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Analysis of finite-time convergence by the method of Lyapunov functions in systems with second-order sliding modes☆

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ABSTRACT

A method for constructing Lyapunov functions for analysing of control systems with second-order sliding modes is proposed. It is based on solving a special partial differential equation and enables Lyapunov functions to be constructed that prove that a system transfers into a sliding mode after a finite time and give an explicit estimate of this time. The method is illustrated for three known second order sliding algorithms.

The sliding mode method [1–4] is one of the most effective means of controlling complex dynamical systems operating under conditions of indeterminacy. It is based on the maintenance of specified properties of a system by means of a continuous (most commonly, a bang-bang) control that switches at a very high (theoretically infinite) frequency. In theory, a sliding mode does not simply enable one to reduce the actions of external perturbations but to quench them completely. A serious drawback of the method shows up in its practical implementation. The frequent switchings of the control give rise to high-frequency oscillations of the controlled object [4] that can lead to its physical destruction. The development of methods for suppressing this chattering effect is one important area of investigations in the theory of sliding mode systems. [4–7]

There are two basic approaches to solving the problem of reducing the negative oscillations of the system in a real sliding mode. The first is based on the smoothing out of the bang-bang control at the boundaries where switching occurs, that is, in practice, by replacing a continuous control with a discontinuous control.[4] The second method is associated with the concept of the *order of a sliding mode*. [8–10]

Higher-order sliding modes retain all the theoretically established properties of classical sliding systems while at the same time reducing the amplitude of the high-frequency oscillations [6] and increasing the accuracy of the control when there are discrete measurements and delays in the system at the time of the switchings.[9] This factor has already influenced the activity in the use of higher order sliding modes in systems for controlling mechanical systems. [11–13]

This paper is concerned with developing a method of constructing Lyapunov functions for analysing of systems with higher order sliding modes. The extension of the method of Lyapunov functions to the class of systems with higher-order sliding modes is important from a practical point of view, since it not only enables one to establish the fact that a sliding mode has occurred but also to estimate the time of entry into it. [1,14]

1. Analysis of finite-time convergence by the method of Lyapunov functions

Differential equation with a discontinuous right-hand side. Consider a vector system of the form

$$\dot{x} = g(t, x, u(t, x)), \quad t > 0 \tag{1.1}$$

where $x \in \mathbf{R}^n$ is the state vector of the system, the vector function $u(t,x)=(u_1(t,x), u_2(t,x),\ldots, u_m(t,x))^T$ defined in the whole of the space $\mathbf{R}^{n+1}$ is continuous and the vector function $g(t,x,u)$ is defined and continuous in $\mathbf{R}^{n+m+1}$.

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Following the classical approach,[15] we will denote the set of all possible values of the scalar function $u_i(\tau,y)$ when $(\tau,y)\to(t,x)$ by $U_i(t,x)$ and, for each point $g(t,x,u)$, we determine the closed convex set

$$K[g](t,x) = \overline{\text{co}}\{g(t,x,u)\colon u = (u_1, \dots, u_m)^T, u_i \in U_i(t,x)\} \tag{1.2}$$

containing all the limiting values of the function $g(t,x,u(t,x))$, where co$S$ is the convex closure of the set S.

Definition 1. We shall call the solution of the differential inclusion

$$\dot{x} \in K[g](t,x) \tag{1.3}$$

that is, the absolutely continuous vector function $x(t)$, defined in the interval or segment I for which the inclusion $\dot{x}\in K[g](t,x)$ is satisfied almost everywhere on I, the Filippov solution (Filippov trajectory) of Eq. (1.1).

In addition, it is assumed that the set $K[g](t,x)$ satisfies the conditions for the existence of solutions of the differential inclusion (1.3) and the conditions for their non-local extendibility (see Ref. 16, for example).

Finite-time convergence and the method of Lyapunov functions. We will now present some simple generalizations of well-known results,[14,17–19] which will be used below to analyse systems with higher-order sliding modes.

Definition 2. The set D containing all the points of the trajectories of the inclusion that never leave this set, that is,

$$\forall x_0 \in D \Rightarrow x(t) \in D \quad \text{for all} \quad t \geq 0$$

where $x(t)$ is a trajectory of inclusion 91.3) with an initial condition $x(0)=x_0\in D$, is called the invariant set of differential inclusion (1.3).

Suppose $d(z, M)$ is the distance from a point $z\in\mathbf{R}^n$ to the set $M\subset\mathbf{R}^n$, that is,

$$d(z,M) = \inf_{\upsilon \in M} \|z - \upsilon\|$$

Definition 3. We call the invariant set D the attracting set for inclusion (1.3) if

$$d(x(t), D) \to 0 \quad \text{when} \quad t \to +\infty$$

where $x(t)$ is a solution of inclusion (1.3) with an arbitrary initial condition

$$x(0) = x_0\colon \|x_0\| < +\infty \tag{1.4}$$

Special cases of attracting sets are stationary sets that are stable on the whole and points of dichotomous systems.[16]

Definition 4. We shall say that an attracting set D is accessible after a finite time if, for each solution $x(t)$ of inclusion (1.3) with an arbitrary initial condition (1.4), a finite instant $t_r=t_r(x_0)$ is found such that

$$x(t) \in D \quad \text{when} \quad t \geq t_r$$

The sliding manifold [1] can serve as a typical example of an attracting set that is accessible after a finite time.

Theorem 1. *Suppose, for a closed set $D\subset\mathbf{R}^n$, a function $V(x)$, which is defined and continuous in $\mathbf{R}^n$, exists and satisfies the conditions*

1) $V(0) = 0$ *when* $x\in D$ *and* $V(x) > 0$ *for all* $x\in\mathbf{R}^n\setminus D$;
2) $V(x)$ *satisfies the Lipschitz condition in a neighbourhood of each point of the space* $\mathbf{R}^n$;
3) *for each solution* $x(t)$, $t > 0$ *of the differential inclusion* (1.3) *with an arbitrary initial value* (1.4), *the function* $V(x, t)$ *satisfies the inequality*

$$\dot{V}(x(t)) \leq -kV^{\rho}(x(t)) \quad \text{for all almost} \quad t > 0; \quad k > 0, \quad 0 \leq \rho < 1 \tag{1.5}$$

The set D is then an attracting set for the inclusion (1.3) *and is accessible after a finite time with an estimate*

$$t_r(x_0) \leq \frac{1}{k(1-\rho)}[V(x_0)]^{1-\rho} \tag{1.6}$$

Proof. Since the absolutely continuous function $x(t)$ is a solution of differential equation (1.1), then, under the constraints mentioned in the theorem, the function of time $V(x(t))$ will also be absolutely continuous. The validity of the theorem then follows from differential inequality (1.5) which guarantees that the function $V(x(t))$ does not increase for all $t > 0$ and reaches a value $V(x,(t_r))=0$, after a finite time (1.6) and, consequently, $x(t_r)\in D$. By virtue of the non-negativity of the function $V(x)$, we have $V(x(t))\equiv 0$ for all $t\geq t_r$. The assertion of the theorem follows from this when account is taken of the non-local extendibility of the solutions.

Corollary 1. Suppose $0\in K[f](t,0)$ and Theorem 1 holds for the set $D=\{0\}$. Differential inclusion (1.1) then has a null solution $x(t) = 0$ that is globally asymptotically stable and accessible after a finite time.

Note that condition 2 in Theorem 1 can be omitted if, instead of inequality (1.5), we use

$$D^*V(x(t)) \leq -kV^{\rho}(x(t)) \text{ for all } t > 0; \quad D^*V(x(t)) = \overline{\lim_{h\to+0}} \frac{V(x(t+h)) - V(x(t))}{h}$$

(D^* is an upper right Dini derivative number (finite of infinite)).

