Robust output linear controller for a class of nonlinear systems based on invariant ellipsoid technique

S. Gonzalez Garcia, A. Polyakov, A. Poznyak

CINVESTAV-IPN,Mexico

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Problem Statement(1)

Consider the nonlinear dynamic system

$$\begin{aligned} \dot{x} &= f(x,t) + Bu \\ y &= Cx + w_y \end{aligned} \tag{1}$$

where $x \in \mathrm{R}^n$ is the state vector, $u \in \mathrm{R}^m$ is the control input, $y \in \mathrm{R}^k$ is the output, $w_y \in \mathrm{R}^k$ is the output perturbation, $f : \mathrm{R}^n \times \mathrm{R} \to \mathrm{R}^n$ is an unknown nonlinear function, $B \in \mathrm{R}^{n \times m}$, $C \in \mathrm{R}^{k \times n}$ are system matrices:

- the output disturbances w_y are inside of a bounded ellipsoid, that is,

$$\|w_y\|^2_{K_\eta} = w_y^T K_\eta w_y \leq 1 \tag{2}$$

  where $K_\eta > 0$ is given;
- the nonlinear function $f(x,t)$ is quasi-Lipschitz

$$\|f(x,t) - Ax\|^2_{K_f} \leq \delta + \|x\|^2_{K_x}$$

  where $0 < K_f \in \mathrm{R}^{n \times n}$, $0 < K_x \in \mathrm{R}^{n \times n}$, $A \in \mathrm{R}^{n \times n}$ are known matrices, $\delta \geq 0$.
- The pair (A, B) is controllable and the pair (A, C) is observable.

Problem Statement(2)

Using denotation $w_x := f(x,t) - Ax$ the system (1) can be rewritten in the form

$$\dot{x} = Ax + Bu + w_x \quad (3)$$

$$y = Cx + w_y \quad (4)$$

with

$$\|w_x\|^2_{K_f} \leq \delta + \|x\|^2_{K_x} \quad (5)$$

Here we consider only the linear feedback controls

$$u = K\hat{x}, \quad K \in \mathrm{R}^{m\times n} \quad (6)$$

with respect to observed state $\hat{x} \in \mathrm{R}^n$ which are obtained by the classical Luenberger observer having the structure

$$\dot{\hat{x}} = A\hat{x} + Bu + F(y - C\hat{x}), \quad F \in \mathrm{R}^{n\times p} \quad (7)$$

The robust stabilization of the system (3),(4) will be realized using the method of *Invariant Ellipsoids (see Nazin,Polyak,Topunov 2007)*.

Full System

Define the state estimation error as $e := x - \hat{x}$. Then its time derivative satisfies

$$\dot{e} = (A - FC)e + w_x - Fw_y$$

Introduce the extended vector $z := \begin{pmatrix} \hat{x} & e \end{pmatrix}^T$ where $z \in \mathbb{R}^{2n}$. Then it follows

$$\dot{z} = \hat{A}z + \hat{F}w \tag{8}$$

where

$$\hat{A} := \begin{pmatrix} A + BK & FC \\ 0 & A - FC \end{pmatrix}, \hat{F} := \begin{pmatrix} 0 & F \\ I & -F \end{pmatrix} \text{ and } w := \begin{pmatrix} w_x \\ w_y \end{pmatrix}$$

Our aim here is to find the control gain matrix K and the observer gain matrix F providing a "good enough" stabilization as well as state estimation of the system (8), or more exactly, to design K and F such that the corresponding invariant ellipsoid, called below "*quasi-minimal*", would contain the minimal one.

