Control by Interconnection of Physical Systems

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Main Message: Provide a new paradigm, alternative to signal–processing, for control of physical systems.

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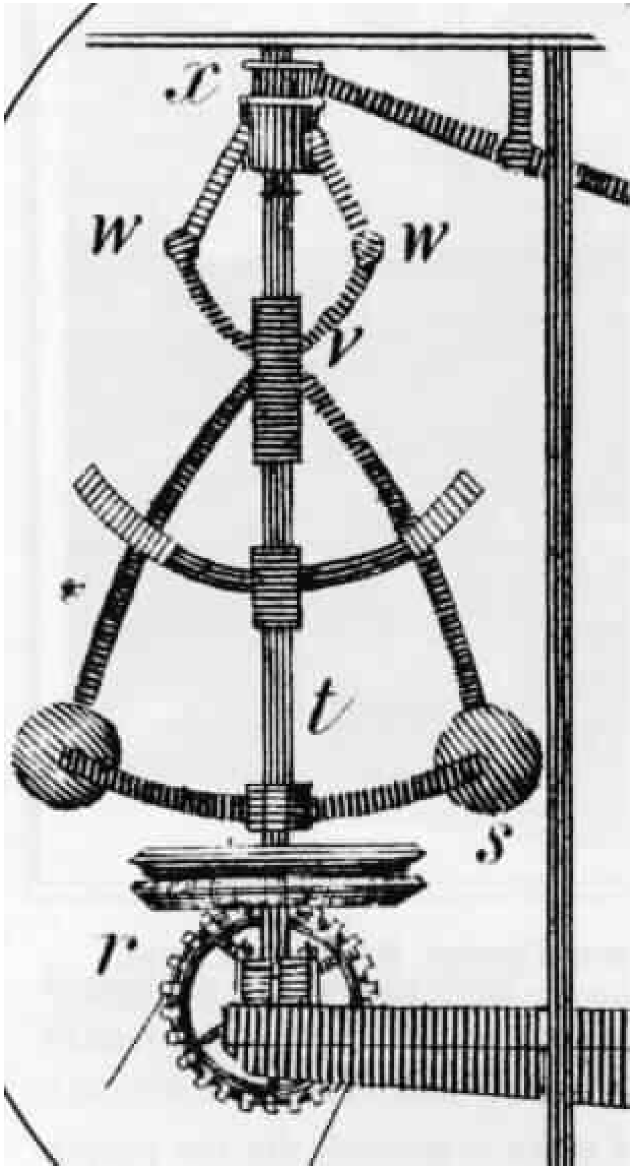
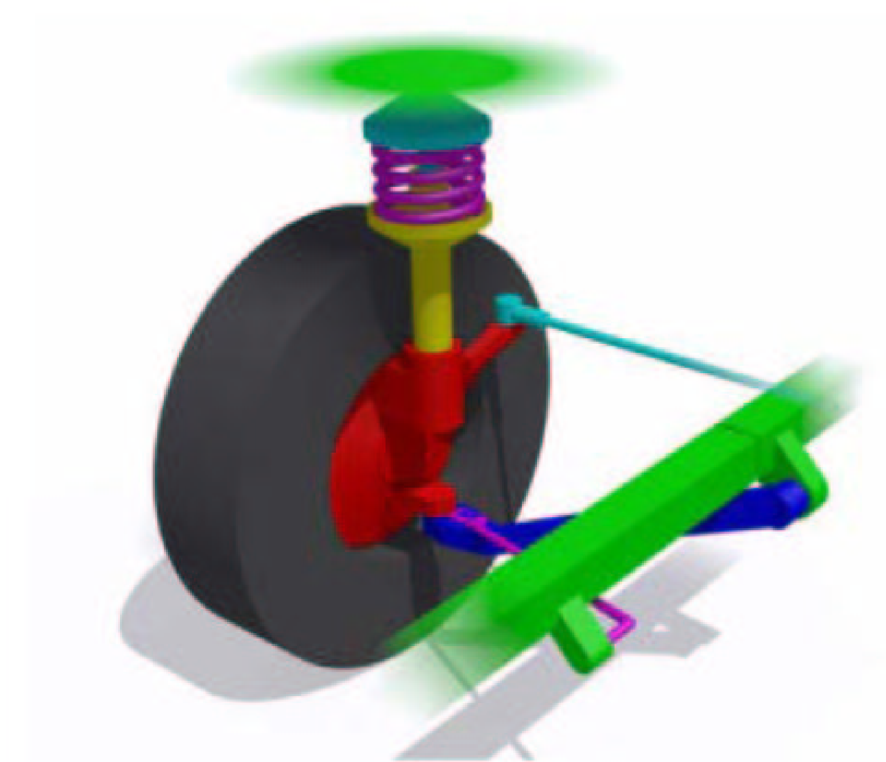
Controllers by Interconnection are as Old as Control Itself