Higher-order sliding modes. Suppose, in system (1.1), the function $u(t, x)$ has the meaning of a control. The scalar output $\sigma = \sigma(t,x)$, $\sigma \in \mathbf{R}$, that it is necessary to stabilize at zero after a finite time, is now introduced into the treatment.

Definition 5. If the continuous output $\sigma(t,x)$ of system (1.1) has derivatives that are continuous with respect to the variable x and total derivatives with respect to time up to the $(r - 1)$-th order inclusive and the set

$$\Sigma = \{x \in \mathbf{R}^n: \sigma(t,x) = \dot{\sigma}(t,x) = \ldots = \sigma^{(r-1)}(t,x) = 0\} \tag{1.7}$$

for system (1.1) is an attracting set and is accessible after a finite time, then its motion on the set Σ is called a *sliding mode of the r-th order* and the set Σ itself is called a sliding manifold of the r-th order.

Below, we shall study second-order sliding modes and consider a control system of the form of (1.1) under the assumption that it is affine with respect to u with a relative order of 2. We shall then have the following equation [10] for the output

$$\ddot{\sigma} = a(t,x) + b(t,x)u(\sigma, \dot{\sigma}) \tag{1.8}$$

where $X \in \mathbf{R}^n$ is a state vector, $u \in \mathbf{R}$ is a scalar input (control) and $\sigma \in \mathbf{R}$ is the scalar output that is available for measurement together with its derivative at each instant. We will assume that the functions a and b are unknown but bounded:

$$|a(t,x)| \le C \text{ и } 0 < b^{\min} \le b(t,x) \le b^{\max} \tag{1.9}$$

where C, $b^{\min}$, $b^{\max}$ are known numbers.

Every solution of Eq. (1.8) satisfies the differential inclusion

$$\ddot{\sigma} \in [-C, C] + [b^{\min}, b^{\max}]u(\sigma, \dot{\sigma}) \tag{1.10}$$

which is obviously autonomous and dependent solely on the output. The problems of synthesizing a sliding control and of analysing of the finite-time convergence for the initial system can therefore be solved by investigating differential inclusions of the form (1.10).

2. A Method of searching for candidates for Lyapunov functions

2.1. Description of the system

Suppose the control system has the form

$$\dot{x} = g(x,y), \quad \dot{y} = a(x,y) + b(t,x,y)u(x,y) + f(t,x,y) \tag{2.1}$$

where $X \in \mathbf{R}^n$ and $y \in \mathbf{R}^k$ are the components of the state vector of the system, g: $\mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^n$, a: $\mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^k$ are known vector functions, u: $\mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^m$ is a known and, in the general case, discontinuous control function, the matrix of the gain coefficients

$$b(t,x,y): \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^{k \times m}$$

is assumed to be unknown but bounded

$$b(t,x,y) = \{b_{ij}(t,x,y)\}_{i=1, j=1}^{k, m}, \quad 0 \le b_{ij}^{\min} \le b_{ij}(t,x,y) \le b_{ij}^{\max} \tag{2.2}$$

and the unknown vector function

$$f(t,x,y): \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^k$$

satisfies the inequality

$$|f_i(t,x,y)| \le C_i, \quad i = 1, 2, \ldots, k \tag{2.3}$$

It is also assumed that the numbers C_i, $b_{ij}^{\min}$ and $b_{ij}^{\max}$ are known.

It is required in the case of system (2.1) to propose a method of constructing candidates for the Lyapunov function using which it can shown that the set D is an attracting set which is accessible after a finite time.

2.2. Estimation of the total derivative

Suppose the continuous function $V(x, t)$ is such that, for almost all $t > 0$, it has a total derivative along the trajectory of system (2.1)

$$\dot{V} = \sum_{s=1}^{n} g_s(x,y)\frac{\partial V}{\partial x_s} + \sum_{i=1}^{k}\left(a_i(x,y) + \sum_{j=1}^{m} b_{ij}(t,x,y)u_j(x,y) + f_i(t,x,y)\right)\frac{\partial V}{\partial y_i}$$

The limit

$$\dot{V} \le \sum_{s=1}^{n} g_s(x,y)\frac{\partial V}{\partial x_s} + \sum_{i=1}^{k} h_i(x,y,\mu_{i1},\ldots,\mu_{im},\gamma_i)\frac{\partial V}{\partial y_i}$$

then holds almost everywhere, where

$$h_i(x,y,\mu_{i1},\ldots,\mu_{im},\gamma_i) := a_i(x,y) + \sum_{j=1}^{m}\mu_{ij}u_j(x,y) + \gamma_i, \quad \gamma_i = C_i \operatorname{sign}\left(\frac{\partial V}{\partial y_i}\right)$$

$$\mu_{ij} = \frac{1}{2}b_{ij}^{\min}\left(1 - \operatorname{sign}\left(\frac{\partial V}{\partial y_i}u_j(x,y)\right)\right) + \frac{1}{2}b_{ij}^{\max}\left(1 + \operatorname{sign}\left(\frac{\partial V}{\partial y_i}u_j(x,y)\right)\right)$$

We will seek the Lyapunov function $V(x, y)$ in the form of a continuous solution of a partial differential equation ($q > 0$ and $r > 0$ are certain positive parameters)

$$\sum_{s=1}^{n} g_s(x,y)\frac{\partial V}{\partial x_s} + \sum_{i=1}^{k} h_j(x,y,\mu_{i1},\ldots,\mu_{im},\gamma_i)\frac{\partial V}{\partial y_i} = -qV^{\rho} \tag{2.4}$$

If $V(x, y)$ is a suitable solution of Eq. (2.4), then the inequality

$$\frac{dV(x(t),y(t))}{dt} \le -qV^{\rho}(x(t),y(t))$$

holds and, in the case when $0 < \rho < 1$, we obtain a Lyapunov function with a finite convergence time

$$t_r \le \frac{1}{q(1-\rho)}[V(x(0),y(0))]^{1-\rho}$$

and, when $\rho \ge 1$, it only guarantees asymptotic stability (and, for $\rho = 1$, exponential stability).

In order to solve Eq. (2.4), it is necessary to know the piecewise-constant functions μ_{ij} and γ_i which, in their turn, depend on sign($\partial V/\partial y_j$) and sign($u(x, y)$). Since γ_i and μ_{ij} can only take two possible values $\gamma_i \in \{-C_i, C_i\}$ *and* $\mu_{ij} \in \{b_{in}^{\min}, b_{ij}^{\max}\}$, we can consider Eq. (2.4) assuming that γ_i and μ_{ij} are constant parameters. The exact values of the functions γ_i and μ_{ij} can only be determined after discovering the general form of the function $V(x, y)$.

The idea of searching for a Lyapunov function in the form of a solution of a partial differential equation was apparently expressed for the first time by Zubov.[17] The proposed modification of Zubov's method allows of a non-linear dependence of the right-hand side of Eq. (2.4) on the required function V. It is precisely this generalization that enables us to find Lyapunov functions with a finite convergence time.

Remark 1. All the arguments still hold if the second equation of (2.1) is replaced by the differential inclusion

$$\dot{y} \in a(x,y) + [b^{\min}, b^{\max}]u + [-C, C]; \quad b^{\min} = \{b_{ij}^{\min}\}, \quad b^{\max} = \{b_{ij}^{\max}\}, \quad C = (C_1,\ldots,C_k)^T$$

2.3. The method of characteristics

The solution of Eq. (2.4) with constant γ_i and μ_{ij} can be found by the method of characteristics: every function $V(x, y)$ that satisfies the following system of ordinary differential equations (the characteristic equations)

$$\frac{dx_1}{g_1(x,y)} = \ldots = \frac{dx_n}{g_n(x,y)} = \frac{dy_1}{h_1(x,y,\mu_{11},\ldots,\mu_{1m},\gamma_1)} = \frac{dy_n}{h_k(x,y,\mu_{k1},\ldots,\mu_{km},\gamma_k)} = \frac{dV}{-qV^{\rho}}$$

is a solution of Eq. (2.4).

