Robust performance analysis in quadratic separation framework

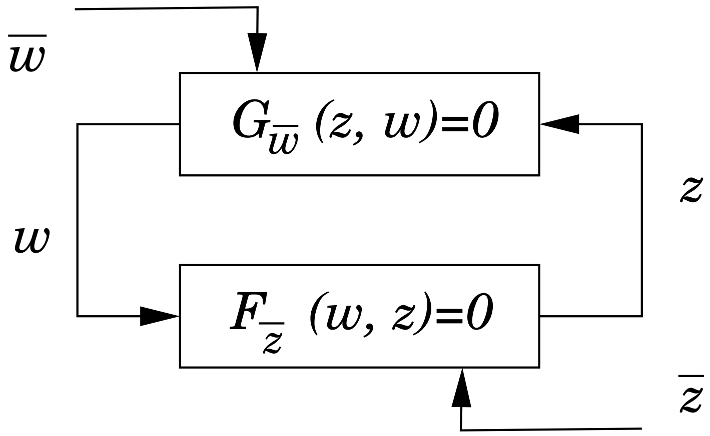
Dimitri PEAUCELLE
LAAS-CNRS - Université de Toulouse - FRANCE
joint work with Denis Arzelier and Frédéric Gouaisbaut

Seminar at Institute of Control Problems, Moscow

July 16th, 2009

1. Topological separation [Safonov 1980]
   - Well-posedness definition and main result
   - Relations with Lyapunov theory
   - The case of linear uncertain systems : quadratic separation
   - The Lur'e problem
   - Relations with IQC framework & μ-theory
   - A $\mathcal{S}$-procedure like result
2. Integral Quadratic Separation (IQS) for the descriptor case
3. Performances in the IQS framework
4. System augmentation : a way to conservatism reduction
5. The Romuald toolbox

Well-posedness



Well-Posedness:

Bounded $(\bar{w}, \bar{z}) \Rightarrow$ unique bounded (w, z)

- In case $\overbrace{z = Aw + \bar{z}}^{F}$ and $\overbrace{w = \Delta z + \bar{w}}^{G}$ are linear applications

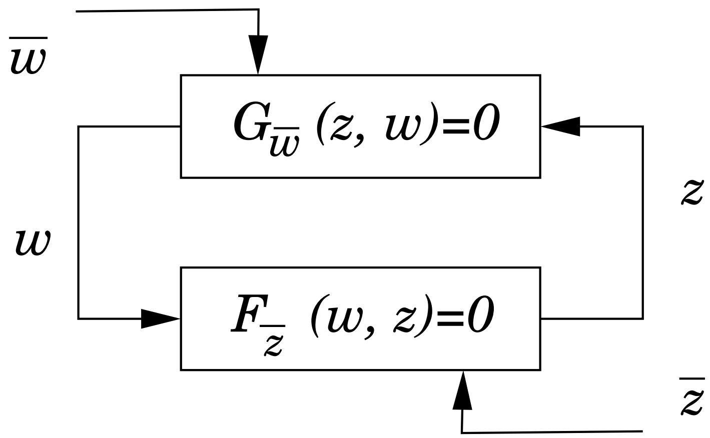
Well-posedness : $(1 - A\Delta)$ non-singular

- What if $\Delta = \Delta \in \boldsymbol{\Delta}$ is uncertain ?
- If $A = T(j\omega)$ is an LTI system ?
- If G is non-linear ?

...

❶ Topological separation

Well-posedness & topological separation



Well-Posedness:

Bounded $(\bar{w}, \bar{z})$

$$\Rightarrow \exists!(w,z)\ ,\ \exists\gamma\ :\ \left\| \begin{array}{c} w \\ z \end{array} \right\| \leq \gamma \left\| \begin{array}{c} \bar{w} \\ \bar{z} \end{array} \right\|$$

- [Safonov 80] $\exists\theta$ topological separator:

$$\mathcal{G}^I(\bar{w}) = \{(w,z)\ :\ G_{\bar{w}}(z,w)=0\} \subset \{(w,z)\ :\ \theta(w,z) \leq \phi_2(||\bar{w}||)\}$$

$$\mathcal{F}(\bar{z}) = \{(w,z)\ :\ F_{\bar{z}}(w,z)=0\} \subset \{(w,z)\ :\ \theta(w,z) > -\phi_1(||\bar{z}||)\}$$

■ For dynamic systems $\dot{x} = f(x)$, topological separation $\equiv$ Lyapunov theory

$$\overbrace{z(t) = f(w(t)) + \bar{z}(t)}^{F} \;,\; \overbrace{\underbrace{w(t)}_{x(t)} = \int_0^t \underbrace{z(\tau)}_{\dot{x}(t)} d\tau + \bar{w}(t)}^{G}$$

▲ $\bar{w}$: contains information on initial conditions ($x(0) = 0$ by convention)

● Well-posedness $\Rightarrow$ for zero initial conditions and zero perturbations :

$w = z = 0$ (equilibrium point).