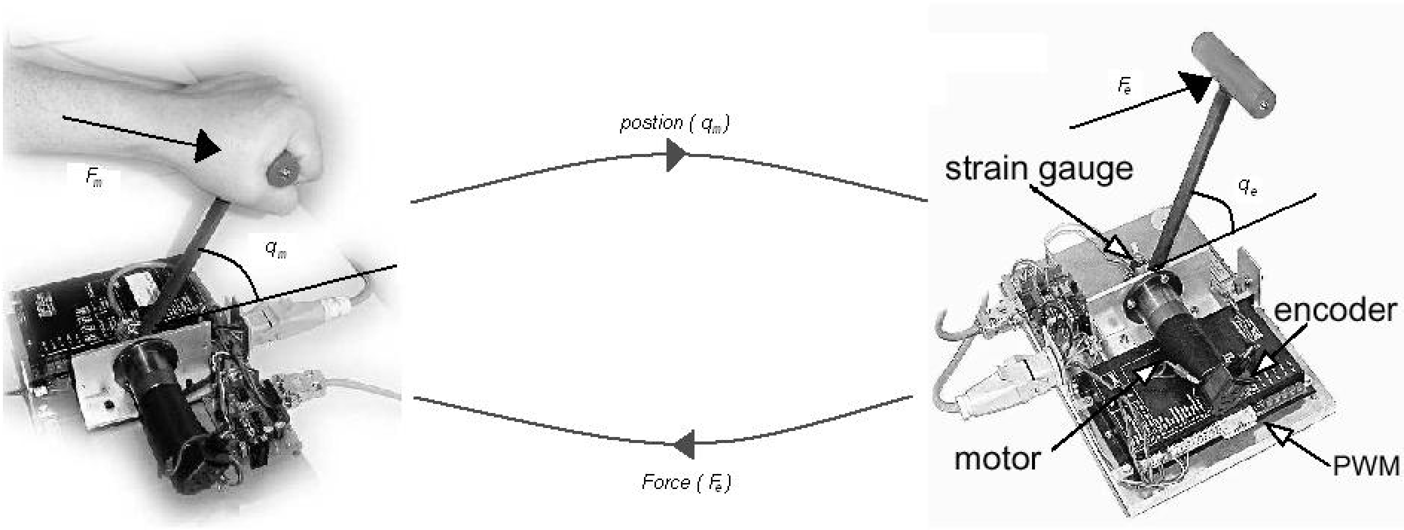
They're Pervasive and Efficient



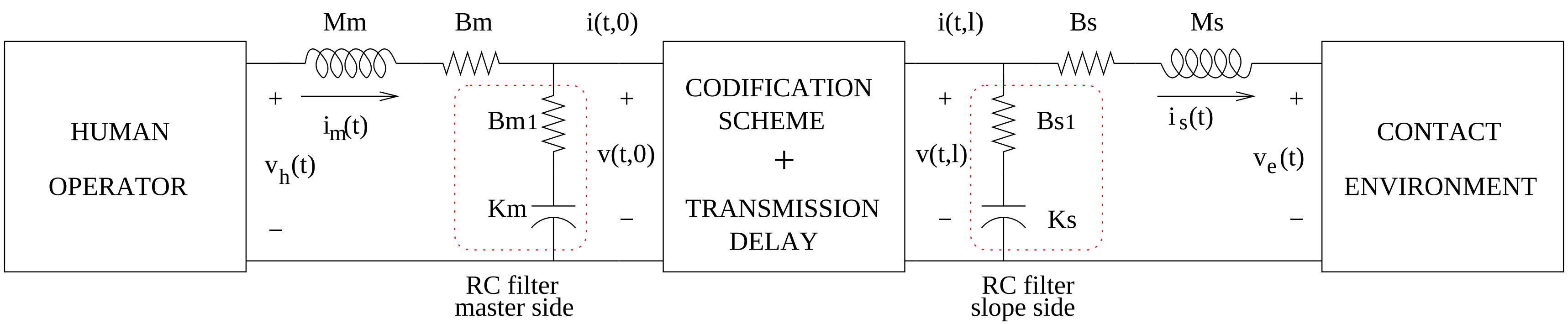
Even in your Privy

Natural Way to Represent Interactions with Environment

Two mechanical systems, a human–controlled master and a teleoperated slave

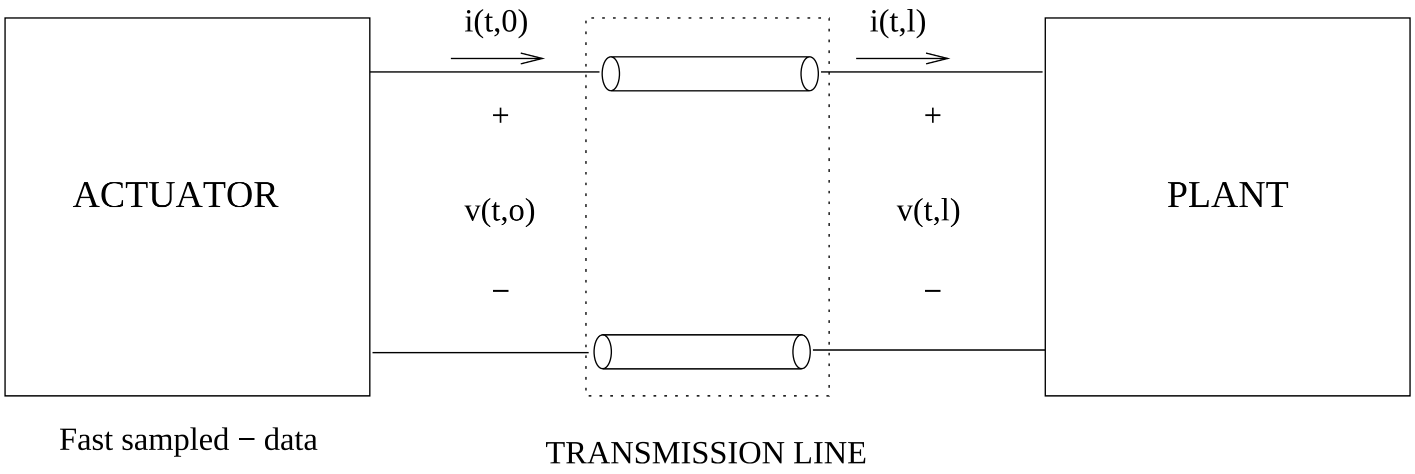


(Anderson/ Spong, '89): Transforming delays into transmission lines

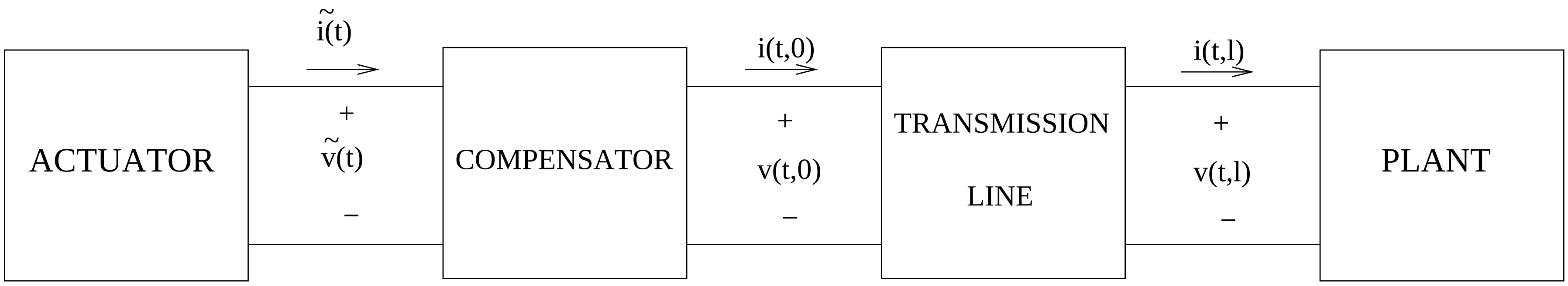


Sometimes the Only Solution

Overvoltage Problem The presence of long cables between a fast–sampling actuator and plant induces oscillations: The cables behave like a transmission line.



(Ortega/Spong, US Patent 07): vice versa



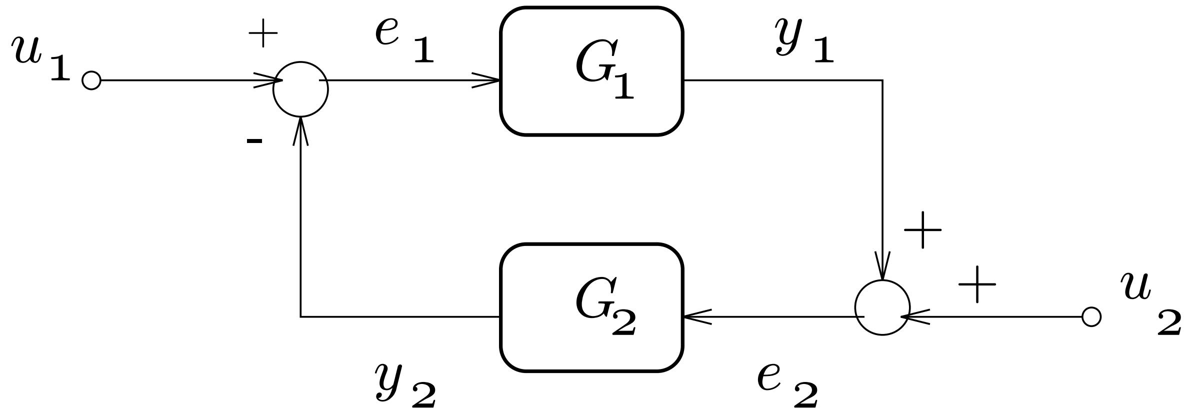
Works Even if We Don't Know Why!

(Spong'06): Synchronization by interconnection, an open problem since the 17th century.

♡

Main Message: Paradigm Shift for Controller Design

Classical formulation: Signal–processing viewpoint



- System model and controller are signal processors: $G_1 : e_1 \to y_1$, $G_2 : e_2 \to y_2$.
- Control specifications in terms of signals: tracking, disturbance attenuation, etc.
- Uncertainty represented via the "$\Sigma - \Delta$ paradigm":
  - discriminated via filtering,
  - very successful for linear time–invariant (LTI) systems
- Control computed from solution of Riccati eqs ($\mathcal{H}_\infty, \mathcal{H}_2$–designs).
- "Impossible" in nonlinear case:
  - nonlinear systems "mix" the frequencies,
  - far from obvious computations (NL filtering, Hamilton-Jacobi-Bellman PDE).

Passivity–Based Control: An Energy–Processing Viewpoint

- View plant as energy–transformation, as opposed to signal–transformation, multiport device
- Consider systems that satisfy (generalized) energy–conservation:

  Stored energy = Supplied energy + Dissipation

- Control objective in PBC: preserve the energy–conservation property but with desired energy and dissipation functions

  Desired stored energy = New supplied energy + Desired dissipation

  In other words

  PBC = Energy Shaping + Damping Assignment

- Two possible formulations:
  - State feedback (also called Standard PBC)
  - Control by Interconnection (CbI)
- Objectives of the talk:
  - Provide a unified framework
  - Explore the relations between the two formulations

Analytical vs Computational Approaches to Control

- Modern analytical (model–based) control theory is not providing solutions to practical control problems with “strong nonlinearities" (phenomena that cannot be captured with linear models)
- Existing analytical designs rely on high gain,
  - E.g. backstepping, sliding modes, Lyapunov domination
  - Intrinsically conservative
  - Amplifies noise
  - Energy consumption...
- Trend in applications (prevailing?): (black–box, data–based) computational “solutions"
  - Expand NL on a basis + some kind of NL inversion
  - Neural networks, fuzzy controllers, genetic algorithms, etc
  - They might work but we will not understand why/when?
  - How to select the fuzzyfication–de-fuzzyfication rules?
  - How many neuron layers? Training?

“New" Computational Trend

- Approach
  - Approximating NL by a (large number of) “linear terms" (piece-wise, LPV,...)
  - Postulating an optimization problem (usually leading to LMI’s)
  - Feasibility checked with particular numerical cases
- Questions
  - Why a performance criterion (with constraints) captures the control objective?
  - Wasn’t the fragility of optimal control the starting point for robust control?
  - Weighting coefficient selection Gordian knot? Is the addition of receding horizons useful?
  - Computational complexity? (Possible for slow systems with “monotonic" behaviors, e.g. process control)
- Analytical vs computational approaches to control
  - Is the objective of control theory to generate code that "solves the problem"?
  - What do we learn about the system by doing this?
  - “Control theory = Number crunching" is a reductionist view
  - NL analysis leads to an understanding of the systems behavior
  - Control action best understood adopting a systems interconnection viewpoint

Passivity–Based Control Programme

- Consider models that capture main physical ingredients:
  - energy, dissipation and interconnection
  - Port–Hamiltonian (PH) systems
- Attain classical control objectives (stability, performance) as by–products of:
  - energy–shaping,
  - interconnection modification and
  - damping assignment.

Applications of PBC

- Mass–balance systems, electrical motors, magnetic levitation systems, power systems, power converters, underwater vehicles, surface vessels, (air)spacecrafts, walking robots, bilateral teleoperation, underactuated mechanical systems....