If the first integrals of the system of characteristic equations

$$\varphi_r(V,x,y,\mu,\gamma,q,\rho) = \text{const}, \quad r = 1,2,\ldots,n+k, \quad \mu = \{\mu_{ij}\}, \quad \gamma = (\gamma_1,\ldots,\gamma_k) \tag{2.5}$$

have been successfully determined, the function $V(x,y,\mu,\gamma,q,\rho)$ can be found as the solution of the non-linear algebraic equation

$$\Phi(\varphi_1(V,x,y,\mu,\gamma,q,\rho),\ldots,\varphi_{n+k}(V,x,y,\mu,\gamma,q,\rho)) = 0 \tag{2.6}$$

where $\Phi(\varphi_1,\ldots,\varphi_{n+k})$ is an arbitrary function. Every solution of the last equation defines a possible candidate for a Lyapunov function. However, the function Φ and the parameters μ, γ, q and ρ have to be chosen such that the resulting function $V(x, y)$ is continuous and satisfies the condition: $V(x, y) = 0$ when $(x,y) \in D$ and $V(x, y) > 0$ when $(x,y) \in \mathbf{R}^{n+k}D$.

This method does not provide a formal algorithm for constructing a Lyapunov function. It only helps to obtain a candidate for a Lyapunov function, while reducing the problem of finding a suitable “energy” function to the problem of the correct determination of the parameters of the candidate function.

2.4. Candidate for a Lyapunov function for second-order sliding systems

Consider the problem of finding a candidate for a Lyapunov function for a second-order sliding system of the form of (1.10) with a bang-bang control.

We put $x=\sigma$ and $y=\dot{\sigma}$. The differential inclusion (1.10) can then be represented in the form

$$\dot{x} = y, \quad \dot{y} \in [-C, C] + [b^{\min}, b^{\max}]u(x, y) \tag{2.7}$$

where $u(x, y)$ is a given piecewise-constant function with a finite set of values (a bang-bang control). Then, when $\rho = 1/2$, Eq. (2.4) takes the form

$$y\frac{\partial V}{\partial x} + \tilde{\gamma}\frac{\partial V}{\partial y} = -q\sqrt{V}$$

$$\tilde{\gamma} = \gamma + \mu u, \quad \gamma = C\,\mathrm{sign}\left(\frac{\partial V}{\partial y}\right), \quad \mu = \frac{1}{2}b^{\min}\left(1 - \mathrm{sign}\left(\frac{\partial V}{\partial y}u\right)\right) + \frac{1}{2}b^{\max}\left(1 + \mathrm{sign}\left(\frac{\partial V}{\partial y}u\right)\right)$$

For the corresponding characteristic equations

$$\frac{dx}{y} = \frac{dy}{\tilde{\gamma}} = \frac{dV}{-q\sqrt{V}}$$

and, when the discontinuous nature of the control u is taken into account, the first integrals

$$\varphi_1(x, y) = x - \frac{y^2}{2\tilde{\gamma}}, \quad \varphi_2(V, y) = -\frac{y}{\tilde{\gamma}} - \frac{2\sqrt{V}}{q}$$

can be easily found.

Choosing the function Φ in the form

$$\Phi(\varphi_1, \varphi_2) = \varphi_2 + p\sqrt{|\varphi_1|}$$

where p is a parameter, we obtain the non-linear equation for finding $V=V(x, y)$

$$\frac{2\sqrt{V}}{q} = \frac{-y}{\tilde{\gamma}} + p\sqrt{\left|x - \frac{y^2}{2\tilde{\gamma}}\right|}$$

from which, subject to the condition that its right-hand side is non-negative

$$p\sqrt{\left|x - \frac{y^2}{2\tilde{\gamma}}\right|} - \frac{y}{\tilde{\gamma}} \geq 0 \tag{2.8}$$

we finally obtain

$$V(x, y) = \frac{q^2}{4}\left(p\sqrt{\left|x - \frac{y^2}{2\tilde{\gamma}}\right|} - \frac{y}{\tilde{\gamma}}\right)^2 \tag{2.9}$$

In view of the fact that the control is of the bang-bang type, the function $V(x, y)$ will be discontinuous. For each specific case it is therefore necessary to choose the parameters q and p so that these discontinuities are eliminated. It is precisely with this aim that the second term in the formula for Φ is taken under the radical sign, thereby ensuring the same power of the terms in the variable y.

Lyapunov functions with a finite convergence time will be constructed below using formula (2.9) for a series of second-order sliding systems with a bang-bang control.

3. Lyapunov functions for second-order sliding systems

3.1. A “twisting” algorithm

A bang-bang control of the form

$$u = -r_1\,\mathrm{sign}(x) - r_2\,\mathrm{sign}(y) \tag{3.1}$$

where $r_1 > r_2 > 0$ are control parameters, is obviously called a “twisting” algorithm [2,10] due to the characteristic behaviour of the trajectories of the closed system (see the left-hand side of Fig. 1).

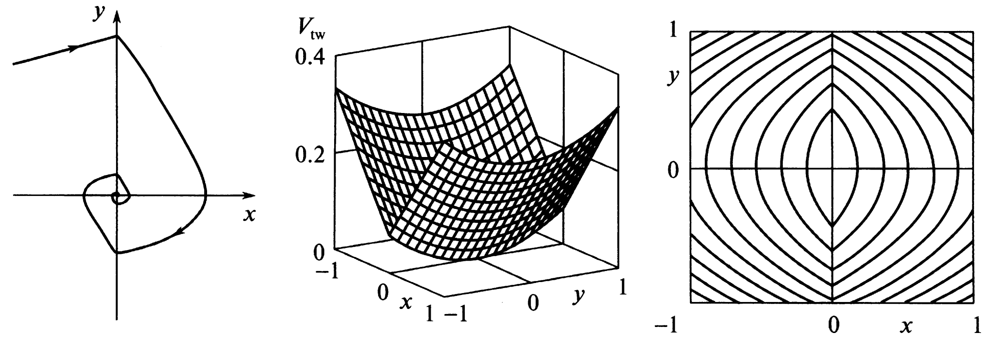

Fig. 1.

It is required to construct a Lyapunov function which proves the asymptotic stability and the attainability of the zeroth solution of system (2.7) with control (3.1) after a finite time and also to obtain an estimate of this time.

A function $V_{tw}(x,y)$ of the form of (2.9), for which positive definiteness is required, can be a candidate for the Lyapunov function in the case considered. Then, when account is taken of inequality (2.8), we obtain the condition

$$p\sqrt{|x - y^2/(2\tilde{\gamma})|} > y/\tilde{\gamma} \text{ when } x^2 + y^2 > 0 \tag{3.2}$$

It can be shown that the constraint $r^1 > r^2 + C/b^{\min}$ is a sufficient condition for the existence of a finite number p that satisfies the inequality (3.2).

It has already been mentioned that, in the general case, the function (2.9) can have discontinuities and the parameters q and p have to be chosen so that they are removed. In the case considered, the function $V_{tw}(x,y)$ is discontinuous on the lines $x = 0$ and $y = 0$ (see the form of Eq. (3.1)). Consideration of the limits of the function $V_{tw}(x,y)$ when $x \to 0$ for arbitrary y and, when $y \to 0$, for arbitrary x:

$$V_{tw}(x,y) \to (p/\sqrt{2|\tilde{\gamma}|} - \operatorname{sign}(y)/\tilde{\gamma})^2 q^2 y^2/4 \text{ when } x \to \pm 0$$

$$V_{tw}(x,y) \to q^2 p^2 |x|/4 \text{ when } y \to \pm 0$$

leads to the system

$$q^2(p/\sqrt{2|\tilde{\gamma}|} - \operatorname{sign}(y)/\tilde{\gamma})^2 = \bar{k}^2, \quad q^2 p^2 = 1$$

where k is a certain positive number. By choosing the parameters p and q in the form

$$p := \frac{\sqrt{2/|\tilde{\gamma}|}\operatorname{sign}(xy)}{\sqrt{2|\tilde{\gamma}|}\bar{k} - 1}, \quad q := \frac{1}{|p|} \tag{3.3}$$

we guarantee the removal of the discontinuities from the function $V_{tw}(x,y)$ on the lines $x = 0$ and $y = 0$.