● Well-posedness (global stability)

$\Rightarrow$ whatever bounded perturbations the state remains close to equilibrium

❶ Topological separation

■ For dynamic systems $\dot{x} = f(x)$, topological separation $\equiv$ Lyapunov theory

$$\overbrace{z(t) = f(w(t)) + \bar{z}(t)}^{F} \quad , \quad \overbrace{\underbrace{w(t)}_{x(t)} = \int_0^t \underbrace{z(\tau)}_{\dot{x}(t)} d\tau + \bar{w}(t)}^{G}$$

● Assume a Lyapunov function $V(0) = 0$, $V(x) > 0$, $\dot{V}(x) < 0$

▲ Topological separation w.r.t. $\mathcal{G}^I(\bar{w})$ is obtained with

$$\theta(w = x, z = \dot{x}) = \int_0^\infty -\frac{\partial V}{\partial x}(x(\tau))\dot{x}(\tau)d\tau = \lim_{t\to\infty} -V(x(t)) < \gamma_1 \|\bar{w}\|$$

▲ Topological separation w.r.t. $\mathcal{F}(\bar{z})$ does hold as well

$$\theta(w, z = f(w)) = \int_0^\infty -\dot{V}(w(\tau))d\tau > -\gamma_2 \|\bar{z}\|$$

■ For linear systems : quadratic Lyapunov function, *i.e. quadratic separator*

$$\overbrace{z(t) = Aw(t) + \bar{z}(t)}^{F_{\bar{z}}(z,w)} \ , \ \overbrace{\underbrace{w(t)}_{x(t)} = \int_0^t \underbrace{z(\tau)}_{\dot{x}(t)} d\tau + \bar{w}(t)}^{G_{\bar{w}}(z,w)}$$

● A possible separator based on quadratic Lyapunov function $V(x) = x^T P x$

$$\theta(w,z) = \int_0^\infty \begin{pmatrix} z^T(\tau) & w^T(\tau) \end{pmatrix} \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix} \begin{pmatrix} z(\tau) \\ w(\tau) \end{pmatrix} d\tau$$

▲ Quadratic separation w.r.t. $\mathcal{G}^I(\bar{w})$:

$$\lim_{t\to\infty} -x^T(t) P x(t) \leq \gamma_1 \|\bar{w}\| \ , \ \textit{i.e.} \ P > 0$$

▲ Quadratic separation w.r.t. $\mathcal{F}(\bar{z})$ guaranteed if

$$\forall t > 0 \ , \ -2w^T(t) P A w(t) > -\gamma_2 \|\bar{z}(t)\| \ , \ \textit{i.e.} \ A^T P + PA < 0$$

Topological separation : alternative to Lyapunov theory

- Needs to manipulate systems in a new form
- Suited for systems described as feedback connected blocs

Any linear system with rational dependence w.r.t. parameters writes as such

$$\dot{x} = (A + B_\Delta \Delta (1 - D_\Delta \Delta)^{-1} C_\Delta) x \quad \overset{\text{LFT}}{\longleftrightarrow} \quad \left\{ \begin{array}{l} \dot{x} = Ax + B_\Delta w_\Delta \\ z_\Delta = C_\Delta x + D_\Delta w_\Delta \\ \\ w_\Delta = \Delta z_\Delta \end{array} \right.$$

- Finding a topological separator is *a priori*

as complicated as finding a Lyapunov function

- Allows to deal with several features simultaneously in a unified way

Quadratic separation [Iwasaki & Hara 1998]