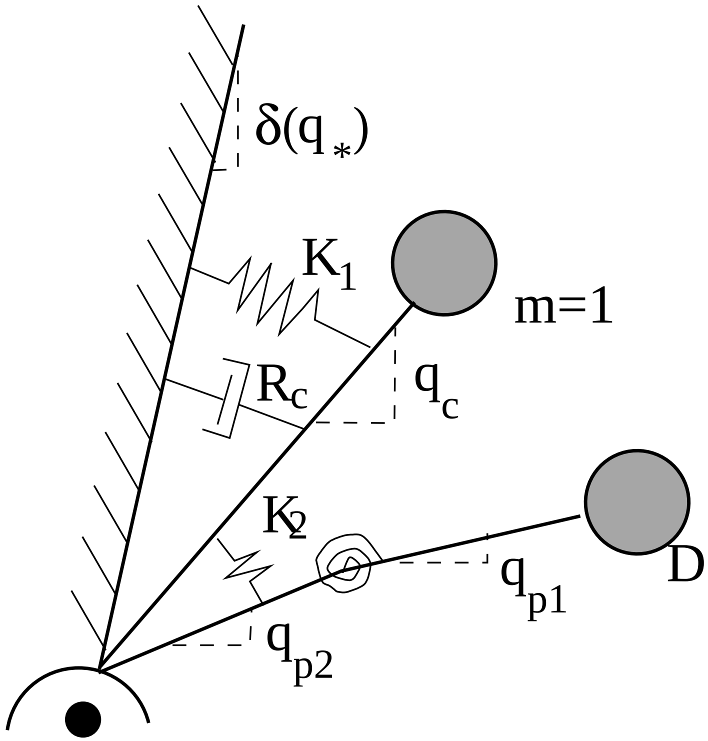
Advantages of PBC

Advantages of energy–shaping (over nonlinearity cancellation and high gain)[a]

- Handle on performance, not just stability
- Respect, and effectively exploit, the structure of the system to
  - incorporate physical knowledge,
  - provide physical interpretations to the control action
- Energy serves as a lingua franca to communicate with practitioners
- There's an elegant geometrical characterization of
  - power–conserving interconnections (via Dirac structures) and
  - passifiable NL systems (in terms of stable invertibility and relative degree)

[a] Euphemistically called "nonlinearity domination".

Example: Control by Interconnection of a Flexible Pendulum



- Plant energy: $H(q_p, p_p) = \frac{1}{2} p_p^\top D^{-1}(q_p) p_p + V(q_p)$
- Controller energy:
  $H_c(q_c, p_c, q_{p2}) = \frac{1}{2}|p_c|^2 + \frac{1}{2}(q_c - q_{p2})^\top K_2 (q_c - q_{p2}) + \frac{1}{2}(q_c - \delta)^\top K_1 (q_c - \delta)$
- Controller Rayleigh dissipation function: $\frac{1}{2}\dot{q}_c^\top R_c \dot{q}_c$

Layout

1. Cyclo–passivity and formulation of PBC stabilization problem
2. Basic control by interconnection (CbI) of Port–Hamiltonian systems
   - Energy–Casimir method
   - Dissipation obstacle
3. Extensions of CbI method
   - Generating new cyclo–passivity properties
   - Overcoming the dissipation obstacle
   - Control by state–modulated interconnection
4. Standard (State–feedback) PBC
   - Energy balancing control (EBC)
   - Interconnection and damping assignment (IDA)
   - Power shaping
5. Comparison of the two methods
   - Domain of applicability
   - Standard PBC as a projection of CbI
6. Conclusions and outlook

List of Acronyms

BIDA	Basic IDA
CBI	Control by interconnection
$CBI_{\mathtt{PS}}$	CBI with power shaping output
$CBI_{\mathtt{PS}}^{\mathtt{SM}}$	CBI with power shaping output and state modulated interconnection
EBC	Energy–balancing control
IDA	Interconnection and damping assignment
NL	Nonlinear
PBC	Passivity–based control
PDE	Partial differential equation
PS	Power shaping
$y_{\mathtt{PS}}$	Power shaping output

1. Key Property: Cyclo–Passivity

Definition We say that the m–port system with state $x \in \mathbb{R}^n$, and power port variables $u, y \in \mathbb{R}^m$

$$\Sigma : \begin{cases} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{cases}$$

is cyclo–passive if there exists storage (energy) function $H : \mathbb{R}^n \to \mathbb{R}$ such that

$$\underbrace{H[x(t)] - H[x(0)]}_{stored\ \ energy} \leq \underbrace{\int_0^t u^\top(s) h(x(s)) ds}_{supplied\ energy}$$

If $H(x) \geqq 0$ then the system is passive with port variables (u, y) and storage function $H(x)$.

Remark For passive systems we have

$$-\int_0^t u^\top(s) y(s) ds \leq H[x(0)] < \infty \quad \Rightarrow$$

amount of energy that can be extracted from a passive system is bounded.

Stabilization via Energy Shaping and Damping Injection

- With $u(t) \equiv 0$, we have

$$H[x(t)] \leq H[x(0)] \qquad \Rightarrow$$

  - Trajectories tend to converge towards points of minimum energy
  - If the minima are strict $H(x)$ qualifies as a Lyapunov function for them

- To operate the system around some desired equilibrium point, say x_*, PBC shapes the energy to assign a strict minimum at this point.

- Furthermore, if we terminate the port with

$$u = -K_{di} y, \quad K_{di} = K_{di}^\top > 0$$

we get

$$\dot{H} \leq -y^\top K_{di} y \leq 0.$$

Hence, $x(t) \to 0$ if $h(x)$ is detectable (for the closed–loop system). That is, if

$$h(x(t)) \equiv 0 \;\Rightarrow\; x(t) \to 0.$$

Port–Hamiltonian (PH) Systems

- PH model of a physical system

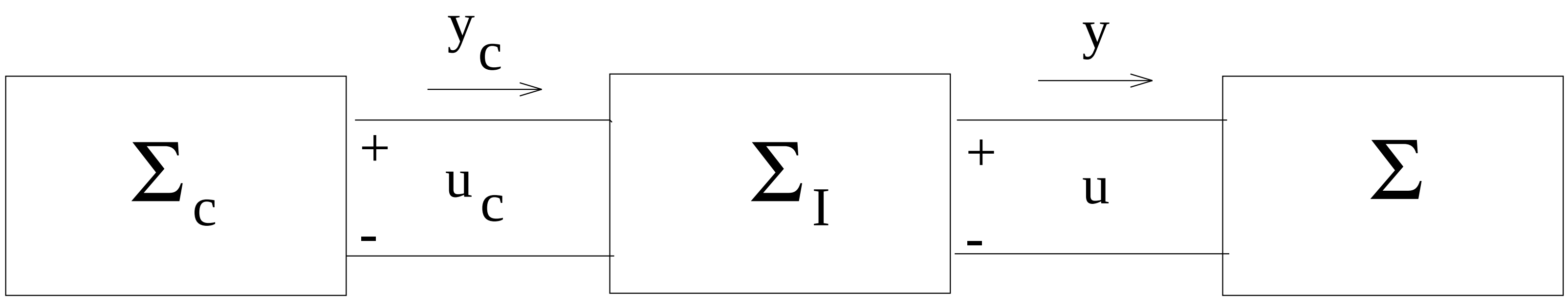
$$\Sigma_{(u,y)} \ : \left\{ \begin{array}{rcl} \dot{x} & = & [\mathcal{J}(x) - \mathcal{R}(x)]\nabla H + g(x)u \\ y & = & g^\top(x)\nabla H \end{array} \right.$$

  - $u^\top y$ has units of power (voltage–current, speed–force, angle–torque, etc.)
  - $\mathcal{J} = -\mathcal{J}^\top$ is the interconnection matrix, specifies the internal power–conserving structure (oscillation between potential and kinetic energies, Kirchhoff's laws, transformers, etc.)
  - $\mathcal{R} = \mathcal{R}^\top \geq 0$ damping matrix (friction, resistors, etc.)
  - g is input matrix.
- PH systems are cyclo–passive

$$\dot{H} = -\nabla H^\top \mathcal{R} \nabla H + u^\top y.$$

- Invariance of PH structure Power preserving interconnection of PH systems is PH.
- Nice geometric structure formalized with notion of Dirac structures.
- Most nonlinear cyclo–passive systems can be written as PH systems. Actually, in (network) modeling is the other way around!

2. Control by Interconnection (CbI)



- Plant (Σ) and controller (Σ_c), with states $x \in \mathbb{R}^n, \zeta \in \mathbb{R}^m$, are cyclo–dissipative, that is, $\exists H : \mathbb{R}^n \to \mathbb{R}$, $H_c : \mathbb{R}^m \to \mathbb{R}$, such that

$$\dot{H} \leq u^\top y, \quad \dot{H}_c \leq u_c^\top y_c.$$

- Interconnection subsystem (Σ_I) is power–preserving (lossless:)

$$y^\top u + y_c^\top u_c = y^\top v \qquad (\Leftarrow u = -y_c + v,\ u_c = y).$$

- Interconnected system satisfies $\dot{H} + \dot{H}_c \leq v^\top y \quad \Rightarrow \quad H + H_c$ is the new energy.

Problem

Although $H_c(\zeta)$ is free, not clear how to “shape" x?

Energy–Casimir Method

Proposition Assume, $\exists\, \mathcal{C} : \mathbb{R}^n \to \mathbb{R}^m$ such that the level sets $\Omega_\kappa \triangleq \{(x,\zeta) | \zeta = \mathcal{C}(x) + \kappa\}$ are invariant, for all $\kappa \in \mathbb{R}$. Then, for all $\Phi : \mathbb{R}^m \to \mathbb{R}$, the function

$$W(x,\zeta) \triangleq H(x) + H_c(\zeta) + \Phi(\mathcal{C}(x) - \zeta).$$

satisfies

$$\dot{W} \leq v^\top y$$

That is, the system is cyclo–passive w.r.t. $W(x,\zeta)$, which (given $\mathcal{C}$) can be shaped selecting H_c and Φ.