Combining the definitions (3.3) with condition (3.2), we obtain the following result.

Lemma 1. *If*

$$r_2 + C/b^{\min} < r_1 < ((b^{\min} + b^{\max})r_2 - 2C)/(b^{\max} - b^{\min}) \tag{3.4}$$

and $\bar{k}$ satisfies the condition

$$(2(b^{\min}(r_1 + r_2) - C))^{-1/2} < \bar{k} < (2(b^{\max}(r_1 - r_2) + C))^{-1/2} \tag{3.5}$$

then $p > 0$ and condition (3.2) *is satisfied for all $xy \neq 0$.*

Proof. We note that, in the case of condition (3.4), the interval (3.5) is non-empty and the equality $sign(\dot{y}) = -sign(x)$ holds. We write inequality (3.2) for p of the form of (3.3)

$$\frac{\sqrt{2/|\tilde{\gamma}|}\operatorname{sign}(xy)}{\sqrt{2|\tilde{\gamma}|}\bar{k} - 1} > -\frac{\operatorname{sign}(xy)}{|\tilde{\gamma}|\sqrt{|x|/y^2 + 1/(2|\tilde{\gamma}|)}}$$

Since

$$|\tilde{\gamma}| = |\gamma + \mu u| = |C\operatorname{sign}(x\partial V_{tw}/\partial y) - \mu(r_1 + r_2\operatorname{sign}(xy))|$$

then, for $xy > 0$, we have $|\tilde{\gamma}| \geq b^{\min}(r_1 + r_2) - C$, and the validity of inequality (3.2) follows from this when the left inequality of (3.5) is taken into account. Similarly, we obtain that the validity of inequality (3.2) follows from the right inequality of (3.5) when $xy < 0$.

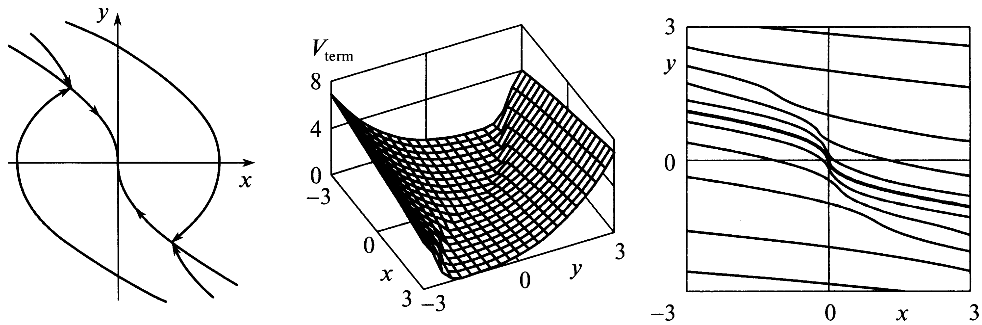

Fig. 2.

Defining γ and μ in the form

$$\gamma := C\text{sign}(y), \quad \mu := \frac{1}{2}b^{\min}(1+\text{sign}(xy)) + \frac{1}{2}b^{\max}(1-\text{sign}(xy))$$

we finally obtain

$$V_{\text{tw}}(x,y) = \begin{cases} q^2(p\sqrt{|x - y^2/(2\tilde{\gamma})|} - y/\tilde{\gamma})^2/4, & xy \neq 0 \\ \bar{k}^2 y^2/4, & x = 0 \\ |x|/4, & y = 0 \end{cases} \tag{3.6}$$

The numbers q and p are given by formula (3.3), $\tilde{\gamma} = \gamma + \mu u$ and the number $\bar{k}$ satisfies condition (3.5).

Theorem 2. *When condition* (3.4) *is satisfied, the function* $V_{\text{tw}}(x,y)$ *possesses the following properties:*

1) *it is positive definite and Lipschitzian in the neighbourhood of each point of the space* $\mathbf{R}^2$ *and also continuously differentiable when* $xy \neq 0$;
2) *the derivative of the function* $V_{\text{tw}}(x(t), y(t))$, *where* $(x(t), y(t))$ *is an arbitrary solution of the inclusion* (2.7), (3.1), *satisfies the inequality*

$$\dot{V}_{\text{tw}}(x(t), y(t)) \le -q_{\min}\frac{b^{\min}(r_1 - r_2) - C}{b^{\max}(r_1 - r_2) + C}\sqrt{V_{\text{tw}}(x(t), y(t))} \tag{3.7}$$

for almost all $t > 0$, *where*

$$q_{\min} := \min_{\delta \in \{-1, 1\}} \frac{|\bar{k}\xi^2(\delta) - \xi(\delta)|}{2},$$

$$\xi(\delta) = \sqrt{[b^{\min}(1+\delta) + b^{\max}(1-\delta)](r_1 + r_2\delta) - 2C\delta}$$

The proof of Theorem 2 and subsequent theorems is given in the Appendix.

Note that the constraints imposed on the system parameters and controls are exactly the same as the results obtained from a geometric analysis of a "twisting" system.[9]

The Lyapunov function $V_{\text{tw}}(x,y)$ and its contour lines for the case when

$$b^{\min} = 3/4, \quad b^{\max} = 1, \quad r_1 = 2, \quad r_2 = 1 \text{ and } C = 1/2$$

are shown in the middle and right parts of Fig. 1.

3.2. Nested algorithm

Suppose the bang-bang control u in system (2.7) has the form

$$u(x,y) = -\alpha\,\text{sign}(z(x,y)); \quad z(x,y) = y + \beta\sqrt{|x|}\,\text{sign}(x) \tag{3.8}$$

where $\alpha > 0$ and $\beta > 0$ are the control parameters. This control algorithm is called the nested controller of a sliding mode.[10]

Both the switching curve

$$z(x,y) = 0 \tag{3.9}$$

as well as the origin of coordinates (0, 0) can be the attracting set for system (2.7) with the control (3.8). In the first case (see the left part of Fig. 2), a first-order terminal sliding mode arises on the curve (3.9) [13] (that is, a sliding mode in which the trajectory of the system during

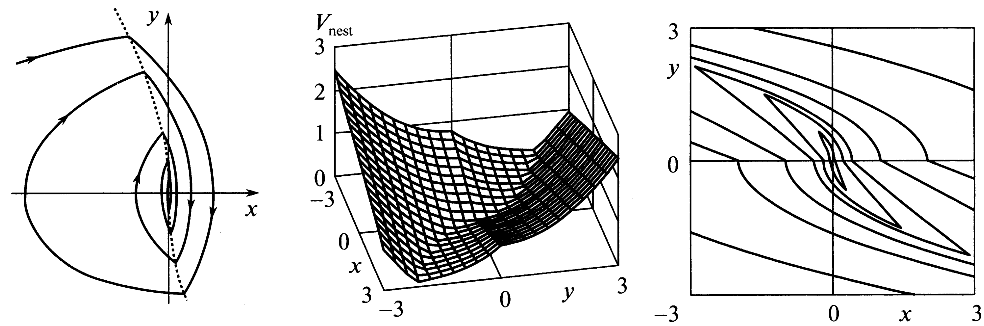

Fig. 3.

the sliding process reaches a specified value (the origin of coordinates, for example) after a finite time and, in the second case (see the left part of Fig. 3), the trajectory at once converges to the origin of coordinates after a finite time in a similar manner to the twisting algorithm considered above. [10]