Main Result(1)

$$\mathrm{tr}(X_1) + \mathrm{tr}(H) \to \min \tag{9}$$

$$\begin{pmatrix} R_1 & Y_2^T C & 0 & Y_2^T & X_2 X_1 \\ C^T Y_2 & \begin{matrix} A^T X_2 + X_2 A - \\ Y_2^T C - C^T Y_2 + \tau_1 X_2 \end{matrix} & X_2 & -Y_2^T & I \\ 0 & X_2 & -\tau_2 K_f & 0 & 0 \\ Y_2 & -Y_2 & 0 & -\tau_3 K_\eta & 0 \\ X_1 X_2 & I & 0 & 0 & -\frac{1}{\tau_2} K_x^{-1} \end{pmatrix} \leq 0 \tag{10}$$

$$\begin{pmatrix} -R_1 - 2X_2 & I \\ I & X_1 A^T + A X_1 + Y_1 B^T + B Y_1^T + \tau_1 X_1 \end{pmatrix} \leq 0$$

$$\begin{pmatrix} H & I \\ I & X_2 \end{pmatrix} \geq 0, X_1 \geq 0, X_2 \geq 0, H \geq 0, \tau_1 \geq \delta\tau_2 + \tau_3, \; \tau_2 \geq 0, \tau_3 \geq 0$$

with respect to the matrix variables $H, X_1, X_2, R_1, R_2 \in \mathrm{R}^{n \times n}$, $Y_1 \in \mathrm{R}^{n \times m}$, $Y_2 \in \mathrm{R}^{k \times n}$ and the scalar variables τ_1, τ_2 and τ_3

$$P = \begin{pmatrix} X_1 & 0 \\ 0 & X_2^{-1} \end{pmatrix} \quad (11)$$

is the matrix of quasi-minimal invariant ellipsoid of the system (8), (2), (5) with the feedback control gain matrix

$$K = \left(X_1^{-1} Y_1\right)^T \quad (12)$$

and the observer gain matrix

$$F = \left(Y_2 X_2^{-1}\right)^T \quad (13)$$

The ellipsoid

$$\varepsilon(0, P) = \left\{ z \in \mathrm{R}^{2n} : z^T P^{-1} z \leq 1 \right\},$$

with center in the origin and shape matrix P is said to be *state-invariant* for the system (3),(4) under the disturbances (2) and nonlinearities (5) if
1) the condition $z(0) \in \varepsilon(0, P)$ implies $z(t) \in \varepsilon(0, P)$ for all $t \geq 0$;
2) if $z(0) \notin \varepsilon(0, P)$ then $z(t) \to \varepsilon(0, P)$ for $t \to \infty$.

Lemma

The set of variables satisfying (10) contains the set of ones satisfying

$$\begin{pmatrix} R_1 & Y_2^T C & 0 & Y_2^T & 0 \\ C^T Y_2 & A^T X_2 + X_2 A - Y_2^T C - C^T Y_2 + \tau_1 X_2 & X_2 & -Y_2^T & I \\ 0 & X_2 & -\tau_2 K_f & 0 & 0 \\ Y_2 & -Y_2 & 0 & -\tau_3 K_\eta & 0 \\ 0 & I & 0 & 0 & R_2 \end{pmatrix} \leq 0 \tag{14}$$

$$\begin{pmatrix} -R_1 - 2X_2 & I \\ I & X_1 A^T + A X_1 + Y_1 B^T + B Y_1^T + \tau_1 X_1 + \Lambda \end{pmatrix} \leq 0$$

$$\begin{pmatrix} -\frac{1}{\tau_2} K_x^{-1} - R_2 & X_1 \\ X_1 & -\Lambda \end{pmatrix} \leq 0, \begin{pmatrix} H & I \\ I & X_2 \end{pmatrix} \geq 0$$

$$\tau_1 \geq \delta\tau_2 + \tau_3, \ \tau_i \geq 0, \ i = 1, 2, 3$$

Numerical Example: Model(1)

Consider the model of a spacecraft with two dynamic elastic elements (rods) (having dissipative properties) as it is shown in the figures below:

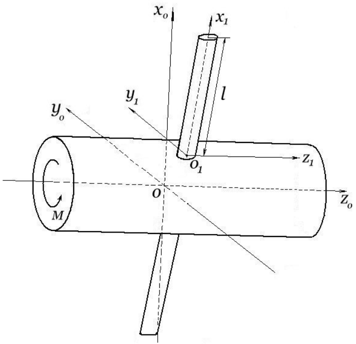


Fig.1. The spacecraft scheme.