Proof Invariance of Ω_κ is equivalent to

$$\dot{\zeta} - \frac{d}{dt}\mathcal{C}(x) = 0.$$

Hence,

$$\dot{\Phi} = \nabla\Phi(\dot{\mathcal{C}} - \dot{\zeta}) = 0 \;\Rightarrow\; \dot{W} = \dot{H} + \dot{H}_c.$$

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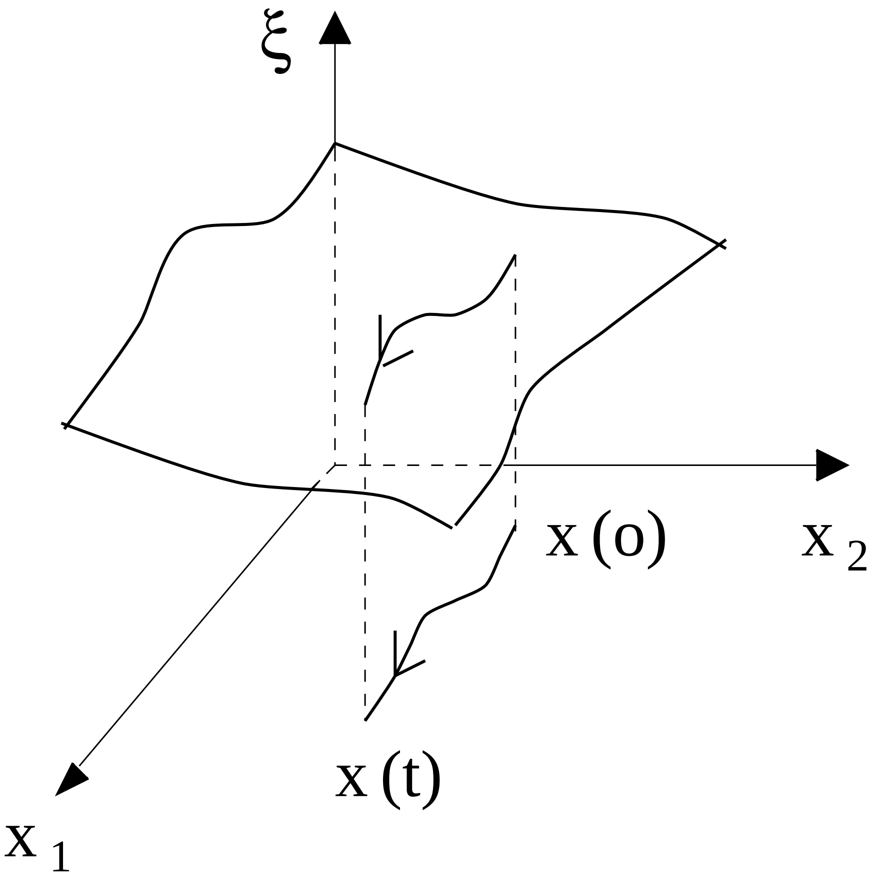
Caveat: Achieving Asymptotic Stability



All level sets

$$\Omega_\kappa = \{(x,\zeta) | \zeta = \mathcal{C}(x) + \kappa\},$$

$\kappa \in \mathbb{R}$ are invariant. In principle, we must set

$$\zeta(0) = \zeta_\star + \mathcal{C}(x(0)) - \mathcal{C}(x_\star)$$

to ensure that the trajectory starts (and remains) in $\Omega_{\kappa_\star}$, with

$$\kappa_\star \triangleq \zeta_\star - \mathcal{C}(x_\star),$$

that contains the desired equilibrium. A more practical solution is to estimate κ, adding an integrator.

Basic CbI for PH Systems

- Given a PH system,

$$\Sigma_{(u,y)} \begin{cases} \dot{x} &= F(x)\nabla H(x) + g(x)u \\ y &= g^\top(x)\nabla H(x), \end{cases} \quad \Rightarrow \quad \dot{H} \leq u^\top y$$

  where we defined $F(x) := \mathcal{J}(x) - \mathcal{R}(x)$, $\mathcal{J} = -\mathcal{J}^\top$, $\mathcal{R} = \mathcal{R}^\top \geq 0$.
- PH controller (nonlinear integrators), $\zeta \in \mathbb{R}^m$

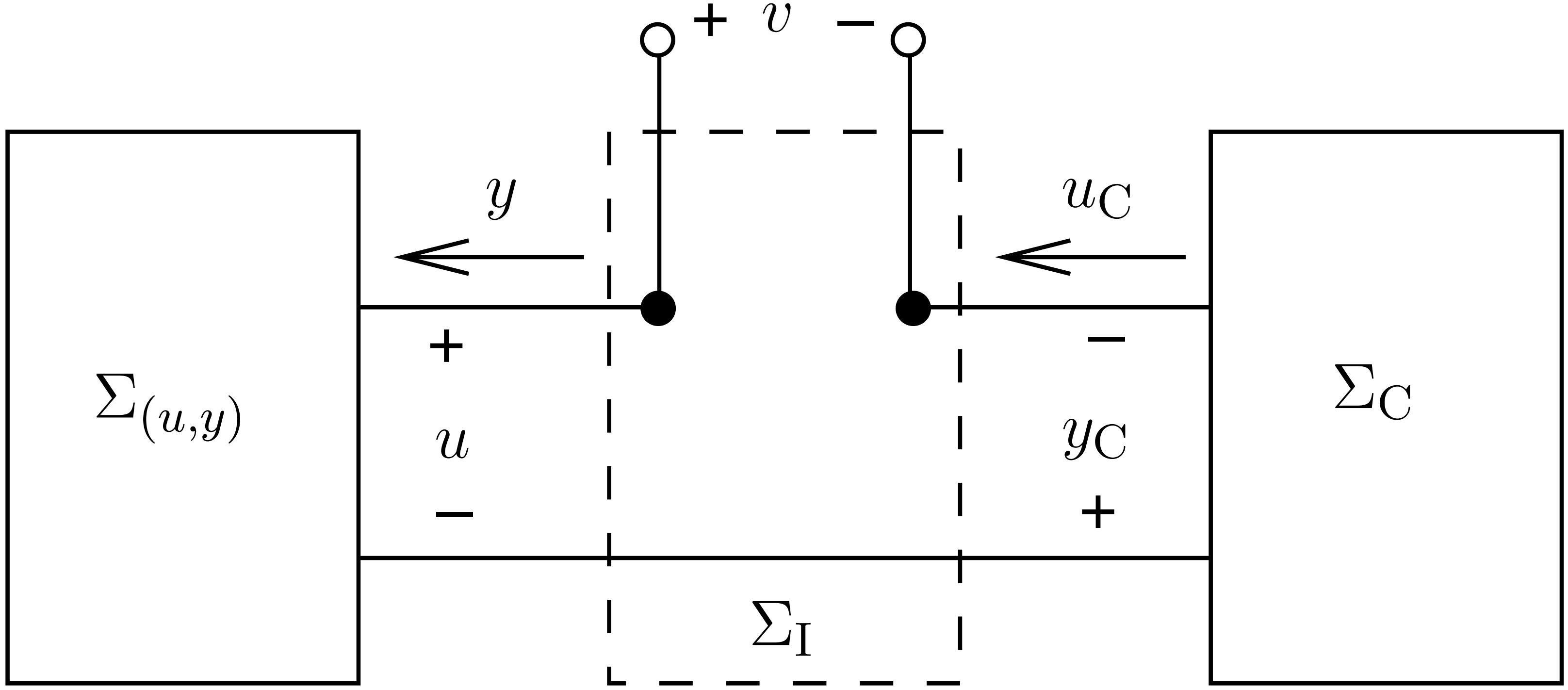
$$\Sigma_c : \begin{cases} \dot{\zeta} &= u_c \\ y_c &= \nabla_\zeta H_c(\zeta), \end{cases} \quad \Rightarrow \quad \dot{H}_c = u_c^\top y_c$$

- Standard negative feedback interconnection

$$\Sigma_I : \left\{ \begin{bmatrix} u \\ u_c \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ y_c \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix} \right. \quad \Rightarrow \quad \dot{H} + \dot{H}_c \leq v^\top y$$

- For ease of presentation, and with loss of generality, we have taken $\zeta \in \mathbb{R}^m$.

Negative Feedback Interconnection Subsystem Σ_I



Casimir Functions for PH Systems

- Recalling the total energy function

$$W(x,\zeta) = H(x) + H_c(\zeta) + \Phi(\mathcal{C}(x) - \zeta),$$

where H is given, $\mathcal{C}$ will be computed and H_c, Φ selected to shape W.

- We want $\mathcal{C}$ to be independent of H and H_c – these are called Casimir functions.
- The dynamics of the interconnected system is given by

$$\begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} F & -g \\ g^\top & 0 \end{bmatrix} \begin{bmatrix} \nabla H \\ \nabla H_c \end{bmatrix} + \begin{bmatrix} g \\ 0 \end{bmatrix} v.$$

We are looking for $\mathcal{C}$ such that $\dot{\mathcal{C}} - \dot{\zeta} = 0$ for all H and H_c. Thus, we get the PDEs

$$\begin{bmatrix} (\nabla \mathcal{C})^\top & -I_m \end{bmatrix} \begin{bmatrix} F & -g \\ g^\top & 0 \end{bmatrix} = 0.$$

- Note that $(\nabla \mathcal{C})^\top g = 0$ ensures $\dot{\mathcal{C}} - \dot{\zeta} = 0$ even with $v \neq 0$.