In view of the bang-bang character of the nested control, the function (2.9) can again be taken as a candidate for the Lyapunov function for system (2.7), (3.8). In this case, it is necessary to remove the discontinuities in the function $V_{term}(x,y)$ on the curve (3.9). We have

$$V_{term} \to q^2\left(p\sqrt{\left|1/\beta^2 + \mathrm{sign}(y)/(2\tilde{\gamma})\right|} - \mathrm{sign}(y)/\tilde{\gamma}\right)^2 y^2/4 \quad \text{when } z \to \pm 0$$

In the case when a terminal sliding mode arises in a system with a nested controller, the Lyapunov function $V_{term}(x,y)$ must vanish on the curve (3.9). Hence, for $z(x,y) \to \pm 0$, we must have

$$p = \frac{\mathrm{sign}(y)}{\tilde{\gamma}\sqrt{\left|1/\beta^2 + \mathrm{sign}(y)/(2\tilde{\gamma})\right|}}$$

However, on choosing the parameter p in the proposed manner for all $(x,y) \in \mathbf{R}^2$, we obtain that $V_{term}(x,y)$ not only vanishes on the curve (3.9) but also on the curve $y - \beta\sqrt{|x|}sign(x) = 0$. It is more judicious to define p in the form

$$p := \frac{\sqrt{-\mathrm{sign}(\varphi)}}{\tilde{\gamma}\sqrt{\left|1/\beta^2 - \mathrm{sign}(\varphi)/(2\tilde{\gamma})\right|}}; \quad \varphi = \varphi(x,y) := x - y^2/(2\tilde{\gamma})$$

The following assertion can serve as a basis of the correctness of this choice.

Lemma 2. *If*

$$\beta^2 < 2(b^{\min}\alpha - C) \tag{3.10}$$

then

$$\mathrm{sign}(y) = -\mathrm{sign}(\varphi) \quad \text{when } z(x,y) \to \pm 0$$

Proof. When $z \to 0$ we have $y \to \beta\sqrt{|x|}sign(x)$ and sign(x)=−sign(y). Then,

$$\mathrm{sign}(\varphi) = \mathrm{sign}(x - \beta^2|x|/(2\tilde{\gamma})) = \mathrm{sign}(\mathrm{sign}(x) - \beta^2/(2\tilde{\gamma}))$$

from which the assertion of the lemma follows by virtue of the inequalities

$$\beta^2 < 2(b^{\min}\alpha - C) \le 2|\tilde{\gamma}|$$

Then, specifying that

$$q := 1, \quad \gamma := C\mathrm{sign}(z(x,y)), \quad \mu := b^{\min} \tag{3.11}$$

we finally obtain

$$V_{term}(x,y) = \begin{cases} \left(p\sqrt{\left|x - y^2/(2\tilde{\gamma})\right|} - y/\tilde{\gamma}\right)^2/4 & \text{when } z \ne 0 \\ 0 & \text{when } z = 0 \end{cases} \tag{3.12}$$

Theorem 3. *In the case of condition* (3.10), *the function* $V_{term}(x,y)$ *possesses the following properties:*

1) *it is continuous in* $\mathbf{R}^2$ *and continuously differentiable when* $\varphi(x,y) \ne 0$;
2) *it vanishes when* $z(x,y) = 0$ *and is positive when* $z(x,y) \ne 0$;
3) *the right upper Dini derivative of the function* $V_{term}(xg/\varepsilon)$, where $(x(t), y(t))$ *is an arbitrary solution of system* (2.7), (3.8), *satisfies the inequality*

$$D^* V_{\text{term}}(x(t), y(t)) \le -\sqrt{V_{\text{term}}(x(t), y(t))} \tag{3.13}$$

for all $t > 0$.

The Lyapunov function and its contour lines for system (2.7), (3.8) with the parameters

$$b^{\min} = 3/4, \quad b^{\max} = 1, \quad \alpha = 2, \quad \beta = 1, \quad C = 1/2$$

is shown in the middle and right parts of Fig. 2.

The Lyapunov function constructed enables us to estimate the finite time of incidence on the sliding curve (3.9)

$$t_r \le t_r^0 := 2\sqrt{V_{\text{term}}(x(0), y(0))}$$

The time of incidence at the origin of the coordinates can be estimated by studying the first-order sliding equation on the curve (3.9)

$$\dot{x} = -\beta\sqrt{|x|}\,\text{sign}(x) \quad \text{when} \quad t > t_r$$

The estimate of the time of entrance into a second-order sliding mode (incidence at the origin of the coordinates) has the form

$$t_0 \le t_r + 2\beta^{-1}\sqrt{\left|x(0) + y(0)t_r + t_r^2(C - b^{\min}\alpha)\,\text{sign}(z(x(0), y(0)))/2\right|} \tag{3.14}$$

In the case when a terminal sliding mode may or may not arise in system (2.7) with a nested controller (3.8) (see the left part of Fig. 3), the Lyapunov function is constructed by the same method as in the case of the twisting algorithm, but with the sole difference that the discontinuities in the curve (3.9) and the line $y = 0$ have to be eliminated.

The Lyapunov function $V_{\text{nest}}(x,y)$ for this case has the form of (3.6) with the conditions $xy \ne 0$ and $x = 0$ replaced by $yz(x,y) \ne 0$ and $z(x, y) = 0$. Here,

$$\gamma := C\,\text{sign}(z), \quad \mu := b^{\min}, \quad q := 1/|p|, \quad p := \frac{\text{sign}(y)/\tilde{\gamma}}{\sqrt{\left|1/\beta^2 + \text{sign}(y)/(2\tilde{\gamma})\right|} - \bar{k}}$$

and

$$\sqrt{1/(2|\tilde{\gamma}|) - 1/\beta^2} < \bar{k} < \sqrt{1/(2|\tilde{\gamma}|) + 1/\beta^2} - 1/\sqrt{2|\tilde{\gamma}|}$$

Theorem 4. *When $4(b^{\min}\alpha - C)/\sqrt{3} > \beta^2 > 2(b^{\min}\alpha - C)$, the function $V_{\text{nest}}(x,y)$ possesses the following properties:*

1) *it is positive definite and Lipschitzian in the neighbourhood of each point of the space $\mathbf{R}^2$ and also continuously differentiable when $yz(x,y) \ne 0$;*
2) *the derivative of the function $V_{\text{nest}}(x(t), y(t))$ along the trajectories of system (2.7), (3.8) satisfies the inequality*

$$\dot{V}_{\text{nest}}(x(t), y(t)) \le -\min\{\bar{k}\beta^2/2, q_{\min}\}\sqrt{V_{\text{nest}}(x(t), y(t))} \tag{3.15}$$

for almost all $t > 0$, where

$$q_{\min} = |\tilde{\gamma}| \min_{\delta \in \{-1, 1\}} \sqrt{\left|1/\beta^2 - \delta/(2|\tilde{\gamma}|)\right|}$$

The function $V_{\text{nest}}(x,y)$ and its contour lines are shown in Fig. 3 for the parameter values

$$b^{\min} = 4/5, \quad b^{\max} = 1, \quad \alpha = 1, \quad \beta = 5/4, \quad C = 1/10$$

"*The super-twisting*" *algorithm.* We now consider the "super-twisting" system

$$\dot{x} = -\alpha\sqrt{|x|}\,\text{sign}(x) + y, \quad \dot{y} \in [-L, L] - \beta\,\text{sign}(x) \tag{3.16}$$

where $x,y \in \mathbf{R}$ are scalar variables and α, β and L are positive parameters. A characteristic trajectory of system (3.16) is shown in the left part of Fig. 4.

The need to study such a second-order sliding system is caused by the numerous applications,[10–12,20] one of which we consider below.

The problem involves constructing a Lyapunov function that proves the convergence of any solution of system (3.16) to the origin of coordinates (0,0) after a finite time. A second-order sliding mode will obviously arise on the line $x = 0$.