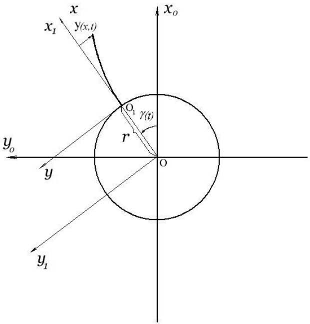


Fig. 2. The spacecraft plane view.

Numerical Example: Model(2)

$$J\ddot{\gamma}(t) + 2\int_0^l m(x+r)\frac{\partial^2 y(x,t)}{\partial t^2}dx = u(t) + \bar{f}_0\,(t,x) \qquad (15)$$

$$m(x+r)\ddot{\gamma} + m\frac{\partial^2 y(x,t)}{\partial t^2} + EI\frac{\partial^4 y(x,t)}{\partial x^4} + EI\chi\frac{\partial^5 y(x,t)}{\partial t \partial x^4} = \bar{f}\,(t,x) \qquad (16)$$

$$y(0,t) = \left.\frac{\partial y(x,t)}{\partial x}\right|_{x=0} = \left.\frac{\partial^2 y(x,t)}{\partial x^2}\right|_{x=l} = \left.\frac{\partial^3 y(x,t)}{\partial x^3}\right|_{x=l} = 0$$

where r is the distance from the longitudinal axis to the point where the rod is fastened, l is the rod length, EI is the flexural stiffness of the rod, χ is the coefficient of internal viscous friction, m is the mass per unit length of the rod, J_0 is the moment of inertia of the spacecraft with respect to the OZ-axis and u is the control moment, J is the moment of inertia of the whole system given by

$$J = J_0 + 2\int_0^l m(x+r)^2 dx$$

Using the Galerkin's method we can suppose approximately that

$$y(x,t)=\sum_{i=1}^{k} q_i(t)\Phi_i(x) \tag{17}$$

where $\Phi_i(x)$ is the natural form corresponding to the positive eigenvalue λ_i of the positive self-conjugate operator

$$L\Phi(x)=\frac{d^4\Phi(x)}{dx^4},\quad \Phi(0)=\Phi'(0)=\Phi''(l)=\Phi'''(l)=0$$

where

$$\frac{d^4\Phi_i(x)}{dx^4}\equiv\lambda_i\Phi_i(x),\quad 0\le x\le l$$

Numerical Example: Model Approximation

After substitution (17) into (15), (16), multiplication of (16) on $\Phi_i(x)$ and an integration on the interval $[0, l]$, we get

$$J\ddot{\gamma} + 2\sum_{i=1}^{k} p_i \ddot{q}_i = u(t) + f_0 \tag{18}$$

$$p_i\ddot{\gamma} + a_i\ddot{q}_i + b_i\dot{q}_i + c_i q_i = f_i, \quad i = 1, 2, ..., k \tag{19}$$

with

$$p_i = \int_0^l m(x+r)\Phi_i(x)dx, \; a_i = \int_0^l m\Phi_i^2(x)dx, b_i = \lambda_i EI\chi \int_0^l \Phi_i^2(x)dx,$$

$$c_i = \lambda_i EI \int_0^l \Phi_i^2(x)dx, \; f_i(t) = \int_0^l \bar{f}(t, x)\,\Phi_i(x)dx$$

and f_0 describes (in general, bounded) disturbances and unmodelled dynamics of the rigid body.

Numerical Example: Simulation(2)

Consider the case of one tone of oscilations $(k = 1)$ in (18) and in (19). Introduce the state space variables $x_1 = \gamma$, $x_2 = \dot{\gamma}$, $x_3 = q_1$ and $x_4 = \dot{q}_1$. For the spacecraft with the parameters $J_0 = 150$, $l = 7.5$, $m = 0.53$, $r = 3$, $EI = 20$, and $\chi = 0.1$ the state space representation of the dynamic model is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0.0028 & 0.0142 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -0.0825 & -0.4126 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0.0076 \\ 0 \\ -0.1676 \end{pmatrix} u + w_x$$

where

$$w_x = \begin{pmatrix} 0 & 0 \\ 0.0076 & 0.0037 \\ 0 & 0 \\ -0.1676 & -0.0979 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$$

Supposing that only the angle γ and the angular speed $\dot{\gamma}$ are measurable, the state-output mapping is