Conditions for CbI

Proposition Assume there exists a vector function $\mathcal{C} : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\begin{bmatrix} F^\top \\ g^\top \end{bmatrix} \nabla \mathcal{C} = \begin{bmatrix} g \\ 0 \end{bmatrix} \qquad (CbI - PDE)$$

Then, for all functions $\Phi : \mathbb{R}^m \to \mathbb{R}$, the following cyclo–passivity inequality is satisfied

$$\dot{W} \leq v^\top y,$$

where the shaped storage function $W : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is defined as

$$W(x, \zeta) \stackrel{\triangle}{=} H(x) + H_c(\zeta) + \Phi(\mathcal{C}(x) - \zeta).$$

Proof

$$\dot{\Phi} = \nabla \Phi (\dot{\mathcal{C}} - \dot{\zeta}) = 0.$$

$\diamond \diamond \diamond$

The Dissipation Obstacle

Proposition If (CbI–PDE) admits a solution then

$$\mathcal{R}\nabla_x \Phi(\mathcal{C}(x) - \zeta) = 0,$$

for all $\Phi : \mathbb{R}^m \to \mathbb{R}$. Consequently, energy cannot be shaped for coordinates that are affected by physical damping.

Proof (CbI–PDE) $\Leftrightarrow$

$$\begin{aligned} F^\top \nabla\mathcal{C} = g,\ g^\top \nabla\mathcal{C} = 0 \quad &\Rightarrow \quad (\nabla\mathcal{C})^\top F^\top \nabla\mathcal{C} = 0 \\ &\Rightarrow \quad \mathcal{R}\nabla\mathcal{C} = 0 \end{aligned}$$

Proof completed with $\nabla_x \Phi = \nabla\mathcal{C}\nabla\Phi$. $\diamond\diamond\diamond$

Remarks

- OK for mechanical systems where dissipation enters in the momenta equations—that need not be shaped.
- Note that $\mathcal{J}\nabla\mathcal{C} = -g$. Hence, (CbI-PDE) is equivalent to

$$F\nabla\mathcal{C} = -g, \quad g^\top \nabla\mathcal{C} = 0.$$

The Dissipation Obstacle and Power Balance

- We proved that (CbI–PDE) $\Rightarrow F\nabla\mathcal{C} = -g, \mathcal{R}\nabla\mathcal{C} = 0$. Hence,

$$\mathcal{R}(F^\top F)^{-1}F^\top g = 0,$$

which is a necessary condition for the existence of Casimirs.

- Denote $(\cdot)(x_*) := (\cdot)_*$, where x_* is an equilibrium to be stabilized and assume (CBI–PDE) holds. From $\dot{x} = F\nabla H + gu$ we have that

$$\begin{aligned} & 0 = F_*\nabla H_* + g_* u_* \Rightarrow \nabla H_* = -(F_*^\top F_*)^{-1}F_*^\top g_* u_* \\ \Rightarrow \quad & \mathcal{R}\nabla H_* = -\mathcal{R}(F_*^\top F_*)^{-1}F_*^\top g_* u_* \Rightarrow \mathcal{R}\nabla H_* = 0 \Rightarrow \nabla^\top H_* \mathcal{R}_* \nabla H_* = 0. \end{aligned}$$

- From the power balance equation

$$\dot{H} = -\nabla H^\top \mathcal{R}\nabla H + u^\top y$$

we see that, if the system admits a Casimir then the dissipated power at the equilibrium should be zero, i.e., $u_*^\top y_* = 0$. That is, we should be able to stabilize the system extracting a finite amount of energy from the source.

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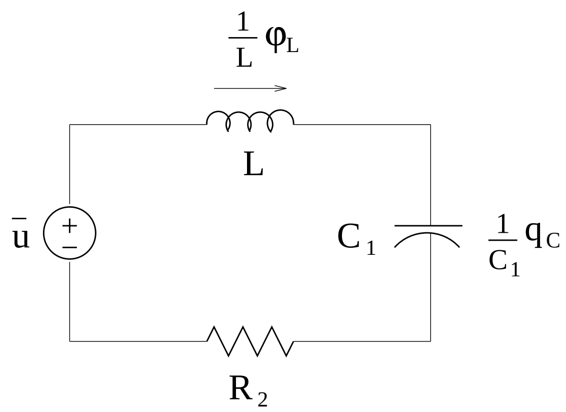
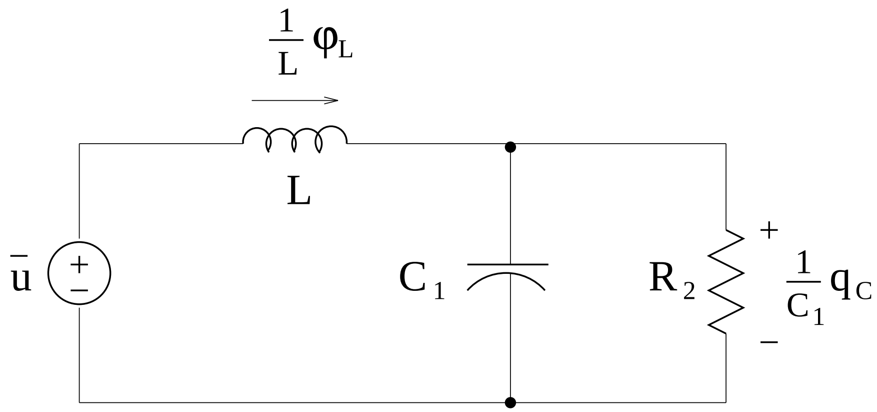
Example without Pervasive Dissipation



- Energy function $H(q_C, \phi_L) = H_E(q_C) + H_M(\phi_L)$. Assume $0 = \arg\min H_M$.
- Co-energy variables $\nabla H =$ col(v_C, i_L)
- PH model $\begin{bmatrix} \dot{q}_C \\ \dot{\phi}_L \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -R_2 \end{bmatrix} \nabla H + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$
- Equilibria: $\nabla_{\phi_L} H(\bar{\phi}_L) = 0 \Rightarrow \bar{\phi}_L = 0 \Rightarrow$ no need to shape H_M
  - No dissipation in the coordinate to be shaped (q_C)
  - (CBI–PDE): $F\nabla\mathcal{C} = -g, g^\top \nabla\mathcal{C} = 0$, a Casimir is $\mathcal{C}(q_C, \phi_L) = q_C$, thus we can add to $H + H_c$ an arbitrary function $\Phi(q_C - \zeta) \Rightarrow$ stabilizable via CbI
- Power balance: $\dot{H} = -R_2(\nabla_{\phi_L} H)^2 + u\nabla_{\phi_L} H \Rightarrow$ dissipation zero at equilibria.

Example with Pervasive Dissipation



- Same energy function, but the dissipation has changed: $\mathcal{J} - \mathcal{R} = \begin{bmatrix} \frac{-1}{R_2} & 1 \\ -1 & 0 \end{bmatrix}$
- Equilibria: $(\nabla_{q_C} H, \nabla_{\phi_L} H) = (u_*, R_2 u_*) \neq (0,0) \Rightarrow$ Dissipation in a coordinate to be shaped (q_C)
- Not stabilizable via CbI!
- We have $\mathcal{R} F^{-1} g =$ col$(\frac{-1}{R_2}, 0)$. Therefore, the necessary condition for the existence of Casimirs, $\mathcal{R} F^{-1} g = 0$, is not satisfied
- Power balance: $\dot{H} = -\frac{1}{R_2}(\nabla_{q_C} H)^2 + u \nabla_{\phi_L} H \Rightarrow$ dissipation not zero at equilibria.

3. Generating New Cyclo–Passivity Properties

Idea To find new cyclo–passive outputs look for full rank matrices $F_d : \mathbb{R}^n \to \mathbb{R}^{n \times n}$, with

$$F_d + F_d^\top \leq 0 \qquad (SYM)$$

and storage functions $H_{\rm PS} : \mathbb{R}^n \to \mathbb{R}$ such that

$$F \nabla H = F_d \nabla H_{\rm PS}.$$

Proposition For all solutions F_d of the PDE

$$\nabla \left(F_d^{-1} F \nabla H \right) = \left[\nabla \left(F_d^{-1} F \nabla H \right) \right]^\top \qquad (PO - PDE)$$

verifying (SYM) there exists a storage function $H_{\rm PS}$ such that

$$\Sigma_{(u, y_{\rm PS})} \begin{cases} \dot{x} & = & F \nabla H + g u \\ y_{\rm PS} & = & -g^\top F_d^{-\top} (F \nabla H + g u) \end{cases} \Rightarrow \dot{H}_{\rm PS} \leq u^\top y_{\rm PS}.$$

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Proof

Poincare's Lemma Given $f:\mathbb{R}^n \to \mathbb{R}^n$, $f \in \mathcal{C}^1$. There exists $\psi:\mathbb{R}^n \to \mathbb{R}$ such that $\nabla\psi = f$ if and only if

$$\nabla f = (\nabla f)^\top.$$

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Consequently,

$$(PO-PDE) \qquad \Leftrightarrow \qquad \nabla H_{\text{PS}} = F_d^{-1} F \nabla H,$$

We then have the following chain of implications

$$\begin{aligned} F_d \nabla H_{\text{PS}} = F \nabla H \quad & \Rightarrow \quad \dot{x} = F_d \nabla H_{\text{PS}} + gu \\ & \Leftrightarrow \quad F_d^{-1}\dot{x} = \nabla H_{\text{PS}} + F_d^{-1} gu \\ & \Rightarrow \quad \dot{x}^\top F_d^{-1}\dot{x} = \dot{H}_{\text{PS}} + \dot{x}^\top F_d^{-1} gu \\ & \Rightarrow \quad 0 \geq \dot{H}_{\text{PS}} + \dot{x}^\top F_d^{-1} gu \\ & \Leftrightarrow \quad y_{\text{PS}}^\top u \geq \dot{H}_{\text{PS}}, \end{aligned}$$

where we used $\dot{x}^\top F_d^{-1}\dot{x} \leq 0$.

