this can be done easily, since the values $\mu = \pm(2\lambda)^s$ suffice. Let us take $\lambda = 1$, then none of the cycles contains the point 0.5 and Theorem 1 applies (it is seen from the proof that $f(x)$ need only be differentiable at the points of the cycle). The quantities n_s (the number of s-cycles) and the respective values of the multiplicators μ are known, see Table 1 below.

Table 1. The number of s-cycles and the values of multiplicators

s	1	2	3	4	5	6
n_s	2	1	2	3	6	9
μ	± 2	-4	± 8	± 16	± 32	± 64

For $s = 1$, the fixed point $x^* = 0$ is stabilized if $\mu > 0$, and the point $x^* = 2/3$ if $\mu < 0$. For $s = 2$ we detect a 2-cycle with $\mu < 0$, and two cycles in each of the cases $s = 3$ and $s = 4$. Also, 5-cycles can be stabilized; all six of them were detected (for each of the two signs of μ, three 5-cycles are stabilized simultaneously). In the experiments, the value of p was chosen from the condition $ps \sim 25$, and we obtained $\varepsilon \sim 10^{-8}$.

In Examples 1 and 2, the cycles of the original (open-loop) system were subject to stabilization. However, sometimes the closed-loop system may possess extra cycles (i.e., there are cycles of $F(x)$ which are not cycles of $f(x)$), and they are stabilizable. The example below illustrates this rather exotic situation.

Example 3: cubic map

We consider the mapping

$$f(x) = x^3 - 2x + c. \tag{8}$$

For $c = c^* = 1/\sqrt{3} \approx 0.57735$ this mapping is shown to have a 3-cycle, see Li (2003); however, there is no such a cycle for other values of the parameter, even for those arbitrarily close to c^*. The experiments with this mapping were conducted for the values $s = 3$, $p = 3$, $c = 0.57$, $\varepsilon = 0.002$; in that case, the function $F(x)$ has a 3-cycle, which is quite close to the cycle of $f(x)$ with $c = c^*$. Moreover, this cycle is stable, and we observe quite fast convergence (in no more than 50 steps) of the iterations of algorithm (4) to this cycle for any initial conditions from the segment $[-0.6 \quad 1.5]$. Although the value of $\varepsilon = 0.002$ is relatively large, the plots of $f_3(x)$ and $F_3(x)$ are almost coincide; see Fig. 3, where the zoomed area around one of the points of the cycle is also depicted. It is inter-

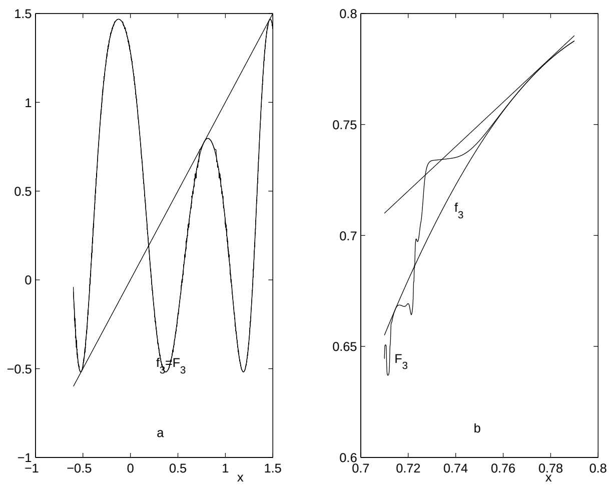

Figure 1.3: *Comparison of the functions $f_3(x)$ and $F_3(x)$ for the cubic map.*

esting to analyze the behavior of iterations (4) for the same value of the parameter $c = 0.57$ in the absence of control, see Fig. 4. The “phantom” cycle is seen to have a definite effect on the trajectories, which are attracted to it from time to time with the subsequent prevalence of the chaotic behavior. This effect is called *intermittence*.

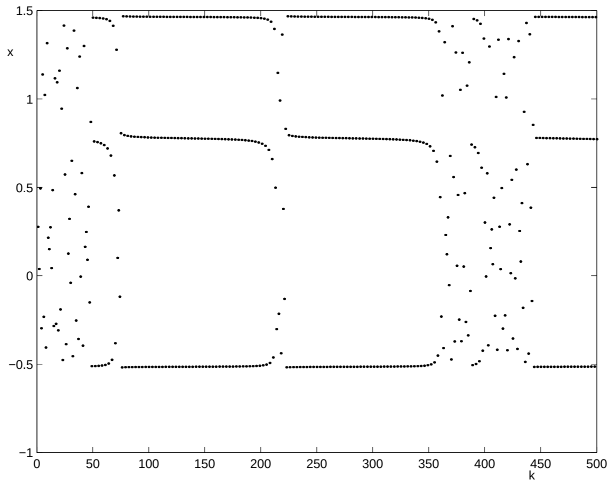

Figure 1.4: *Uncontrollable iterations of the cubic map.*

2 The Vector Case

We turn to the n-dimensional counterpart of system (3):