Numerical Example: Simulation(2)

Let the output noise be estimated as follows

$$w_y^T K_\eta w_y \le 1, K_\eta = \begin{pmatrix} 530 & 25 \\ 25 & 1960 \end{pmatrix}$$

Assume also that system disturbances are bounded as

$$|f_0| \le 0.1 + 0.05|\gamma| + 0.1|\dot{\gamma}| + 0.06|q| + 0.11|\dot{q}|$$

and

$$|f_1| \le 0.1 + 0.011|\gamma| + 0.02|\dot{\gamma}| + 0.07|q| + 0.012|\dot{q}|$$

One can rearrange them to the form (5) with

$$K_f = 10^6 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, K_x = \begin{pmatrix} 0.0001 & 0.0042 & 0 & -0.0946 \\ 0.0042 & 2.0308 & 0.0072 & -46.106 \\ 0 & 0 & 0.0001 & -0.1691 \\ 0 & 0 & 0 & 1047.4 \end{pmatrix}$$

Numerical Example: Simulation(3)

For sufficiently large deviation of the spacecraft from the origin we will apply a minimum-time optimal control which will be designed in the phase plane (β, ν) with $\nu = \dot{\beta}$ and a local stabilization scheme based on linear feedback. As it is well known, the solution of the minimum-time optimal control problem is a bang-bang control given by

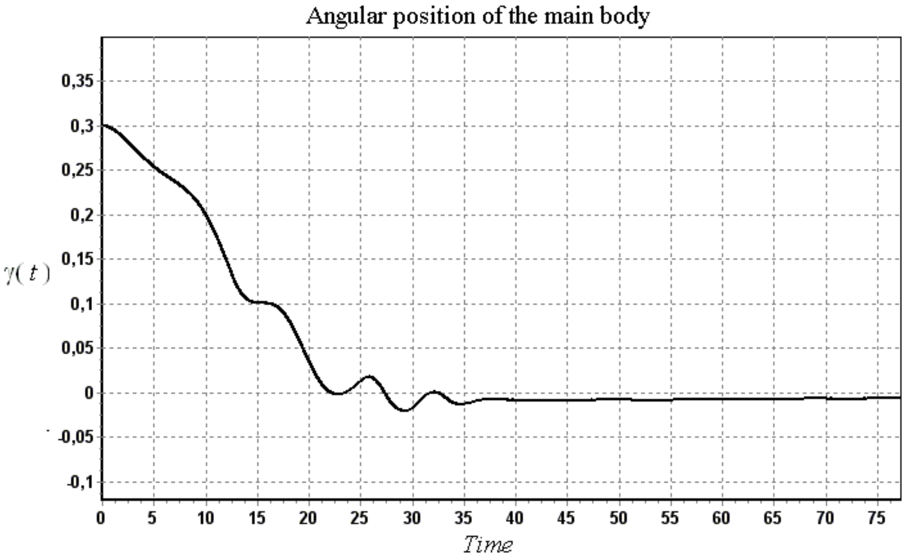
$$u = \begin{cases} 1, & 0 \leq t \leq T_1 = \dot{\gamma}(0) + \sqrt{\gamma(0)/2} \\ -1, & T_1 < t \leq T_2 = \dot{\gamma}(0) + 2\sqrt{\gamma(0)/2} \\ 0, & t > T_2 \end{cases}$$

For the local stabilization part $(t > T_2)$, a linear feedback is designed according to the main results of the previous section. The obtained parameters K and F realizing the robust output linear controller are as follows