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Remarks

- If F is full rank $F_d = F$ solves (PO–PDE). In this case, $H_d = H$ and we obtain the new power–balance equation

$$\dot{H} = \dot{x}^\top F^{-1} \dot{x} + u^\top y_{\mathrm{PS}}.$$

  Comparing with $\dot{H} = u^\top y + \nabla^\top H F \nabla H$, we see that the new passive output is obtained swapping the damping.

- The new cyclo–passive output y_{PS} is equal to $g^\top \nabla H_{\mathrm{PS}}$ if and only if the dissipation obstacle for the PH system with port variables $(u, g^\top \nabla H_{\mathrm{PS}})$ is absent, that is

$$\mathcal{R}_d F_d^{-1} g = 0 \quad \Leftrightarrow \quad y_{\mathrm{PS}} = -g^\top F_d^{-\top}(F_d \nabla H_{\mathrm{PS}} + gu) = g^\top \nabla H_{\mathrm{PS}}.$$

- Setting $F_d = F$ in (PO–PDE) we obtain

$$\mathcal{R} F^{-1} g = 0 \Leftrightarrow y = y_{\mathrm{PS}}.$$

- For (underactuated) mechanical systems $\mathcal{R} F^{-1} g = 0$.

Solving (PO–PDE)

Proposition (Ortega et al. '04) For all matrices $M : \mathbb{R}^n \to \mathbb{R}^{n \times n}$, with $M(x) = M^\top(x)$ and all $\lambda \in \mathbb{R}$, such that

$$\tilde{M} \triangleq \frac{1}{2}[(\nabla^2 H)M + \nabla(M\nabla H) + 2\lambda I_n]$$

is full rank,

$$F_d^{-1} = \tilde{M} F^{-1}$$

solves (PO–PDE). The resulting storage function being

$$H_{\text{PS}} = \lambda H + (\nabla H)^\top M \nabla H.$$

◇ ◇ ◇

Remarks

- Generates a family of solutions of (PO–PDE) parameterized in terms of (M, λ).
- Case of constant M reported in (Brayton/Moser '64)

CbI with $\Sigma_{(u,y_{\text{PS}})}$ Overcomes the Dissipation Obstacle

Proposition Assume (PO–PDE) admits a solution F_d verifying (SYM) and such that

$$F_d \nabla \mathcal{C} = -g \qquad\qquad (CbI_{\text{PS}} - PDE)$$

for some vector function $\mathcal{C} : \mathbb{R}^n \to \mathbb{R}^m$. Consider the PH system $\Sigma_{(u,y_{\text{PS}})}$ coupled with the PH controller Σ_c through the power–preserving interconnection subsystem

$$\Sigma_I^{\text{PS}} : \left\{ \begin{bmatrix} u \\ u_c \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_{\text{PS}} \\ y_c \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix} \right. .$$

Then, for all functions $\Phi : \mathbb{R}^m \to \mathbb{R}$, the following cyclo–passivity inequality is satisfied

$$\dot{W}_{\text{PS}} \leq v^\top y_{\text{PS}},$$

where

$$W_{\text{PS}}(x,\zeta) \stackrel{\triangle}{=} H_{\text{PS}}(x) + H_c(\zeta) + \Phi(\mathcal{C}(x) - \zeta),$$

with $H_{\text{PS}} = \int (F_d^{-1} F \nabla H) dx$.

Remark The condition $g^\top \nabla \mathcal{C} = 0$ is absent.

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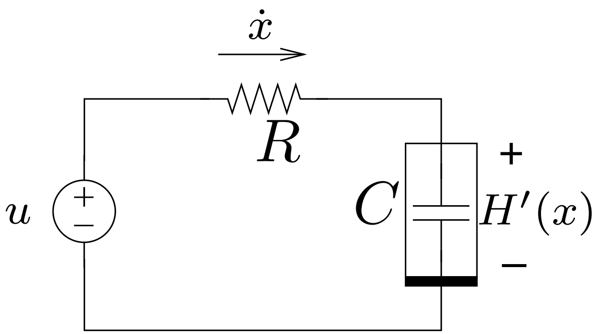
An Example of Stabilization via CbI_{PS}

Consider the nonlinear RC circuit with x the capacitor charge, $\dot{x} = i$ and $H' = v$



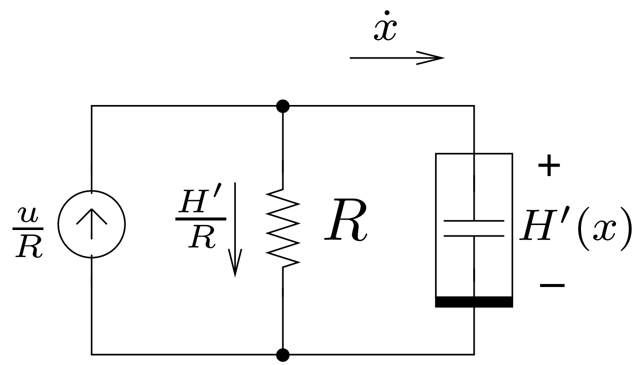
One PH model $u \to y := \frac{1}{R} H'$:

$$\begin{aligned} \dot{x} &= -\frac{1}{R} H' + \frac{1}{R} u \\ y &= \frac{1}{R} H'. \end{aligned}$$

Power balance equation

$$\dot{H} = -\frac{1}{R} (H')^2 + H' \frac{1}{R} u.$$

Best understood from the equivalent circuit:



obtained applying the Thevenin–Norton transformation A more physically sensible way of viewing the system is $u \to \dot{x}$, that is,

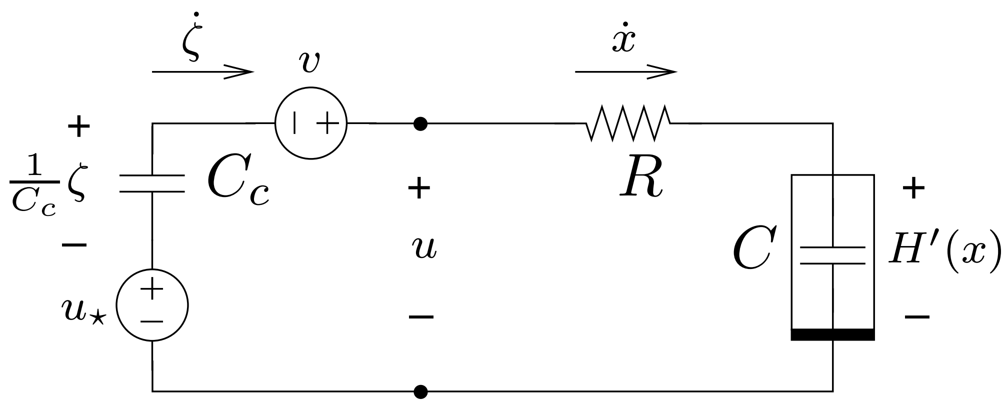
$$\Sigma_{(u, y_{\text{PS}})} : \begin{cases} \dot{x} = -\frac{1}{R} H' + \frac{1}{R} u \\ y_{\text{PS}} = -\frac{1}{R} H' + \frac{1}{R} u = \dot{x}, \end{cases}$$

cont’d

- Total energy function $W_{\mathtt{PS}}(x,\zeta) = H(x) + \frac{1}{2C_c}(\zeta - C_c u_\star)^2 - u_\star(x-\zeta)$.
- The controller is given by

$$\Sigma_c + \Sigma_I^{\mathtt{PS}} : \begin{cases} \dot{\zeta} & = & \frac{1}{R}(-H' + u_\star - \frac{1}{C_c}\zeta + v) \\ u & = & u_\star - \frac{1}{C_c}\zeta + v. \end{cases}$$

- Physical realization



Remark It can be implemented without distinction of “inputs" and “outputs".