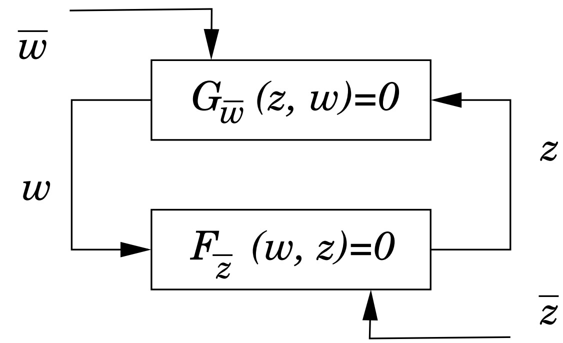
- If $F(w) = Aw$ is a linear transformation and $G = \Delta$ is an uncertain operator defined as $\Delta \in \mathbf{\Delta}$ convex set it is necessary and sufficient to look for a quadratic separator

$$\theta(z,w) = \int_0^\infty \begin{pmatrix} z^T & w^T \end{pmatrix} \Theta \begin{pmatrix} z \\ w \end{pmatrix} d\tau$$

- If $F(w) = A(\omega)w$ is a linear parameter dependent transformation and $G = \Delta$ is an uncertain operator defined as $\Delta \in \mathbf{\Delta}$ convex set necessary and sufficient to look for a parameter-dependent quadratic separator

$$\theta(z,w) = \int_0^\infty \begin{pmatrix} z^T & w^T \end{pmatrix} \Theta(\omega) \begin{pmatrix} z \\ w \end{pmatrix} d\tau$$

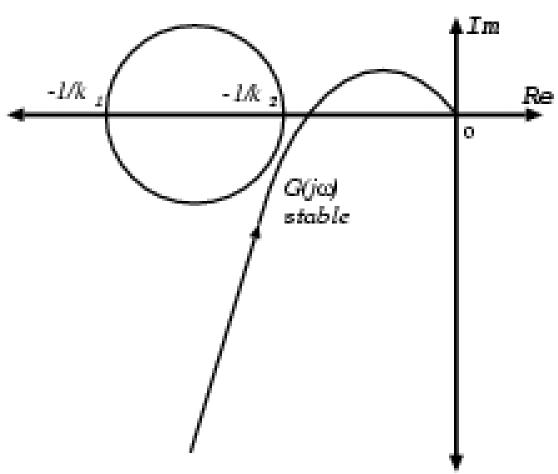
A well-known example : the Lur'e problem



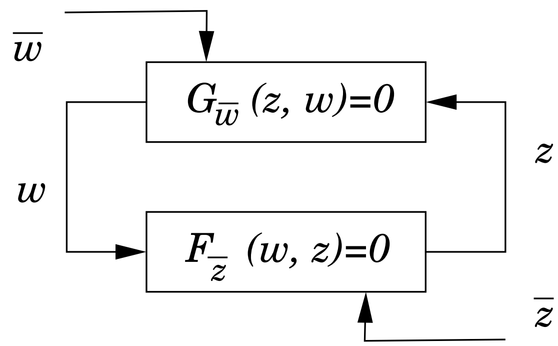
- $F = T(j\omega)$ is a transfer function
- $G(z)/z \in [\ -k_1,\ \ -k_2\ ]$ is a sector-bounded gain

  (*i.e.* the inverse graph of G is in $[\ -1/k_1\ ,\ \ -1/k_2\ ]$)

- Circle criterion : exists a quadratic separator (circle) for all ω



Another example : parameter-dependent Lyapunov function



- $F = A(\delta)$ parameter-dependent LTI state-space model ($\dot{\delta}$)
- $G = \mathcal{I}$ is an integrator

- Necessary and sufficient to have

$$\Theta(\delta) = \begin{bmatrix} 0 & -P(\delta) \\ -P(\delta) & 0 \end{bmatrix}$$

Direct relation with the IQC framework

- $F = T(j\omega)$ is a transfer matrix
- $G = \Delta$ is an operator known to satisfy an Integral Quadratic Constraint (IQC)

$$\int_{-\infty}^{+\infty} \begin{bmatrix} 1 & \Delta^*(j\omega) \end{bmatrix} \Pi(\omega) \begin{bmatrix} 1 \\ \Delta(j\omega) \end{bmatrix} d\omega \leq 0$$

- Stability of the closed-loop is guaranteed if for all ω

$$\begin{bmatrix} T^*(j\omega) & 1 \end{bmatrix} \Pi(\omega) \begin{bmatrix} T(j\omega) \\ 1 \end{bmatrix} > 0$$

- Knowing $\mathbf{\Delta}$ the set of Δ how to choose $\Pi = \Theta$?