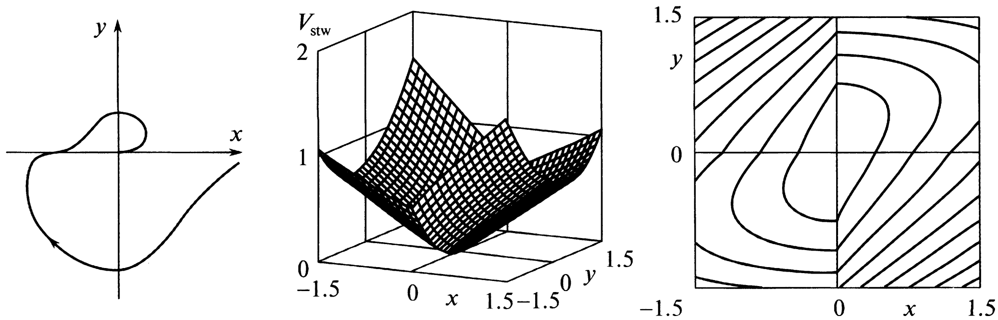

Fig. 4.

Following the proposed method of searching for candidates for the Lyapunov function, we obtain (the approach previously described[21] is used before and the details of the proofs are therefore omitted)

$$V_{stw}(x,y) = \frac{k^2}{4}\left(\frac{y\,\mathrm{sign}(x)}{\tilde{\gamma}} + k_0 e^{m(x,y)}\sqrt{s(x,y)}\right)^2$$

$$\tilde{\gamma} = \beta - \gamma, \quad \gamma = L\,\mathrm{sign}(x\partial V/\partial y)$$

$$m(x,y) = \frac{1}{\sqrt{g-1}}\mathrm{arctg}\left(\frac{\alpha g\sqrt{|x|}\,\mathrm{sign}(x) - 2y}{2\sqrt{g-1}y}\right), \quad \gamma = \frac{8\tilde{\gamma}}{\alpha^2} > 1$$

$$s(x,y) = 2\tilde{\gamma}|x| - \alpha\sqrt{|x|}\,\mathrm{sign}(x)y + y^2 \tag{3.17}$$

where k and k_0 are parameters.

As in the preceding cases, the discontinuities in the function (3.17) are removed by means of a special choice of the parameters k and k_0:

$$k_0 = \frac{\mathrm{sign}(xy)}{\sqrt{\tilde{\gamma}}|k|}\exp\left(-\frac{\pi\,\mathrm{sign}(xy)}{2\sqrt{g-1}}\right), \quad k = \sqrt{\tilde{\gamma}}|\bar{k}\sqrt{g} - \psi(-\mathrm{sign}(xy))|$$

$$\psi(\delta) = \exp\left(\frac{\pi}{2\sqrt{g-1}}\delta - \frac{\mathrm{arctg}(1/\sqrt{g-1})}{\sqrt{g-1}}\right)$$

$$\bar{k} \in I(g^-) \cap I(g^+), \quad I(g) = \left(\frac{2}{g} + \frac{\psi(-1)}{\sqrt{g}}, \frac{\psi(1)}{\sqrt{g}}\right), \quad g^- = \frac{8(\beta - L)}{\alpha^2}, \quad g^+ = \frac{8(\beta + L)}{\alpha^2}$$

($\bar{k}$ is a numerical parameter).

Theorem 5. *if* $\beta > 5L$ *and* $32L < \alpha^2 < 8(\beta - L)$, *the function* $V_{stw}(x,y)$ *possesses the following properties:*

1) *it is continuous and Lipschitzian in the neighbourhood of each point of the space* $\mathbf{R}^2$ *and is also continuously differentiable when* $xy \neq 0$;
2) *the derivative of the function* $V(x(t), y(t))$, *where* $(x(t), y(t))$ *is an arbitrary solution of system* (3.16), *satisfies the inequality*

$$\dot{V}_{stw}(x(t), y(t)) \le -k_{min}\sqrt{V_{stw}(x(t), y(t))} \tag{3.18}$$

for almost all $t > 0$, *where*

$$k_{min} = \frac{\alpha}{\sqrt{8}}\min_{g \in \{g^-, g^+\}}\{\sqrt{g}|\bar{k}\sqrt{g} - \psi((\beta - \alpha^2 g)/8)|\}$$

The Lyapunov function $V_{stw}(x,y)$ and its contour lines are shown in Fig. 4 for the case when $\alpha = \beta = 1$ and $L = 0.2$.

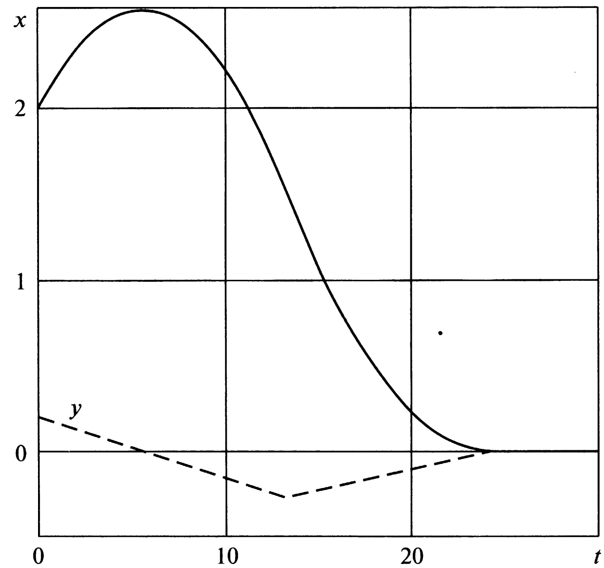

Fig. 5.

4. Examples of the use of the results obtained in problems of identification and control under conditions of bounded perturbations

4.1. Stabilization of an object with a variable mass

We will now consider the equation of the dynamics of a body with a variable mass (the Meshcherskii equation)

$$m(t)\ddot{x}(t) = \dot{m}(t)\upsilon + f(t,x,\dot{x}); \quad |f(t,x,\dot{x})| \le f_0, \quad 0 < m_0 \le m(t) \le m_1 \tag{4.1}$$

where x is the scalar coordinate of the centre of mass, $m(t)$ is the mass of the body at an instant t, υ is the velocity of the separating material with respect to the body and $f(t, x, \dot{x})$ are the bounded perturbations of the system. It is also assumed that the mass of the body at each instant is unknown and only the range over which it changes is known.

We shall assume that system (4.1) can be stabilized by controlling the reactive force $F = \dot{m}(t)\upsilon$, which can take the values $F^0 \in \{-\alpha, 0, +\alpha\}$ with a certain relative error δ, that is, $F = (1 \pm \delta)F^0$. Reducing the initial system (4.1) to the form

$$\dot{x} = y, \quad \dot{y} = \frac{(1 \pm \delta)F^0}{m(t)} + \frac{f(t,x,\dot{x})}{m(t)}$$

and treating F^0 as a control, we can solve the problem of stabilizing of the given system using the second order sliding algorithms considered above. Using the results for a terminal controller, we define the control in the form

$$F^0 = -\alpha\,\mathrm{sign}(y + \beta\sqrt{|x|}\,\mathrm{sign}(x))$$

$$\alpha > m_1 f_0/(m_0(1-\delta)), \quad 0 < \beta < \sqrt{2(\alpha(1-\delta)/m_1 - f_0/m_0)}$$

The trajectory of system (4.1), obtained by numerical modelling of the control process for an initial state (2, 0.2) and the parameters

$$m_0 = 8, \quad m_1 = 10, \quad f_0 = 0.2, \quad \delta = 0.03, \quad \alpha = 0.5155, \quad \beta = 0.2214$$

is shown in Fig. 5.

The estimate of the time of incidence of the system at the origin of the coordinates $t_r \le 29.32$, obtained using a Lyapunov function according to formula (3.14), was confirmed by numerical modelling when this time was found to be equal to 24.4.