Control by State–Modulated Interconnection with $\Sigma_{(u,y)}$

Proposition Assume the PDE

$$\begin{bmatrix} g^{\perp} F^{\top} \\ g^{\top} \end{bmatrix} \nabla \mathcal{C} = 0,, \qquad (CbI^{\mathtt{SM}} - PDE)$$

admits a solution for some vector function $\mathcal{C} : \mathbb{R}^n \to \mathbb{R}^m$. The PH system $\Sigma_{(u,y)}$ with the PH controller Σ_c and the state–modulated power–preserving interconnection subsystem

$$\Sigma_I^{SM} : \left\{ \begin{bmatrix} u \\ u_c \end{bmatrix} = \begin{bmatrix} 0 & -\alpha(x) \\ \alpha^{\top}(x) & 0 \end{bmatrix} \begin{bmatrix} y \\ y_c \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix} \right.,$$

where $\alpha : \mathbb{R}^n \to \mathbb{R}^{m \times m}$ is defined as

$$\alpha = -(g^{\top} g)^{-1} g^{\top} F \nabla \mathcal{C}.$$

Then, for all functions $\Phi : \mathbb{R}^m \to \mathbb{R}$, $\dot{W} \leq v^{\top} y$, where

$$W(x, \zeta) \triangleq H(x) + H_c(\zeta) + \Phi(\mathcal{C}(x) - \zeta).$$

Control by State–Modulated Interconnection with $\Sigma_{(u,y_{\tt PS})}$

Proposition Assume (PO–PDE) admits a solution F_d verifying (SYM) and such that

$$g^{\perp} F_d \nabla \mathcal{C} = 0, \qquad (CbI^{\tt SM}_{\tt PS} - PDE)$$

for some vector function $\mathcal{C} : \mathbb{R}^n \to \mathbb{R}^m$, where $g^{\perp} \in \mathbb{R}^{(n-m)\times n}$ is a full rank left annihilator of g, that is, $g^{\perp} g = 0$ and rank $g^{\perp} = n - m$. The PH system $\Sigma_{(u,y_{\tt PS})}$ with the PH controller Σ_c and the state–modulated power–preserving interconnection subsystem

$$\Sigma_I^{SM} : \left\{ \begin{bmatrix} u \\ u_c \end{bmatrix} = \begin{bmatrix} 0 & -\alpha(x) \\ \alpha^{\top}(x) & 0 \end{bmatrix} \begin{bmatrix} y_{\tt PS} \\ y_c \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix} \right.,$$

where $\alpha : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is defined as

$$\alpha = -(g^{\top} g)^{-1} g^{\top} F_d \nabla \mathcal{C}.$$

Then, for all functions $\Phi : \mathbb{R}^m \to \mathbb{R}$, $\dot{W}_{\tt PS} \leq v^{\top} y_{\tt PS}$, where

$$W_{\tt PS}(x,\zeta) \triangleq H_{\tt PS}(x) + H_c(\zeta) + \Phi(\mathcal{C}(x) - \zeta).$$

Summary of CbI

- (CbI)

$$\begin{bmatrix} F \\ g^\top \end{bmatrix} \nabla\mathcal{C} = \begin{bmatrix} -g \\ 0 \end{bmatrix}.$$

- $(CbI^{\mathtt{SM}})$

$$\begin{bmatrix} g^\perp F \\ g^\top \end{bmatrix} \nabla\mathcal{C} = 0.$$

- (Basic $CbI_{\mathtt{PS}}$)

$$F\nabla\mathcal{C} = -g.$$

- (Basic $CbI_{\mathtt{PS}}^{\mathtt{SM}}$)

$$g^\perp F\nabla\mathcal{C} = 0.$$

- $(CbI_{\mathtt{PS}})$

$$F_d\nabla\mathcal{C} = -g,$$

- $(CbI_{\mathtt{PS}}^{\mathtt{SM}})$

$$g^\perp F_d\nabla\mathcal{C} = 0,$$

cont'd

Implication diagram (from the point of view of solvability of the PDEs)

$$
\begin{array}{ccccc}
\text{CbI} & \longrightarrow & \text{Basic CbI}_{\mathtt{PS}} & \longrightarrow & \text{CbI}_{\mathtt{PS}} \\
\downarrow & & \downarrow & & \downarrow \\
\text{CbI}^{\mathtt{SM}} & \longrightarrow & \text{Basic CbI}_{\mathtt{PS}}^{\mathtt{SM}} & \longrightarrow & \text{CbI}_{\mathtt{PS}}^{\mathtt{SM}}
\end{array}
$$

Notation: $A \rightarrow B$ means that the set of solutions of the PDEs of B is strictly larger than the one of A, consequently the set of plants to which B is applicable is also strictly larger.

Also, $A \leftrightarrow B$ if the PDEs are the same.

4. Stabilization of PH Systems via Standard PBC

Passivation Objective Consider the PH system

$$\Sigma_{(u,y)} \left\{ \begin{array}{rcl} \dot{x} & = & F\nabla H + gu \\ y & = & g^\top \nabla H, \end{array} \right. \qquad \Rightarrow \qquad \dot{H} = u^\top y + \underbrace{\nabla H^\top F \nabla H}_{-d \leq 0}$$

with x_* an equilibrium to be stabilized. Select a control action

$$u = \hat{u}(x) + v,$$

so that the closed–loop system satisfies the desired dissipation equality (DDE)

$$\dot{H}_d = v^\top z - d_d \qquad\qquad (DDE)$$

- $H_d(x)$ has a strict minimum at x_*, ($\Rightarrow v \rightarrow z$ is passive)
- $d_d(t) \geq 0$ desired damping, ($\int d_d$ is the dissipated energy), and
- z is the new passive output (to be defined.)

Remark State feedback, for ease of presentation. Must derivations applicable for (f, g, h) systems.

Satisfying the Desired Dissipation Equality

Fact (Hill/Moylan, '76) Consider $\Sigma_{(u,y)}$ with $u = \hat{u}(x) + v$. Then (DDE) holds iff

$$\begin{aligned} \nabla H_d^\top (F \nabla H + g\hat{u}) &= -d_d \qquad (HM1) \\ z &= g^\top \nabla H_d \end{aligned}$$

Approach

Given (F, g, H). Select desired damping $d_d \geq 0$ to be able to characterize a set of assignable energy functions and controls, $(H_d, \hat{u})$, that solve (HM1).

Remarks

- For LTI systems, $\dot{x} = Ax + Bu$, with

$$\hat{u} = Kx, \quad H_d = \frac{1}{2} x^\top P_d x, \quad d_d = \frac{1}{2} x^\top R_d x$$

  (HM1) becomes the Lyapunov equation $P_d(A + BK) + (A + BK)^\top P_d = -R_d$.
- In (HM1) the data are H, F and g and unknowns H_d, d_d and $\hat{u}$.
- Relative degree zero outputs do not help because (HM1) is the same.

Energy Balancing Control

Proposition Let

$$d_d = d = -\nabla H^\top F \nabla H.$$

Denote $\hat{u} = \hat{u}_{\text{EB}}$ and define the added energy function $H_a \triangleq H_d - H$.

- All solutions of the PDEs $\begin{bmatrix} g^\perp F^\top \\ g^\top \end{bmatrix} \nabla H_a = 0 \ (EB - PDE)$ define assignable energy functions with

$$\hat{u}_{\text{EB}} = -(g^\top g)^{-1} g^\top F^\top \nabla H_a.$$

- The added energy equals the energy supplied by the controller, that is,

$$\dot{H}_a = -y^\top \hat{u}_{\text{EB}}.$$

- EBC suffers from the dissipation obstacle. More precisely,

$$(EB - PDE) \quad \Rightarrow \quad \mathcal{R} \nabla H_a = 0.$$

Basic IDA–PBC

Proposition Let

$$d_d = -\nabla H_d^\top F \nabla H_d$$

and denote $\hat{u} = \hat{u}_{\mathtt{BIDA}}$.

- All solutions of the PDE

$$g^\perp F \nabla H_a = 0 \qquad\qquad (BIDA - PDE)$$

  define assignable energy functions with

$$\hat{u}_{\mathtt{BIDA}} = (g^\top g)^{-1} g^\top F \nabla H_a.$$

- If $\mathcal{R}\nabla H_a = 0$ and $v = 0$ then

$$\dot{H}_a = -y^\top \hat{u}_{\mathtt{BIDA}}.$$

Remark The closed–loop system is $\dot{x} = F\nabla H_d + gv$.

IDA–PBC

Proposition Let

$$d_d = -\nabla H_d^\top F_d \nabla H_d,$$

with $F_d + F_d^\top \leq 0$, and denote $\hat{u} = \hat{u}_{\mathtt{IDA}}$.

- All solutions of the PDE

$$g^\perp F_d \nabla H_a = g^\perp (F - F_d)\nabla H \qquad (IDA - PDE)$$

  define assignable energy functions with

$$\hat{u}_{\mathtt{IDA}} = (g^\top g)^{-1} g^\top [F_d \nabla H_a + (F_d - F)\nabla H].$$

- If $\mathcal{R} = \mathcal{R}_d =: -\frac{1}{2}(F_d + F_d^\top)$, $\mathcal{R}\nabla H_a = 0$ and $v = 0$ then $\dot{H}_a = -y^\top \hat{u}_{\mathtt{IDA}}$.

Remarks

- The closed–loop is $\dot{x} = F_d \nabla H_d + gv$, hence the name IDA.
- Clearly EBC ⇒ BIDA–PBC ⇒ IDA–PBC but the converses are not true.

Power Shaping PBC

In PS–PBC the solution of (IDA-PDE)

$$g^{\perp} F_d \nabla H_a = g^{\perp}(F - F_d)\nabla H$$

is split in two parts. Note that, using $H_d = H + H_a$, the latter is equivalent to

$$g^{\perp} F_d \nabla H_d = g^{\perp} F \nabla H \qquad (\heartsuit)$$

First, we solve (PO–PDE) $F\nabla H = F_d \nabla H_{\rm PS}$, which replaced in $(\heartsuit)$ yields

$$g^{\perp} F_d \nabla \tilde{H}_a = 0 \qquad (PS - PDE)$$

where we defined $\tilde{H}_a \triangleq H_d - H_{\rm PS}$.