$$x_{k+1} = f(x_k), \quad x_k \in \mathbb{R}^n, \quad k = 1, \dots. \tag{9}$$

The definitions of s-cycle and multiplicator are the same as in the scalar case with the difference that the multiplicator is now represented by the $n \times n$ Jacobi matrix $M = f'(x_s^*) \cdot \ldots \cdot f'(x_1^*)$. We stress that in multi-dimensional case, the multiplicator depends on the order of the points x_i^*, i.e., it is important which of them is taken as the initial point of the cycle. For instance, if x_i^* is chosen as the starting point, we obtain $M_i = f'(x_{i-1}^*) \cdot \ldots \cdot f'(x_i^*)$, where the subscript at x changes in cyclic order, i.e., $i-1, i-2, \dots, 1, s, s-1, \dots, i$. Hence, we have $M = M_1$ and, generally speaking, $M_i \neq M_1$ for $i \neq 1$, but the matrices $M_1, \dots, M_s$ have the same eigenvalues (indeed, for any $n \times n$ matrices A, B, their products AB and BA have the same eigenvalues: given $ABe = \lambda e$, premultiplying by B yields $BABe = \lambda Be$ so that $BAf = \lambda f$, where $f = Be$). Let μ_i, $i = 1, \dots, n$, denote the eigenvalues of any of the matrices M_j. The cycle is stable if the spectral radius $\rho \doteq \max_i |\mu_i| < 1$, and unstable if $\rho > 1$. Below, we use the representation of the matrix M_i in the form $M_i = A_i B_i$, where $A_i = f'(x_{i-1}^*) \cdot \ldots \cdot f'(x_1^*)$, $B_i = f'(x_s^*) \cdot \ldots \cdot f'(x_i^*)$, $A_1 = I$, $B_1 = M$,

$B_i A_i = M$. We apply the same control as in the scalar case:

$$x_{k+1} = F(x_k), \quad F(x) = f(x) - \varepsilon\Big(f_{(p+1)s+1}(x) - f_{ps+1}(x)\Big), \tag{10}$$

$$\frac{|\varepsilon - \varepsilon^*|}{|\varepsilon^*|} < \frac{1}{|\mu|^{1/s}}, \quad \varepsilon^* = \frac{1}{\mu^p(\mu - 1)},$$

and the choice of μ will be detailed later. We now formulate a simplest result on the stabilization of cycles.

Theorem 2
Assume that $f \in C^1$ and system (9) possesses an unstable s-cycle with multiplicator M, and $\rho > 1$. Let $\mu_n = \mu$ be real, $|\mu| = \rho$, and $|\mu_i| < 1, i = 1, \ldots, n-1$. Then for p large enough, this cycle is a stable cycle of system (10).

Proof: It follows the logic of that of Theorem 1; the only difference is that the matrix product is non-commutative. In order to calculate the matrix multiplicator $N = F'(x_s^*) \cdot \ldots \cdot F'(x_1^*)$ for the cycle $x_1^*, x_2^*, \ldots, x_s^*$ of the mapping F, we calculate each term of the product. Using the chain rule $f_m'(x_i^*) = f_{m-1}'(x_{i+1}^*) f'(x_i^*)$ and the definition of the multiplicator M_i, we find $f_{ps}'(x_i^*) = M_i^p$, $f_{ps+1}'(x_i^*) = M_{i+1}^p f'(x_i^*) = f'(x_i^*) M_i^p$, $M_i^p = A_i M^{p-1} B_i$, whence $F'(x_i^*) = f'(x_i^*)(I - \varepsilon A_i (M^p - M^{p-1}) B_i)$. By induction, we obtain $F'(x_{i-1}^*) \cdot \ldots \cdot F'(x_1^*) = A_i (I - \varepsilon M^p (M - I))^{i-1}$ and arrive at the expression $N = F'(x_s^*) \cdot \ldots \cdot F'(x_1^*) = A_{s+1}(I - \varepsilon M^p (M - I))^s = M(I - \varepsilon M^p (M - I))^s$. The eigenvalues ν_i of the multiplicator N are expressed via the eigenvalues μ_i of the multiplicator M in the following way:

$$\nu_i = \mu_i (1 - \varepsilon \mu_i^p (\mu_i - 1))^s.$$

Next, for $i = n$ we have $\mu_n = \mu$, and in accordance with (10) we obtain $|\nu_n| < 1$ similarly to the scalar case, while for $i \neq n$ we have $|\nu_i| \leq |\mu_i| \left(1 + \frac{|\mu_i|^p}{|\mu|^p} \frac{|1 - \mu_i|}{|\mu - 1|}\right)$. Since $|\mu_i| < 1$ by the conditions of the theorem, the quantity $|\mu_i|^p / |\mu|^p$ tends to zero as p increases, i.e., $|\nu_i| < 1$ for p large enough. We conclude the proof by noting that $r = \max_{1 \leq i \leq n} |\nu_i| < 1$ for such p; i.e., $x_1^*, x_2^*, \ldots, x_s^*$ is a stable cycle of the mapping F. □

The behavior of ε is the same as in the scalar case, i.e., the value of ε decreases as p grows. The boundedness of the function f (which is assumed for chaotic systems) implies the smallness of control. Moreover, keeping in mind local stability and the mixing property of chaotic systems, one may expect method (10) to have global rather than only local convergence.

It is instructive to analyze the structure of the method as applied to linear problems. For example, let $f(x) = Ax$ with nonsingular A; then $x^* = 0$ is the only fixed point, and there are no higher order cycles. Assume that μ is a unique unstable real eigenvalue of A having the property $|\mu| > 1$, and the rest of the eigenvalues are less than one by absolute value. Then method (10) can be slightly modified (simplified for this special case) to take the form

$$x_{k+1} = Ax_k - \varepsilon A^{p+1} x_k, \quad \varepsilon^* = \frac{1}{\mu^p}, \quad \frac{|\varepsilon - \varepsilon^*|}{|\varepsilon^*|} < \frac{1}{|\mu|}. \tag{11}$$

These iterations converge to zero for p large enough (while the original iterations $x_{k+1} = Ax_k$ diverge), and such a method seems to be new. However, in contrast with the nonlinear case (in which the function $f(x)$ was assumed to be bounded), the term $\varepsilon A^{p+1} x_k$ is no longer small at the initial iterations though it tends to zero as k grows.

Example 4: the Hénon map

This classical two-dimensional example was first analyzed in Henon (1976); at present, it is the subject of discussion in all books on chaos. We consider the mapping

$$y_{k+1} = 1 - 1.4y_k^2 + z_k, \quad z_{k+1} = 0.3y_k, \quad k = 1, \ldots. \tag{12}$$

For various initial x_1 picked from the uniform grid on $S = [-1.4 \quad 1.4] \times [-0.4 \quad 0.4]$, the points x_{40}, $x = (y, z)^{\mathrm{T}}$, are shown in Fig. 5; the structure of the "strange attractor" is also visible.

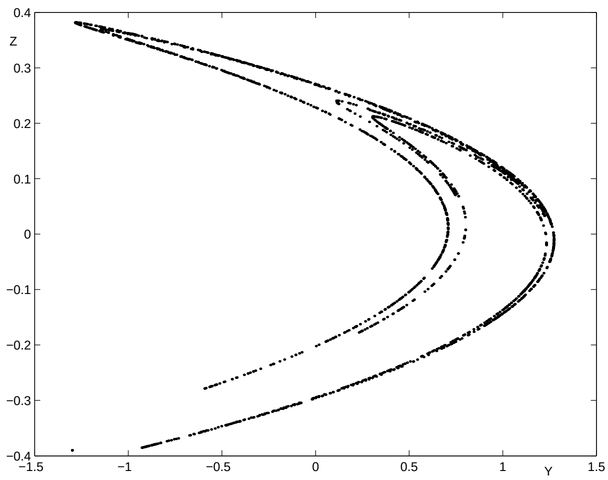

Figure 1.5: *Strange attractor of the Hénon map.*

Figure 6 depicts the trajectory of the system for a fixed initial x_1; an intricate quasirandom walk over the points of the strange attractor is typical.