(*i.e.* the quadratic separator)

■ Some choices of quadratic separators Θ

▲ Δ is full-bloc complex norm-bounded : $\mathbf{\Delta} = \{ \Delta^* \Delta \leq \bar{k}^2 1 \}$

Θ should be such that

$$\begin{bmatrix} 1 & \Delta^* \end{bmatrix} \begin{bmatrix} -\bar{k}^2 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \Delta \end{bmatrix} \leq 0 \Rightarrow \begin{bmatrix} 1 & \Delta^* \end{bmatrix} \Theta \begin{bmatrix} 1 \\ \Delta \end{bmatrix} \leq 0$$

● $\mathcal{S}$-procedure ! [Yakubovitch 70's]

$$\exists \tau > 0 \ : \ \Theta \leq \tau \begin{bmatrix} -\bar{k}^2 1 & 0 \\ 0 & 1 \end{bmatrix}$$

● Closed-loop well-posedness losslessly assessed by

$$\begin{bmatrix} F^* & 1 \end{bmatrix} \begin{bmatrix} -\tau \bar{k}^2 1 & 0 \\ 0 & \tau 1 \end{bmatrix} \begin{bmatrix} F \\ 1 \end{bmatrix} > 0$$

Some choices of quadratic separators Θ

▲ $\Delta = \delta 1$ is scalar complex norm-bounded : $\mathbf{\Delta} = \{ \delta 1 \ : \ |\delta| \leq \bar{k} \}$

- D-scaling

$$\exists D > 0 \ : \ \Theta \leq \begin{bmatrix} -\bar{k}^2 D & 0 \\ 0 & D \end{bmatrix}$$

- Closed-loop well-posedness losslessly assessed by

$$\begin{bmatrix} F^* & 1 \end{bmatrix} \begin{bmatrix} -\bar{k}^2 D & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} F \\ 1 \end{bmatrix} > 0$$

Some choices of quadratic separators Θ

▲ $\Delta = \delta 1$ is scalar real norm-bounded : $\mathbf{\Delta} = \{ \delta 1 \ : \ |\delta| \leq \bar{k} \ , \ \delta = \delta^* \}$

- DG-scaling

$$\exists D > 0 \ , \ G = -G^* \ : \ \Theta \leq \begin{bmatrix} -\bar{k}^2 D & G \\ G^* & D \end{bmatrix}$$

- Closed-loop well-posedness losslessly assessed by

$$\begin{bmatrix} F^* & 1 \end{bmatrix} \begin{bmatrix} -\bar{k}^2 D & G \\ G^* & D \end{bmatrix} \begin{bmatrix} F \\ 1 \end{bmatrix} > 0$$

Some choices of quadratic separators Θ

▲ $\Delta = j\omega 1$ with $\omega \in$ R

- Naturally suggested scaling

$$\exists P = P^* \quad : \quad \Theta \leq \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix}$$

- Closed-loop well-posedness losslessly assessed by (KYP lemma)

$$\begin{bmatrix} F^* & 1 \end{bmatrix} \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} F \\ 1 \end{bmatrix} > 0$$

Some choices of quadratic separators Θ

▲ $\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \delta_2 1 \end{bmatrix}$

Δ_1 if full-bloc complex norm-bounded in $\{ \Delta_1{}^* \Delta_1 \leq \bar{k}_1^2 1 \}$

δ_2 is scalar real norm-bounded in $\{ \delta_2 1 \ : \ |\delta_2| \leq \bar{k}_2 \ , \ \delta_2 = \delta_2{}^* \}$

● One can take (full-block $\mathcal{S}$-procedure [Scherer], etc.)

$$\Theta = \left[\begin{array}{cc|cc} -\tau \bar{k}_1^2 1 & 0 & 0 & 0 \\ 0 & -\bar{k}_2^2 D & 0 & G \\ \hline 0 & 0 & \tau 1 & 0 \\ 0 & G^* & 0 & D \end{array}\right]$$

μ-theory is a special case of IQC framework

- $F = T(j\omega)$ is a transfer matrix
- Δ is bloc-diagonal composed of
  - m_F full-bloc complex norm-bounded uncertainties
  - m_c scalar complex norm-bounded uncertainties
  - m_r scalar real norm-bounded uncertainties
- All uncertainties bounded by same $\bar{k}$ (at the expense of modifying $T(j\omega)$)
- Goal : $k_m = \max \bar{k}$

- If $\mu = \frac{1}{k_m} < 1$ stability is proved.
- Convex problem with DG-scalings

(LMI for fixed $\bar{k}$, else Generalized Eigenvalue Problem)

❶ Topological separation

- [Meinsma 97] DG-scalings are lossless if

$$2(m_c + m_r) + m_F \leq 3$$

▲ In μ-theory needed to test for all $\omega \in$ R : griding techniques and ...