Remarks

- (IDA–PDE) may have solutions even though $F_d^{-1} F \nabla H$ is not a gradient of some function—as required by (PO–PDE). In other words PS–PBC $\Rightarrow$ IDA–PBC but the converse is not true.
- PS–PBC originated, and is a natural option, for electrical circuits. See (PhD Jeltsema'05).

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cont'd

Proposition Denote $\hat{u} = \hat{u}_{\rm PS}$ and consider the solutions F_d, with $F_d + F_d^\top \leq 0$, of

$$\nabla\left(F_d^{-1} F \nabla H\right) = \left[\nabla\left(F_d^{-1} F \nabla H\right)\right]^\top. \qquad (PO - PDE)$$

Let

$$d_d = -(F\nabla H + g\hat{u}_{\rm PS})^\top F_d^{-1}(F\nabla H + g\hat{u}_{\rm PS})$$

All solutions of the PDE

$$g^\perp F_d \nabla \tilde{H}_a = 0 \qquad (PS - PDE)$$

define assignable energy functions with

$$\hat{u}_{\rm PS} = (g^\top g)^{-1} g^\top F_d \nabla \tilde{H}_a.$$

5. Comparison of CbI and Standard PBC: Applicability

- (CbI) $\begin{bmatrix} F \\ g^\top \end{bmatrix} \nabla \mathcal{C} = \begin{bmatrix} -g \\ 0 \end{bmatrix}$
- $(CbI^{\mathtt{SM}})$ $\begin{bmatrix} g^\perp F \\ g^\top \end{bmatrix} \nabla \mathcal{C} = 0$
- (Basic $CbI_{\mathtt{PS}}$) $F \nabla \mathcal{C} = -g$
- $(CbI_{\mathtt{PS}})$ $F_d \nabla \mathcal{C} = -g$ plus (PO–PDE) $(F \nabla H = F_d \nabla H_{\mathtt{PS}})$
- (Basic $CbI^{\mathtt{SM}}_{\mathtt{PS}}$)

$$g^\perp F \nabla \mathcal{C} = 0$$

- $(CbI^{\mathtt{SM}}_{\mathtt{PS}})$

$$g^\perp F_d \nabla \mathcal{C} = 0$$

plus (PO-PDE).

- (EBC) $\begin{bmatrix} g^\perp F \\ g^\top \end{bmatrix} \nabla H_a = 0$
- (Basic IDA)

$$g^\perp F \nabla H_a = 0$$

- (PS)

$$g^\perp F_d \nabla H_a = 0$$

plus (PO–PDE)

- (IDA)

$$g^\perp F_d \nabla H_a = g^\perp (F - F_d) \nabla H$$

Final Implication Diagram

$$
\begin{array}{ccccccc}
\mathrm{CbI} & \longrightarrow & \text{Basic CbI}_{\texttt{PS}} & \longrightarrow & \mathrm{CbI}_{\texttt{PS}} & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{CbI}^{\texttt{SM}} & \longrightarrow & \text{Basic CbI}^{\texttt{SM}}_{\texttt{PS}} & \longrightarrow & \mathrm{CbI}^{\texttt{SM}}_{\texttt{PS}} & & \\
\updownarrow & & \updownarrow & & \updownarrow & & \\
\mathrm{EBC} & \longrightarrow & \text{Basic IDA} & \longrightarrow & \texttt{PS} & \longrightarrow & \mathrm{IDA}
\end{array}
$$

Standard PBC and CbI: Connections

- CbI
  - Dynamic feedback control $u = -y_c + v = -\nabla_\zeta H_c(\zeta) + v$,
  - ζ controllers state with energy $H_c(\zeta)$ free,
  - Generate Casimir functions, $\mathcal{C}$, that make $\Omega = \{(x, \zeta) | \zeta = \mathcal{C}(x)\}$ invariant
  - $\Rightarrow$ For arbitrary Φ

$$\dot{H}(x) + \dot{H}_c(\zeta) + \Phi(\mathcal{C}(x) - \zeta) \leq v^\top y$$

- Standard PBC
  - Solve some PDE on H_a and define a static state feedback, $\hat{u}(x)$, that ensures

$$\dot{H} + \dot{H}_a \leq v^\top y$$

- Questions
  - Is there a connection between the two methods?
  - What happens if we restrict to Ω?
  - Is there an advantage of dynamic extension from minimum assignment viewpoint?

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Restricting a CbI Controller Yields an EBC

Proposition Assume (CbI–PDE) admit a solution. Then, for all $H_c : \mathbb{R}^m \to \mathbb{R}$, the PH system $\Sigma_{(u,y)}$ in closed–loop with the static state–feedback control $u = \hat{u}_{\mathtt{EB}}(x) + v$, where

$$\hat{u}_{\mathtt{EB}}(x) = -\nabla_{\mathcal{C}} H_c(\mathcal{C}(x)),$$

satisfies the cyclo–passivity inequality

$$\dot{H} + \frac{d}{dt} H_c(\mathcal{C}(x)) \leq v^\top y.$$

Furthermore,

$$\frac{d}{dt} H_c(\mathcal{C}(x)) = -y^\top \hat{u}_{\mathtt{EB}}.$$

Proof Define $H_a(x) \stackrel{\triangle}{=} H_c(\mathcal{C}(x))$

$$\begin{aligned} \dot{H}_a &= (\nabla_{\mathcal{C}} H_c(\mathcal{C}))^\top (\nabla \mathcal{C})^\top (F \nabla H + g u) \\ &= (\nabla_{\mathcal{C}} H_c(\mathcal{C}))^\top g^\top \nabla H \qquad (\Leftarrow F^\top \nabla \mathcal{C} = g, g^\top \nabla \mathcal{C} = 0) \\ &= -\hat{u}_{\mathtt{EB}}^\top y \qquad (\Leftarrow \hat{u}_{\mathtt{EB}} = -\nabla_{\mathcal{C}} H_c(\mathcal{C}), y = g^\top \nabla H). \end{aligned}$$

Restricting a $CbI_{\mathtt{PS}}$ Controller Yields an IDA–PBC

Proposition Assume the conditions for $CbI_{\mathtt{PS}}$ are satisfied. Then, for all $H_c : \mathbb{R}^m \to \mathbb{R}$, the state–feedback controller

$$\hat{u}_{\mathtt{IDA}}(x) = -\nabla_{\mathcal{C}} H_c(\mathcal{C}(x)),$$

ensures that the IDA–PBC matching condition

$$F\nabla H + g\hat{u}_{\mathtt{IDA}} = F_d \nabla H_d$$

is satisfied with $H_d = H_{\mathtt{PS}} + H_a$ and $H_a(x) \stackrel{\triangle}{=} H_c(\mathcal{C}(x))$.

Proof

Conditions for $CbI_{\mathtt{PS}}$:

- (PO–PDE) $\Leftrightarrow F\nabla H = F_d \nabla H_{\mathtt{PS}}$,
- ($CbI_{\mathtt{PS}}$–PDE) $\Leftrightarrow F_d \nabla \mathcal{C} = -g$.

Replacing in the matching equation yields

$$F_d(\nabla H_{\mathtt{PS}} - (\nabla \mathcal{C})\hat{u}_{\mathtt{IDA}}) = F_d \nabla H_d \quad \Leftrightarrow \quad \nabla H_a = -(\nabla \mathcal{C})\hat{u}_{\mathtt{IDA}},$$

which is satisfied with H_a and $\hat{u}_{\mathtt{IDA}}$ above. $\diamond\diamond\diamond$

Dynamic Extension and Stabilization

- We have concentrated our attention on the ability of the various PBCs to modify the energy function, without particular concern to stabilization.
- Stability will be ensured if a (desired) strict minimum is assigned to the total energy function

Proposition In the single input case, the use of a dynamic extension does not provide any additional freedom for minimum assignment to the corresponding static state–feedback solutions.

Proof Define

$$\begin{aligned} W(x,\zeta) &\triangleq H(x) + H_c(\zeta) + \Phi(\mathcal{C}(x) - \zeta) \\ H_d(x) &\triangleq H(x) + H_c(\mathcal{C}(x)). \end{aligned}$$

We can prove that

$$\nabla W_\star = 0 \;\text{ and }\; \nabla^2 W_\star > 0 \;\Rightarrow\; (\nabla H_d)_\star = 0 \;\text{ and }\; (\nabla^2 H_d)_\star > 0.$$

Future Research

- Is there a CbI version of IDA? What is the modification that is needed to add this degree of freedom?
- We have fixed the order of the dynamic extension to be m. There are some advantages for increasing their number. Also, we have taken simple nonlinear integrators.
- Dynamic extension does not help for minimum assignment, but certainly has an impact on performance and simplicity.
- Will dynamic extension enlarge the domain of applicability of IDA–PBC?
- We have chosen the “standard" interconnection $u = -y_c,\ u_c = y$. If we consider $u = u_c,\ y = -y_c$, it is also power–preserving hence shapes the energy—adding an algebraic constraint. New way to shape kinetic energy in mechanical systems.