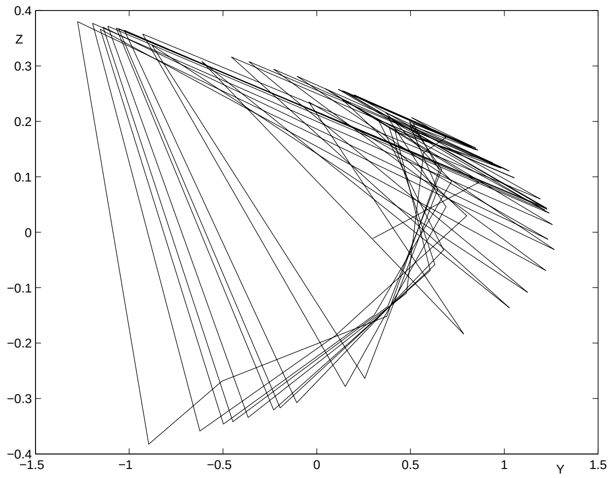

Figure 1.6: *An individual trajectory of the Hénon map.*

This mapping is known to have an unstable fixed point $x^* = (0.6314 \quad 0.1894)$; the eigenvalues of the associated matrix M are equal to $(-1.92 \quad 0.15)$ so that the conditions of Theorem 2 are satisfied with $\mu = -1.92$. Figure 7 shows the behavior of the y component for a typical trajectory in course of stabilization of the fixed point by method (10) (with multiplicator $\mu = -1.92$); this point possesses the global stability.

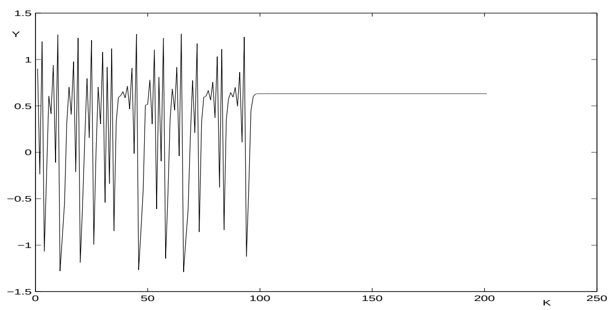

Figure 1.7: *Stabilization of the fixed point of the Hénon map.*

There is also one 2-cycle $x_1^* = (-0.4758 \quad 0.2927)$, $x_2^* = (0.9758 \quad -0.1427)$, which is unstable. Similar results were observed when the 2-cycle was stabilized. For $s = 4$ the existence of cycles and the values of their multiplicators μ are not known. By trial and error, we managed to obtain the value $\mu = -9$

such that a 4-cycle becomes stable. The results of simulation are presented in Fig. 8, where the first component for a typical trajectory is depicted, and Fig. 9, which shows the last 20 iterations of the same trajectory on the x-plane; it is seen that all of them are within the 4-cycle.

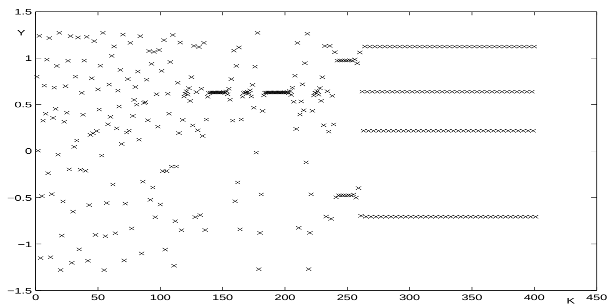

Figure 1.8: *Stabilization of a 4-cycle of the Hénon map; the y coordinate.*

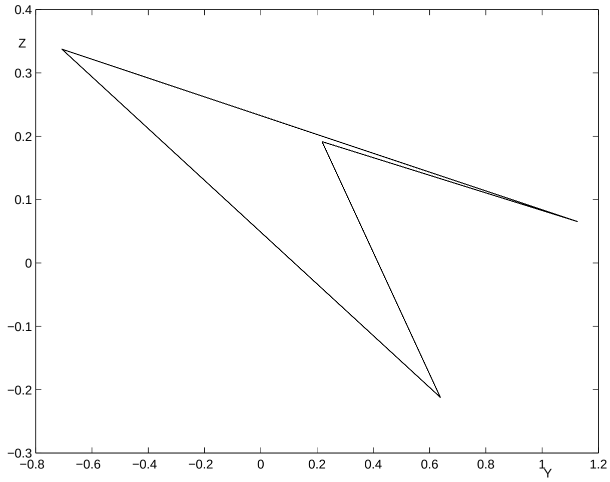

Figure 1.9: *Stabilization of a 4-cycle of the Hénon map; the x plane.*

In all the experiments, the typical value of ε was found to be $\varepsilon \sim 10^{-4} \div 10^{-5}$.

Theorem 2 assumes the presence of a single dominating eigenvalue of M, which is greater than one by absolute value and is real-valued, while the absolute values of the rest of the eigenvalues are less than one. Such a situation is quite typical, though the multiplicators with arbitrarily located eigenvalues are also encountered. Theorem 2 can be extended to cover this latter case at the expense of sophisticating the algorithm, since the whole matrix M need to be known.