Other examples of the use of second-order sliding algorithms for controlling mechanical systems are available. [11,13]

4.2. Identification of a state using a “super-twisting” algorithm

Consider the system

$$\dot{x} = y, \quad \dot{y} = f(t,x,y) + \varepsilon(t,x,y) \tag{4.2}$$

where $x,y \in \mathbf{R}$ are scalar state variables (for a mechanical system, x has the meaning of a coordinate and y the meaning of a velocity), the function $f(t,x,y)$ is assumed to be known and $\varepsilon(t,x,y)$ is an unknown function characterizing the bounded perturbations and the indeterminacy in the model parameters.

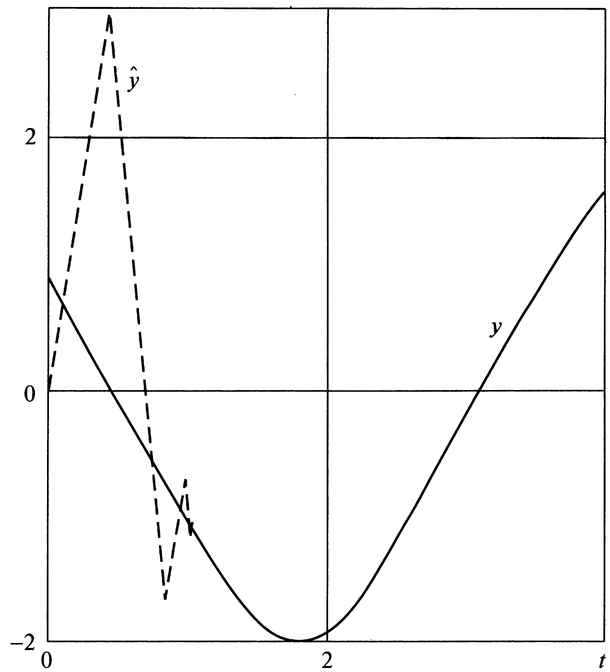

Fig. 6.

It is assumed that only the quantity x is accessible for measurement and it is required to construct an identifier of the second state variable (y). For this purpose, we make use of a “super-twisting” algorithm and introduce into the treatment an observer’s equation in the form

$$\dot{\hat{x}} = \hat{y} + \alpha\sqrt{|x-\hat{x}|}\,\mathrm{sign}(x-\hat{x}), \quad \dot{\hat{y}} = f(t,x,\hat{y}) + \beta\,\mathrm{sign}(x-\hat{x})$$
$$\hat{x}(0) = \hat{y}(0) = 0 \tag{4.3}$$

where $x,y \in \mathbf{R}$ are the state variables of the observation system, and α and β are positive parameters. For the equations in the deviations $\bar{x} = x - \hat{x}$, $\bar{y} = y - \hat{y}$, we shall then have

$$\dot{\bar{x}} = -\alpha\sqrt{|\bar{x}|}\,\mathrm{sign}(\bar{x}) + \bar{y}, \quad \dot{\bar{y}} = g(t,x,y,\hat{y}) - \beta\,\mathrm{sign}(\bar{x})$$

$$g(t,x,y,\hat{y}) = f(t,x,y) - f(t,x,\hat{y}) + \varepsilon(t,x,y)$$

If the functions $g(t,x,y,\hat{y})$ is bounded:

$$|g(t,x,y,\hat{y})| \le L \quad \text{for all} \quad x,y,\hat{y} \in R, \quad t > 0$$

then, on the basis of the results from the super-twisting algorithm, the choice of the parameters from the conditions

$$\beta > 5L, \quad \sqrt{32L} < \alpha < \sqrt{8(\beta - L)}$$

guarantees the finite-time convergence of the error of observation to zero after a finite time.

As an example, we consider system (4.2) when

$$f(t,x,y) = -0.05\,\mathrm{sign}(y) - 2\sin(x), \quad |\varepsilon(t,x,y)| < 0.1$$

Then, $l = 0.2$ and, for $\beta = 9$ and $\alpha = 3$, the observer’s variables, obtained by numerical integration of system (4.3), will converge after a finite time to the exact values of the state variables of the system. Results of the numerical modelling of the corresponding transient are shown in Fig. 6.

A Lyapunov function of the form (3.17) can be used to estimate the guaranteed time of convergence. Since the estimate of the time

$$t_r \le 2\sqrt{V_{\mathrm{stw}}(\bar{x}(0), \bar{y}(0))}/k_{\min}$$

requires a knowledge of the unknown value $\bar{y}(0)$, it is not possible in practice to obtain this estimate without prior estimation of the maximum possible value $|y(0)|$. Assuming that $|y(0)|<1$, we have $t_r \le 3.38$ and this is confirmed by a numerical experiment, according to which t_r=1.05.

The construction of an observer was carried out earlier [12] for more general mechanical systems. The use of higher-order sliding modes to construct state observers is also discussed in Ref. 22.

Acknowledgement

We thank F. L. Chernous'ko and participants in the seminar supervised by him for their internet and remarks.

Appendix A.

A.1. Proof of Theorem 2

Property 1. *Since all the parameters of the function $V_{tw}(x,y)$ depend solely on sign(x) and sign(y), this function can only have discontinuities on the lines $x = 0$ and $y = 0$. At the same time, the parameters p and q were chosen such that the discontinuities on the above-mentioned lines were eliminated. The constructed function $V_{tw}(x,y)$ is therefore continuous.*

By virtue of inequality (3.2), the positive definiteness of the function $V_{tw}(x,y)$ follows from Lemma 2 and the continuous differentiability when $xy \neq 0$ is checked by direct calculations of the partial derivatives.

We also note that the function $V_{tw}(x,y)$ is continuously differentiable right up to the boundaries $x = 0$ and $y = 0$ and has finite jumps in the partial derivatives on them for any finite x and y.

Consequently, it is Lipschitzian in the neighbourhood of each point of the space $\mathbf{R}^2$.

Property 2. *We calculate the total derivative of the function $V_{tw}(x,y)$ along the trajectory $(x(t), y(t))$ of system* (2.7), (3.1) *for all* $t > 0$, $x(t) \neq 0$, $y(t) \neq 0$

$$\dot{V} = q\sqrt{V_{\text{tw}}}\left(-\frac{\bar{\tilde{\gamma}}}{\underline{\tilde{\gamma}}} + \frac{p|y|\operatorname{sign}(xy)}{2\sqrt{|x| + y^2/|2\tilde{\gamma}|}}\left(1 - \frac{\bar{\tilde{\gamma}}}{\underline{\tilde{\gamma}}}\right)\right);\ \ \tilde{\gamma} = \bar{\gamma} + \bar{\mu}u,\ \ \bar{\gamma} \in [-C, C],\ \ \bar{\mu} \in [b^{\min}, b^{\max}]$$