- Alternative is to consider s^{-1} as a scalar uncertainty

(treated as additional complex scalar bloc)

▲ Lossless conditions may only be achieved for $m_F = 1,\ m_c = 0,\ m_r = 0$.

Outline

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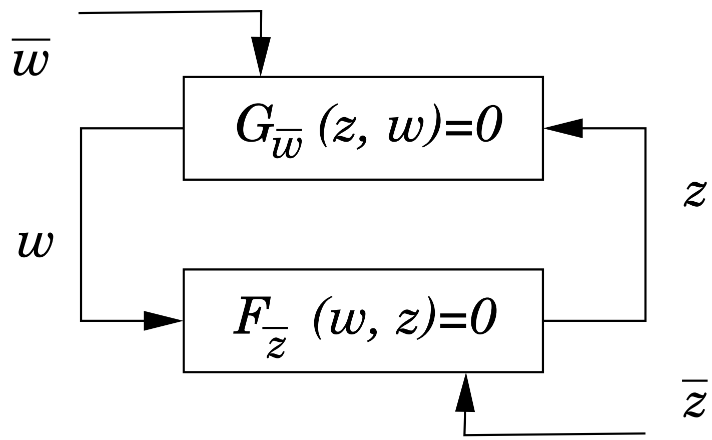
❷ The descriptor case



■ Linear implicit application in feedback loop with an uncertain operator

$$\underbrace{\mathcal{E}z(t) = \mathcal{A}w(t)}_{F} \quad , \quad \underbrace{w(t) = [\nabla z](t)}_{G} \qquad \nabla \in \mathbb{W}$$

● ∇ is bloc-diagonal contains scalar, full-bloc, LTI and LTV uncertainties and other operators such as integrator etc.

■ Integral Quadratic Separation [Automatica'08, CDC'08, ROCOND'09]

● For the case of linear application with uncertain operator

$$\mathcal{E}z(t) = \mathcal{A}w(t) \ , \ w(t) = [\nabla z](t) \quad \nabla \in \mathbb{W}$$

where $\mathcal{E} = \mathcal{E}_1\mathcal{E}_2$ with $\mathcal{E}_1$ full column rank,

● Integral Quadratic Separator (IQS) : $\exists\Theta$, matrix, solution of LMI

$$\begin{bmatrix} \mathcal{E}_1 & -\mathcal{A} \end{bmatrix}^{\perp *} \Theta \begin{bmatrix} \mathcal{E}_1 & -\mathcal{A} \end{bmatrix}^{\perp} > 0$$

and Integral Quadratic Constraint (IQC) $\forall\nabla \in \mathbb{W}$

$$\int_0^\infty \begin{pmatrix} \mathcal{E}_2 z(t) \\ [\nabla z](t) \end{pmatrix}^* \Theta \begin{pmatrix} \mathcal{E}_2 z(t) \\ [\nabla z](t) \end{pmatrix} dt \leq 0$$

■ Integral Quadratic Separation [Automatica'08, CDC'08, ROCOND'09]

▲ Proof of sufficiency is starting from

$$\mathcal{E}z(t) = \mathcal{A}w(t) + \bar{z} \ , \ \ w(t) = [\nabla z](t) + \bar{w}$$

to prove using the LMI and IQC constraints that

$$\exists\lambda \ : \ \begin{array}{l} \forall \bar{z} \in L_2, \bar{w} \in L_2 \\ \forall \nabla \in \mathbb{W} \end{array} \ , \ \left\| \begin{array}{c} \mathcal{E}z \\ w \end{array} \right\| \leq \lambda \left\| \begin{array}{c} \bar{z} \\ \bar{w} \end{array} \right\|$$

▲ Note that z is not required to be unique and bounded, only $\mathcal{E}z$.