First of all, without loss of generality, we let $s = 1$ and restrict our attention to the case of the fixed point x^* (indeed, in the general situation, by replacing the function f with f_s, we reduce the problem to seeking a fixed point of the mapping f_s). In that case, the multiplicator is given by the $n \times n$ Jacobi matrix

$$M = f'(x^*)$$

with eigenvalues $\mu_1, \ldots, \mu_n$. The fixed point is stable if $\rho = \max_i |\mu_i| < 1$ and unstable if $\rho > 1$.

We make use of the control law

$$x_{k+1} = F(x_k), \quad F(x) = f(x) - E(f_{p+2}(x) - f_{p+1}(x)), \tag{13}$$

which differs from (10) in that the scalar ε is changed for the matrix E. Let us represent $M = T\Lambda T^{-1}$, where $T \in \mathbb{R}^{n\times n}$, $\Lambda = \mathrm{diag}(\lambda_i)$, $\lambda_i = \mu_i$ for $\mu_i \in \mathbb{R}$, $i = 1, \ldots, t$, and $\lambda_i = \begin{pmatrix} u_i & v_i \\ -v_i & u_i \end{pmatrix}$ for $\mu_i = u_i \pm jv_i$, $i = t+1, \ldots, n$, $j = \sqrt{-1}$. In other words, by means of a real linear transformation T, the multiplicator M is converted to the real block diagonal form, where the real eigenvalues are represented by diagonal entries, and every pair of complex conjugate eigenvalues $\mu_i = u_i \pm jv_i$ is represented by a real 2×2 block, which is also located on the diagonal. Then the matrix E is taken in the following form:

$$E = T\widetilde{\Lambda}T^{-1}, \quad \widetilde{\Lambda} = \mathrm{diag}\,(\varepsilon_i),$$

where $\varepsilon_i = 0$ for $|\mu_i| < 1$ and

$$\varepsilon_i^* = \frac{1}{\mu_i^p(\mu_i - 1)}, \quad \frac{|\varepsilon_i - \varepsilon_i^*|}{|\varepsilon_i^*|} < \frac{1}{|\mu_i|}\,,$$

otherwise. All manipulations over complex numbers μ_i are to be understood as those performed over their realizations in the form of 2×2 real-valued matrices λ_i.

Theorem 3
Assume that x^* is an unstable fixed point of the mapping f, and the eigenvalues of the matrix $M = f'(x^*)$ are all distinct and do not belong to the unit circumference. Then x^* is a stable fixed point of (13).

Proof: It is based on the formula

$$\nu_i = \lambda_i(1 - \varepsilon_i\lambda_i^p(\lambda_i - 1))$$

obtained above, which is seen to be valid for the method under consideration. For $|\lambda_i| < 1$, we take $\varepsilon_i = 0$, i.e., $|\nu_i| = |\lambda_i| < 1$, while for $|\lambda_i| > 1$ we have $|\nu_i| < 1$ by the calculations similar to those in the proof of Theorem 1. □

3 Implementation Matters

3.1 Estimation of μ.

In some of the examples above, the value of the multiplicator of a stabilized cycle was either known in advance or could be easily calculated; for instance, this was the case with the fixed points or 2-cycles as well as with all cycles of the tent map. In the general case, the quantity μ is not available. For example, the value of s may be large; the function f may not be specified in closed form and its values are generated by a certain algorithm, etc. However, the value of μ still can be evaluated efficiently; most straightforwardly this is doable in the scalar case, $n = 1$. Let us introduce the function $g(x) = f_s(x) - x$ and compute its values over the uniform grid $a = x_0 < x_1 < \ldots < x_N = b,\ x_{i+1} - x_i = d$ (the interval $S = [a, b],\ f : S \to S$ is assumed to be known). We next detect the points of change of sign: $g(x_i)g(x_{i+1}) < 0$, which are the candidate zeros of the function g, i.e., the candidate points of s-cycles of the function f. Since the points of t-cycles (for $t < s$ being divisors of s) are also zeros of g, they are excluded from consideration. Hence, the quantities $(g(x_{i+1}) - g(x_i))/d$ can be taken as reasonably accurate estimates of μ provided that d is small enough.

This approach extends to the multi-dimensional case, where the minimization of the function $\|g(x)\|$ can be accomplished either on a grid or using one or another optimization routine such as `fmin` in MATLAB. Let x_0 be a local minimum and $\|g(x_0)\| \approx 0$. We perform m iterations ($m \sim 10$) to obtain $x_1 = f_s(x_0), \ldots, x_m = f_s(x_{m-1})$, and compute $a = (x_m - x_{m-1},\, x_{m-1} - x_{m-2})$, $r_1 = \|x_m - x_{m-1}\|,\ r_2 = \|x_{m-1} - x_{m-2}\|$ and $q = a/(r_1 r_2)$. Then for the values of $|q|$ close to unity, the quantity a/r_2^2 is an acceptable estimate of μ.