We have

$$\frac{\bar{\tilde{\gamma}}}{\underline{\tilde{\gamma}}} = \frac{\bar{\mu}(r_1 + r_2) + \bar{\gamma}\operatorname{sign}(x)}{b^{\min}(r_1 + r_2) - C} \le 1, \quad \sup_{\forall\bar{\gamma},\,\forall\bar{\mu}} \dot{V}_{\text{tw}} \le -q\sqrt{V_{\text{tw}}} \ \ \text{when } xy > 0$$

$$\frac{\bar{\tilde{\gamma}}}{\underline{\tilde{\gamma}}} = \frac{\bar{\mu}(r_1 - r_2) + \bar{\gamma}\operatorname{sign}(x)}{b^{\min}(r_1 - r_2) + C} \ge \sup_{\forall\bar{\gamma},\,\forall\bar{\mu}} \dot{V}_{\text{tw}} \le -q\sqrt{V_{\text{tw}}} \ \text{when } xy < 0$$

$$\beta = \frac{b^{\min}(r_1 - r_2) - C}{b^{\max}(r_1 - r_2) + C}$$

It can be shown that, in the case of the constraints imposed on the parameters r_1, r_2 and C, all trajectories of system (2.7), (3.1) will intersect the lines $x = 0$ and $y = 0$ at isolated instants and, moreover, there will be no motion along these lines. The instants of incidence at the origin of coordinates $t_0 > 0$: $x(t_0) = y(t_0) = 0$, for which the derivative can be calculated by definition, are an exception. By virtue of the system

$$\dot{x}(t_0) = y(t_0), \quad \dot{y}(t_0) = \bar{\gamma} + \bar{\mu}\bar{u}$$

$$\bar{\gamma} \in [-C, C], \quad \bar{\mu} \in [b^{\min}, b^{\max}], \ \bar{u} \in [-r_1 - r_2, r_1 + r_2]$$

we have

$$\dot{V}_{\text{tw}}(x(t), y(t))_{t = t_0} = \lim_{h \to 0} \frac{V_{\text{tw}}(x(t_0 + h), y(t_0 + h)) - V_{\text{tw}}(x(t_0), y(t_0))}{h} =$$

$$= \lim_{h \to 0} \frac{V_{\text{tw}}(x(t_0) + hy(t_0) + o(h), y(t_0) + \eta)}{h} = \lim_{h \to 0} \frac{q^2\left(p\sqrt{|o(h)| + \eta^2/(2\tilde{\gamma})} - \eta/\tilde{\gamma}\right)^2}{4h} = 0$$

$$\eta = h(\bar{\gamma} + \bar{\mu}\bar{u}) + o(h)$$

It has therefore been shown that inequality (3.7) holds for almost all $t > 0$.

Proof of Theorem 3.

Property 1. *The parameters γ, μ and p of the function $V_{term}(x,y)$ depend on sign($\varphi(x,y)$) and sign(z(x, y)) and, consequently, the function $V_{term}(x,y)$ can only have discontinuities when $\varphi(x,y)z(x,y) = 0$. Note that, in the case of the constraints imposed on the system parameters and the control mentioned in this theorem, the curves $z(x, y) = 0$ and $\varphi(x,y) = 0$ only intersect at the origin of coordinates. The continuity of the function $V_{term}(x,y)$ on the curve $z(x, y) = 0$ was attained due to the choice of a suitable p, and the continuity when $\varphi(x,y) = 0$ follows from the limit*

$$V_{\text{term}}(x, y) \to y^2/|\tilde{\gamma}| \ \ \text{when} \ \ \varphi(x, y) \to 0$$

which is independent of sign(φ). Hence, the function $V_{term}(x,y)$ is continuous in the whole space $\mathbf{R}^2$. The continuous differentiability of the function $V_{term}(x,y)$ when $\varphi(x,y)=0$ is verified by direct calculations of the partial derivatives.

Property 2. *The choice of the parameter p in the form of* (3.2) *guarantees that* $V_{term}(x,y)=0$ *when* $z(x,y)=0$.

We will now show that

$$\upsilon(x,y) = p\sqrt{|x-y^2/(2\tilde{\gamma})|} - y/\tilde{\gamma} > 0, \quad (x,y)\in\mathbf{R}^2: z(x,y)\neq 0$$

Actually,

a) when $\varphi=0$ and $z\neq 0$, we obtain $xz<0$, $yz>0$ and $\upsilon(x,y)=|y||\tilde{\gamma}|>0$,
b) for $\varphi z<0$, we have $xz<0$, $yz>0$ and $\bar{\bar{y}}^2/|y|<x<y^2/\beta^2$. It follows from this that $\upsilon(x,y)>0$;
c) the case when $\varphi z>0$ is considered in a similar manner.

Since $V_{term}(x,y)=\upsilon^2(x,y)/4$, the required property of the positive semi-definiteness of the function $V_{term}(x,y)$ is proved.

Property 3. *When* $t>0$: $\varphi(x(t),y(t))$, *we have*

$$\dot{V}_{term} = \sqrt{V_{term}}\left(-\frac{\tilde{\tilde{\gamma}}}{\tilde{\gamma}} + \frac{p|y|\,\mathrm{sign}(y\varphi)}{2\sqrt{|\varphi|}}\left(1-\frac{\tilde{\tilde{\gamma}}}{\tilde{\gamma}}\right)\right)$$
$$\tilde{\tilde{\gamma}} = \bar{\gamma}+\bar{\mu}u, \quad \bar{\gamma}\in[-C,C], \quad \bar{\mu}\in[b^{min},b^{max}]$$

Taking account of the fact that $\bar{\bar{y}}/\bar{y}\geq 1$ and

$$1+\frac{p|y|\,\mathrm{sign}(y\varphi)}{2\sqrt{|\varphi|}}\geq 0, \quad (x,y)\in\mathbf{R}^2$$

we obtain $\dot{V}_{term}\leq-\sqrt{V_{term}}$ when $t>0:\varphi(x(t),y(t))\neq 0$.

For instants $t>0:\varphi(x(t),y(t))=0$, the inequality is verified by calculating the right Dini derivative number which is equal to $-\infty$.

Proof of Theorem 4. To start with, we note that the following equalities

$$|\tilde{\gamma}| = b^{min}\alpha - C, \quad \mathrm{sign}(\varphi) = \mathrm{sign}(z) \text{ and } \varphi(x,y)>0 \text{ when } x^2+y^2>0$$

hold in the case of the constraints specified in the theorem that are imposed on the system parameters and the control.

Property 1. *The continuity in* $\mathbf{R}^2$ *and the continuous differentiability of the function* $V_{nest}(x,y)$ *when* $z(x,y)x\neq 0$ *are verified by analogy with the arguments presented in the proof of* Theorem 2.

Property 2. *When* $t>0:y(t)z(x(t),y(t))\neq 0$, *we have*

$$\dot{V}_{nest}(x(t),y(t)) = q\sqrt{V_{nest}}\left(-\frac{\tilde{\tilde{\gamma}}}{\tilde{\gamma}} + \frac{p|y|\,\mathrm{sign}(yz)}{2\sqrt{|\varphi(x,y)|}}\left(1-\frac{\tilde{\tilde{\gamma}}}{\tilde{\gamma}}\right)\right)$$

where $\tilde{\tilde{\gamma}}=\bar{\gamma}+\bar{\mu}u$, $\bar{\gamma}\in[-C,C]$, $\bar{\mu}\in[b^{min},b^{max}]$. *From the inequalities* $(\bar{\gamma}+\bar{\mu}u)/(\gamma+\mu u)\geq 1$ *and*

$$1+\frac{p|y|\,\mathrm{sign}(yz)}{2\sqrt{|\varphi|}}\geq 0 \quad \text{for all} \quad (x,y)\in\mathbf{R}^2$$

we then obtain $\dot{V}_{nest}(x(t),y(t))\leq -q\sqrt{V_{nest}(x(t),y(t))}$.

If a sliding mode arises on the curve $z(x,y)=0$, then, for the variable y, we obtain the following sliding equation $\dot{y}=-0.5\beta^2 sign(y)$. Calculating, in this case, the derivative $\dot{V}_{nest}(x(t),y(t))$ according to the definition, we obtain

$$\dot{V}_{nest}(x(t),y(t)) = \lim_{h\to 0}\frac{\bar{k}^2(y(t)+h(-0.5\beta^2\,\mathrm{sign}(y(t))))^2/4-\bar{k}^2y^2(t)/4}{h} =$$
$$= -\frac{\bar{k}\beta^2}{2}\sqrt{V_{nest}(x(t),y(t))}$$

In order to complete the proof it merely remains to note that any trajectory of system (2.7) with the control (3.8) always intersects the line $y=0$ at isolated instants and, moreover, there will be no motion along this line. Consequently, inequality (3.15) is satisfied for almost all $t>0$.

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