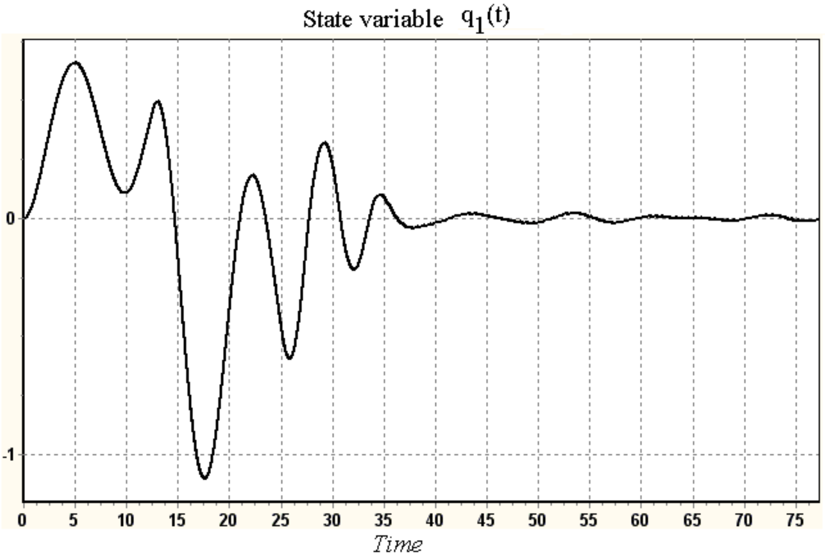
$$K = \begin{pmatrix} -26.8393 & -2689.3 & 82.86 & -74.357 \end{pmatrix}$$

$$F = \begin{pmatrix} 1.6946 & -0.0063 & 0.0596 & -0.0534 \\ -0.6719 & 1.0317 & -0.0325 & 0.0632 \end{pmatrix}^T$$

Numerical Example: Simulation(4)



Numerical Example: Simulation(6)



Proof(1)

Define the Lyapunov function as

$$V(z) := (z, P^{-1}z), \;\; P^{-1} = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$$

where $P > 0$ is the matrix of an invariant ellipsoid should be minimized. Then

$$\begin{aligned} \dot{V} = (z, P^{-1}\dot{z}) + (\dot{z}, P^{-1}z) = (z, P^{-1}(\hat{A}z + \hat{F}w)) + ((\hat{A}z + \hat{F}w), P^{-1}z) \\ = z^T[\hat{A}^T P^{-1} + P^{-1}\hat{A}]z + w^T \hat{F}^T P^{-1} z + z^T P^{-1} \hat{F} w \end{aligned}$$

This ellipsoid $(z, P^{-1}z) \leq 1$ will be invariant if and only if outside of it we have $\dot{V} \leq 0$, that is, for z satisfying

$$z^T P^{-1} z \geq 1 \qquad (20)$$

we have

$$\dot{V} = \begin{pmatrix} z \\ w_x \\ w_y \end{pmatrix}^T \begin{pmatrix} \hat{A}^T P^{-1} + P^{-1}\hat{A} & P^{-1}\hat{F} \\ \hat{F}^T P^{-1} & 0 \end{pmatrix} \begin{pmatrix} z \\ w_x \\ w_y \end{pmatrix} \leq 0 \qquad (21)$$

Proof(2)

Together with (20) and (21) we also have

$$w_x K_f w_x \leq \delta + x^T K_x x \text{ and } \|w_y\|^2_{K_\eta} \leq 1 \tag{22}$$

To fullfill all this constrains let us apply the, so-called, S-procedure: if there exist $\tau_1 \geq 0$, $\tau_2 \geq 0$ and $\tau_3 \geq 0$ such that

$$\begin{array}{c} \alpha_0 \geq \tau_1\alpha_1 + \tau_2\alpha_2 + \tau_3\alpha_3 \\ A_0 \leq \tau_1 A_1 + \tau_2 A_2 + \tau_3 A_3 \\ \tau_1 \geq 0, \ \tau_2 \geq 0, \ \tau_3 \geq 0 \end{array} \tag{23}$$

then the conditions (20),(21), (2) and (5) hold and the ellipsoid with the matrix P is an invariant ellipsoid of our system.