3.2 Choice of p.

From expressions (5) and (10) it is seen that the higher p the smaller ε. However, due to computer roundoff errors (remind that we assume them to be the only source of uncertainty!), the value of p should not be chosen too large, since otherwise the function $f_m(x)$ cannot be accurately computed for large values of m. We turn to examples. For $f(x) = 4x(1 - x)$ we have $f_m(0) = 0$ for any $m \geq 1$; however, $f_m(\epsilon) \approx 4^m \epsilon$ for small ϵ and moderate values of m. Therefore, the roundoff error in computing x, equal to the floating-point accuracy `eps` $= \epsilon = 2^{-52}$ induces the error in computing $f_m(x)$, equal to 2^{2m-52}.

Hence, the prediction horizon m should be taken as small as $m \sim 20$ in order not to yield too rough results. In some cases, these limitations on m are not that severe. For instance, if the points x_i, $x_{i+1} = f(x_i)$, $i = 1, \ldots, m$, are distributed approximately uniformly on $[0, 1]$, then $\mathbf{E}|f'(x)| = 2$ and $\mathbf{E}|f'_m(x)| = 2^m$ so that the values of $m \sim 40$ are admissible. This consideration is equally valid for the tent map $f(x) = (1 - |2x - 1|)$, $|f'_m(x)| = 2^m$ for any x and m. We may conclude that choosing $s(p + 1) \sim 25$ is relatively safe for the two examples above; this conclusion was supported by the numerous experiments. On the whole, the growth of roundoff errors depends on the so-called *Lyapunov indices*, which could be efficiently evaluated. Notably, the condition $s(p+1) \sim 25$ imposes limitations on the lengths of cycles under stabilization; e.g., the value $s = 31$ for the logistic mapping discussed above is close to the maximal computable (in the experiments, we had to take $p = 0$).

3.3 The number of iterations K.

Above it was noted that Theorems 1–3 ensure only local stability of the periodic orbits. As a rule, the higher s and p, the narrower the basin of attraction of the stabilized cycle. Because of the chaotic nature of the motion, the trajectories nevertheless enter the basin of attraction of the stable orbit, although after a large number of iterations K. This explains the fact that as s and p get larger, higher values of K are required to stabilize a cycle. Thus, to stabilize globally a 7-cycle in Example 1, we had to perform $K = 10,000$ iterations, while the stabilization of the fixed point required only $K = 150$ iterations.

Highly remarkably, due to the fact that the control is applied at all time instants (not only at the instants of closeness to the cycle, as with all other methods of controlling chaos known from the literature) hitting the domain of local convergence is observed much earlier than in the absence of control. Respectively, the number of iterations required to achieve stability is substantially smaller. Thus, to stabilize the fixed point of the Hénon map (Example 4), the number of iterations was 100 to $1,000$ times as small as compared to the method in the pioneering paper Ott et al. (1990) (for the same level ε of control), see Fig. 10. The upper line relates to OGY method, the lower one — to the presented algorithm; the average number of iterations is denoted $< \tau >$ following Ott et al. (1990). The reason of such acceleration is explained in Gryazina and Polyak (2006) for a particular example, the general nature of the effect remains not completely clarified.

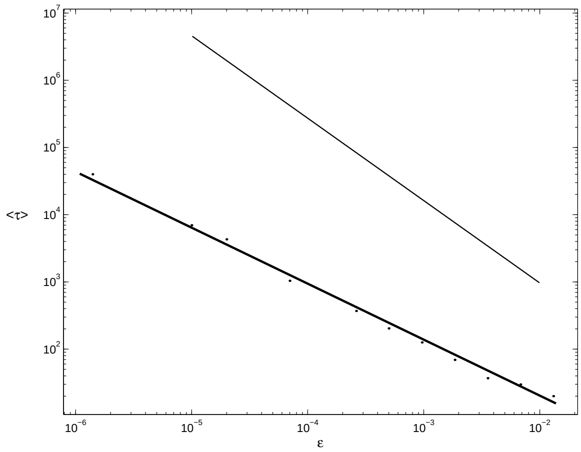

Figure 1.10: *Comparison of number of iterations with OGY method.*

4 Synchronization

Synchronization is a widely known phenomenon in the nature; there are numerous publications relating this subject, see e.g. Fujisaka and Yamada (1983); Pekora and Caroll (1990); Strogatz (2003). There are also many control approaches to achieve synchronization, Lai and Grebogi (1993); **?**); Kaneko (1992); Kurths et al. (2003); Bocaletti et al. (2002); Pecora et al. (1997). We address the single problem in this field: synchronization of n identical discrete-time nonlinear systems (oscillators)

$$x^i_{k+1} = f(x^i_k), \quad i = 1, \dots, n, \quad k = 1, \dots, \tag{14}$$

where x^i is a state of i-th system, k — time instance, $f : [a, b] \subset \mathbb{R} \to [a, b] \subset \mathbb{R}$ is a nonlinear smooth function. It is assumed that each system exhibits chaotic behavior provided there is no interaction. Our goal is to design links (to couple the systems) to achieve synchronization. For this purpose we exploit the same idea as above. Indeed we make prediction of uncontrolled trajectories and use control in the form

$$u(x^i_k) = \varepsilon^i_k (f_m(x^i_k) - \overline{f}). \tag{15}$$

Here ε^i_k is a small step-size, m is the prediction horizon while $\overline{f}$ is the result of averaging for several neighboring oscillators. In numerous works: Lai and

Grebogi (1993); Kaneko (1992); Bocaletti et al. (2002); Cheng et al. (2004) the proposed algorithms can be presented in the form (15) with $m = 1$. However in this case ε_k^i can not be made small enough. For $m > 1$ it happens to be possible.

4.1 Global interaction

One of the versions of (15) has the form:

$$x_{k+1}^i = f(x_k^i) + \varepsilon_k^i (f_m(x_k^i) - \overline{f_k}), \tag{16}$$

$$\overline{f_k} = (1/n) \sum_{i=1}^{n} f_m(x_k^i), \tag{17}$$

$$\varepsilon_k^i = -\left(\prod_{j=1}^{m-1} f'(f_j(x_k^i)) \right)^{-1}, \tag{18}$$

where $i = 1, \dots, n$, $k = 1, \dots$. Its idea is simple: given values $x_k^1, x_k^2, \dots, x_k^n$ at some time instant k; our goal is to make them equal at a future moment $k + m$. For this purpose we take $x_{k+1}^i = f(x_k^i) + u_k^i$ and wish to solve equation

$$f_{m-1}(x_{k+1}^i) = c, \quad i = 1, 2, \dots, n,$$

c being the average of predicted values. If control is small enough one can linearize the above equation and seek the solution by use of Newton's method:

$$f_m(x_k^i) + f'_{m-1}(f(x_k^i)) u_k^i = c, \quad i = 1, 2, \dots, n. \tag{19}$$

Having in mind that $f'_{m-1}(f(x_k^i)) = \prod_{j=1}^{m-1} f'(f_j(x_k^i))$ due to chain rule for differentiation and taking $c = (1/n) \sum_{i=1}^{n} f_m(x_k^i)$ we arrive to (16)–(18). To implement the algorithm one should collect and average all the predicted vales, thus a sort of global interaction is required.

The results of simulation for the algorithm applied to the logistic oscillators and $x_k, k = 1, \dots, 50$, random x_0, $m = 30$, $n = 50$, $\lambda = 3.9$ are presented below. Figure 1.11 demonstrates the behavior of the first 3 coordinates of x. Surprisingly, the synchronization is achieved very fast, in all experiments the number of required iterations does not depend on n and is approximately equal to m. Maximal value of control in this example is $4 \cdot 10^{-4}$ at the first iteration, but after $k = 10$ controls are very small: 10^{-22} for $k = 40 \div 50$.

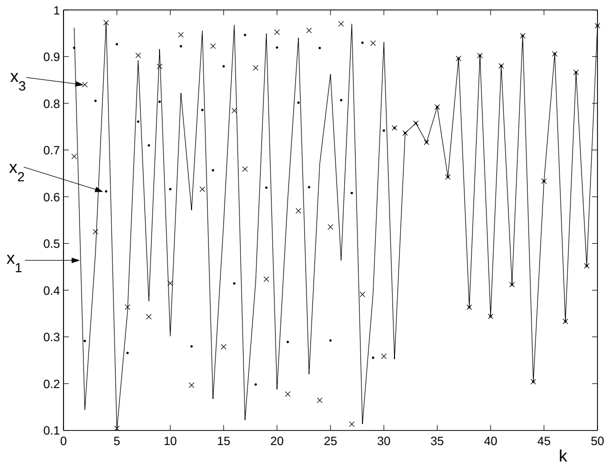

Figure 1.11: *Iterations of logistic oscillators under global interaction*

4.2 Local interaction

The above algorithm is simple and effective, however sometimes just local information is available, and the algorithm is revised as follows:

$$x_{k+1}^i = f(x_k^i) + \varepsilon_k^i (f_m(x_k^i) - \overline{f_k^i}), \tag{20}$$

$$\overline{f_k^i} = \frac{1}{2r+1} \sum_{j=-r}^{j=r} f_m(x_k^{i+j}), \tag{21}$$

$$\varepsilon_k^i = -\left(\prod_{j=1}^{m-1} f'(f_j(x_k^i)) \right)^{-1}. \tag{22}$$