Proof(3)

In our case

$$A_0 := \begin{pmatrix} \hat{A}^T P^{-1} + P^{-1}\hat{A} & P^{-1}\hat{F} \\ \hat{F}^T P^{-1} & 0 \end{pmatrix}, \ \alpha_0 := 0$$

$$A_1 := \begin{pmatrix} -P^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \alpha_1 := -1$$

Form $w_x K_f w_x \leq \delta + x^T K_x x = \delta + z^T K_z z, K_z = \begin{pmatrix} K_x & K_x \\ K_x & K_x \end{pmatrix}$

$$A_2 := \begin{pmatrix} -K_z & 0 & 0 \\ 0 & K_f & 0 \\ 0 & 0 & 0 \end{pmatrix}, \alpha_2 := \delta, K_z = \begin{pmatrix} K_x & K_x \\ K_x & K_x \end{pmatrix}$$

$$A_3 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K_\eta \end{pmatrix}, \alpha_3 := 1$$

Proof(4)

$$Q = \begin{pmatrix} A_K^T P_1 + P_1 A_K + \tau_2 K_x & P_1 FC + \tau_2 K_x & 0 & P_1 F \\ C^T F^T P_1 + \tau_2 K_x & A_F^T P_2 + P_2 A_F + \tau_2 K_x & P_2 & -P_2 F \\ 0 & P_2 & -\tau_2 K_f & 0 \\ F^T P_1 & -F^T P_2 & 0 & -\tau_3 K_\eta \end{pmatrix}$$

$$A_K := A + BK + \frac{\tau_1}{2} I, A_F := A - FC + \frac{\tau_1}{2} I$$

From S-procedure: $Q = A_0 - \tau_1 A_1 - \tau_2 A_2 - \tau_3 A_3 \leq 0$, $\tau_1 \geq \delta\tau_2 + \tau_3$, $\tau_i \geq 0$, $i = 1, 2, 3$.

$$Q_1 = T_1 Q T_1^T \leq 0, \text{ where } T_1 = \begin{pmatrix} P_1^{-1} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

Proof(5)

$$Q_1 = \begin{pmatrix} \Psi_1 & FC + \tau_2 P_1^{-1} K_x & 0 & F \\ C^T F^T + \tau_2 K_x P_1^{-1} & \Psi_2 & P_2 & -P_2 F \\ 0 & P_2 & -\tau_2 K_f & 0 \\ F^T & -F^T P_2 & 0 & -\tau_3 K_\eta \end{pmatrix}$$

where

$$\Psi_1 := P_1^{-1} A_K^T + A_K P_1^{-1} + \tau_1 P_1^{-1} + \tau_2 P_1^{-1} K_x P_1^{-1}$$
$$\Psi_2 := A_F^T P_2 + P_2 A_F + \tau_1 P_2 + \tau_2 K_x$$

$$Q_1 = \tilde{Q} + \begin{pmatrix} P_1^{-1} & I & 0 & 0 \end{pmatrix}^T (\tau_2 K_x) \begin{pmatrix} P_1^{-1} & I & 0 & 0 \end{pmatrix} \leq 0 \qquad (24)$$

where

$$\tilde{Q} = \begin{pmatrix} P_1^{-1} A_K^T + A_K P_1^{-1} + \tau_1 P_1^{-1} & FC & 0 & F \\ C^T F^T & A_F^T P_2 + P_2 A_F + \tau_1 P_2 & P_2 & -P_2 F \\ 0 & P_2 & -\tau_2 K_f & 0 \\ F^T & -F^T P_2 & 0 & -\tau_3 K_\eta \end{pmatrix}$$

Proof(6)

$$Q_2 = \begin{pmatrix} P_1^{-1}A_K^T + A_K P_1^{-1} + \tau_1 P_1^{-1} & FC & 0 & F & P_1^{-1} \\ C^T F^T & A_F^T P_2 + P_2 A_F + \tau_1 P_2 & P_2 & -P_2 F & I \\ 0 & P_2 & -\tau_2 K_f & 0 & 0 \\ F^T & -F^T P_2 & 0 & -\tau_3 K_\eta & 0 \\ P_1^{-1} & I & 0 & 0 & -\frac{1}{\tau_2} K_x^{-1} \end{pmatrix}$$

$$Q_3 = T_2 Q_2 T_2^T \leq 0, \quad T_2 = \begin{pmatrix} P_2 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix}$$

Proof(7)

$$Q_3 = \begin{pmatrix} P_2(P_1^{-1}A_K^T+A_KP_1^{-1} \\ +\tau_1P_1^{-1})P_2 & P_2FC & 0 & P_2F & P_2P_1^{-1} \\ C^TF^TP_2 & A_F^TP_2+P_2A_F \\ +\tau_1P_2 & P_2 & -P_2F & I \\ 0 & P_2 & -\tau_2K_f & 0 & 0 \\ F^TP_2 & -F^TP_2 & 0 & -\tau_3K_\eta & 0 \\ P_1^{-1}P_2 & I & 0 & 0 & -\frac{1}{\tau_2}K_x^{-1} \end{pmatrix}$$

By Λ-inequality $XY^T + YX^T \leq X\Lambda X^T + Y\Lambda^{-1}Y^T$ valid for any $X \in R^{n\times k}$, $Y \in R^{n\times k}$ and any $0 < \Lambda = \Lambda^T \in R^{k\times k}$. For $X = P_2$ and $Y = I$ we have

$$X + X^T \leq X\Lambda X^T + \Lambda^{-1}$$

$$\Lambda := -(P_1^{-1}A_K^T + A_KP_1^{-1} + \tau_1P_1^{-1})$$

$$P_2(P_1^{-1}A_K^T+A_KP_1^{-1}+\tau_1P_1^{-1})P_2 \leq -P_2-P_2-(P_1^{-1}A_K^T+A_KP_1^{-1}+\tau_1P_1^{-1})^{-1} \leq R_1$$

Proof(8)

$$X_1 := P_1^{-1}, Y_1 := P_1^{-1}K^T, X_2 := P_2, \;\; Y_2 := F^T P_2$$

$$\begin{pmatrix} R_1 & Y_2^T C & 0 & Y_2^T & X_2 X_1 \\ C^T Y_2 & \begin{array}{c} A^T X_2 + X_2 A - Y_2^T C \\ -C^T Y_2 + \tau_1 X_2 \end{array} & X_2 & \text{-}Y_2^T & I \\ 0 & X_2 & \text{-}\tau_2 K_f & 0 & 0 \\ Y_2 & \text{-}Y_2 & 0 & \text{-}\tau_3 K_\eta & 0 \\ X_1 X_2 & I & 0 & 0 & \text{-}\frac{1}{\tau_2} K_x^{\text{-}1} \end{pmatrix} \leq 0$$

$$\begin{pmatrix} -R_1 - 2X_2 & I \\ I & X_1 A^T + A X_1 + Y_1 B^T + B Y_1^T + \tau_1 X_1 \end{pmatrix} \leq 0 \tag{25}$$

$$\operatorname{tr}(X_1) + \operatorname{tr}(X_2^{-1}) \to \min \tag{26}$$

Introduce the following addtitional constraint

$$H \geq X_2^{-1} \Leftrightarrow \begin{pmatrix} H & I \\ I & X_2 \end{pmatrix} \geq 0$$

Proof of Lemma

By the Λ-inequality $XY^T + YX^T \leq X\Lambda X^T + Y\Lambda^{-1}Y^T$ with

$$X^T := \begin{pmatrix} 0 & 0 & 0 & 0 & X_1 \end{pmatrix} \text{ and } Y := \begin{pmatrix} X_2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

we have

$$Q_3 = G + XY^T + YX^T \leq G + X\Lambda X^T + Y\Lambda^{-1}Y^T := \tilde{Q}_3 \leq 0$$

$$\tilde{Q}_3 :=$$

$$\begin{pmatrix} \begin{matrix} P_2(P_1^{-1}A_K^T + A_K P_1^{-1} \\ +\tau_1 P_1^{-1} + \Lambda)P_2 \end{matrix} & P_2 F C & 0 & P_2 F & 0 \\ C^T F^T P_2 & \begin{matrix} A_F^T P_2 + P_2 A_F \\ +\tau_1 P_2 \end{matrix} & P_2 & -P_2 F & I \\ 0 & P_2 & -\tau_2 K_f & 0 & 0 \\ F^T P_2 & -F^T P_2 & 0 & -\tau_3 K_\eta & 0 \\ 0 & I & 0 & 0 & \begin{matrix} -\frac{1}{\tau_2} K_x^{-1} + \\ P_1^{-1}\Lambda^{-1}P_1^{-1} \end{matrix} \end{pmatrix}$$

THANK YOU FOR YOUR ATTENTION