Here $i = 1, \dots, n$, $k = 1, \dots,$ and r is the number of neighboring oscillators from each side. The only difference with (16)–(18) is the calculation of averaged prediction (21). For $m = 1$, $\varepsilon_k^i = \varepsilon$, $r = 1$ the method has been proposed in Lai and Grebogi (1993); Kaneko (1992); Bocaletti et al. (2002); Cheng et al. (2004). However the choice of ε remained open, and moreover synchronization can be achieved for $\varepsilon > \epsilon > 0$ only, while in (20), (22) control can be made arbitrary small for large m. In contrast with global interaction algorithm the number of iterations in (20)–(22) strongly depends on n and r. For instance in the same example with $\lambda = 3.9$, $n = 17$, $m = 30$, $r = 4$ as many as 800 iterations were needed for global synchronization. Moreover for $n > 4r + 1$ the algorithm failed to achieve synchronization.

4.3 Master and slaves

In the above considerations all the oscillators were equivalent. Sometimes one of them is the leading one (“master”) while others are subordinate (“slaves”), see e.g. Pekora and Caroll (1990). To synchronize “slaves” with “master” we can modify the algorithm:

$$x_{k+1}^1 = f(x_k^1), \tag{23}$$

$$x_{k+1}^i = f(x_k^i) + \varepsilon_k^i (f_m(x_k^i) - f_m(x_k^{i-1}))/2, \quad i = 2, \dots, n, \tag{24}$$

$$\varepsilon_k^i = -\left(\prod_{j=1}^{m-1} f'(f_j(x_k^i)) \right)^{-1}, \tag{25}$$

$k = 1, \dots, x_k^{n+1} = x_k^1$; the first oscillator is the leading one. Figure 1.12 depicts the total deviation $\Delta = \frac{1}{m-1} \sum_{i=2}^{m} (x_k^i - x_k^1)^2$ of the subordinate oscillators for the logistic maps with $\lambda = 3.9$, $n = m = 30$. Synchronization occurs very fast.

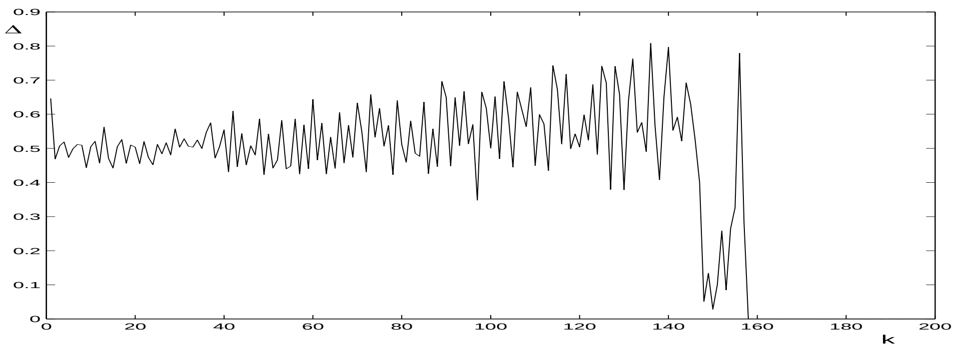

Figure 1.12: *Synchronization to master oscillator*

5 Conclusions

In this paper, we proposed a simple and efficient method of stabilization of unstable s-cycles in nonlinear discrete time systems, which uses small additive controls. It is based on predicting the trajectory by m and $m + s$ iterations ahead, where m is of the form $ps + 1$ and p is sufficiently large. The cornerstone assumption of the approach is the ability to perform such a prediction accurately enough. Said another way, the function $f(x)$ is assumed to be known (or, alternatively, specified by a certain algorithm) and free of perturbations. The method can as well be used for detecting and counting all cycles in the system.

Among the directions for future research in the framework of the approach, we mention the study of global behavior of the proposed algorithms, stabilization in continuous-time systems (i.e., those described by ordinary differential equations), analysis of the role of uncertainties and noises, and a great body of applications of the method to the problems of mechanics, economics, physics, communication theory, etc. The author intends to address these issues in the publications to follow.

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