● For some given $\mathbb{W}$, $\exists$ LMI conditions for Θ solution to IQC [ECC'09]

(improved DG-scalings, build out of IQS for elementary blocs of ∇)

(tedious construction but can be automatized)

- Integral Quadratic Separator : all signals are assumed L_2: $\|z\|^2 < \infty$

$$\|z\|^2 = \text{Trace} \int_0^\infty z^*(t)z(t)dt \ , \quad < z|w >= \text{Trace} \int_0^\infty z^*(t)w(t)dt$$

- Notation

$$\|z\|_T^2 = \text{Trace} \int_0^T z^*(t)z(t)dt \ , \quad < z|w >_T = \text{Trace} \int_0^T z^*(t)w(t)dt$$

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■ Induced L_2 norm (H_∞ in the LTI case)

$$E\dot{x} = Ax + Bv \ , \ \ g = Cx + Dv$$

▲ Prove that system is asymptotically stable

▲ and $\|g\| < \gamma\|v\|$ for zero initial conditions $x(0) = 0$

(strict upper bound on the L_2 gain attenuation)

● Equivalent to well-posedness with respect to

Integrator with zero initial conditions $x(t) = [\mathcal{I}_1\dot{x}](t) = \int_0^t \dot{x}(\tau)d\tau$

and signals such that $\|v\| \leq \frac{1}{\gamma}\|g\|$

■ Induced L_2 norm

$$E\dot{x} = Ax + Bv \ , \ \ g = Cx + Dv$$

▲ Define ∇_{n2n} the fictitious non-causal uncertain operator such that

$$v = \nabla_{n2n} g \quad \text{iff} \quad \|v\| \leq \frac{1}{\gamma}\|g\|$$

● Induced L_2 norm problem is equivalent to well-posedness of

$$\underbrace{\begin{bmatrix} E & 0 \\ 0 & 1 \end{bmatrix}}_{\mathcal{E}} \underbrace{\begin{pmatrix} \dot{x} \\ g \end{pmatrix}}_{z} = \underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{pmatrix} x \\ v \end{pmatrix}}_{w} \ , \quad \nabla = \begin{bmatrix} \mathcal{I}_1 & 0 \\ 0 & \nabla_{n2n} \end{bmatrix}$$

Induced L_2 norm

$$\underbrace{\begin{bmatrix} E & 0 \\ 0 & 1 \end{bmatrix}}_{\mathcal{E}} \underbrace{\begin{pmatrix} \dot{x} \\ g \end{pmatrix}}_{z} = \underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{pmatrix} x \\ v \end{pmatrix}}_{w} \quad , \quad \nabla = \begin{bmatrix} \mathcal{I}_1 & 0 \\ 0 & \nabla_{n2n} \end{bmatrix}$$

- Elementary IQS for bloc $\mathcal{I}_1$ is

$$\Theta_{\mathcal{I}_1} = \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix} \quad : \quad P > 0$$

Indeed (recall $x(t) = [\mathcal{I}_1 \dot{x}](t) = \int_0^t \dot{x}(\tau) d\tau$ and $x(0) = 0$)

$$\left\langle \begin{pmatrix} \dot{x} \\ \mathcal{I}_1 \dot{x} \end{pmatrix} \middle| \Theta_{\mathcal{I}_1} \begin{pmatrix} \dot{x} \\ \mathcal{I}_1 \dot{x} \end{pmatrix} \right\rangle_T = -x^*(T) P x(T) \leq 0$$

Induced L_2 norm

$$\underbrace{\begin{bmatrix} E & 0 \\ 0 & 1 \end{bmatrix}}_{\mathcal{E}} \underbrace{\begin{pmatrix} \dot{x} \\ g \end{pmatrix}}_{z} = \underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{pmatrix} x \\ v \end{pmatrix}}_{w} \quad , \quad \nabla = \begin{bmatrix} \mathcal{I}_1 & 0 \\ 0 & \nabla_{n2n} \end{bmatrix}$$

- Elementary IQS for bloc ∇_{n2n} is (small gain theorem)

$$\Theta_{\nabla_{n2n}} = \begin{bmatrix} -\tau 1 & 0 \\ 0 & \tau\gamma^2 1 \end{bmatrix} \quad : \quad \tau > 0$$

Indeed (recall $v = \nabla_{n2n} g$ iff $\|v\| \leq \frac{1}{\gamma}\|g\|$)

$$\left\langle \begin{pmatrix} g \\ \nabla_{n2n} g \end{pmatrix} \middle| \Theta_{\nabla_{n2n}} \begin{pmatrix} g \\ \nabla_{n2n} g \end{pmatrix} \right\rangle = \tau(-\|g\|^2 + \gamma^2\|v\|^2) \leq 0$$

❸ Performance analysis in quadratic separation framework

- Apply IQS and get (for non-descriptor case $E = 1$)

$$P > 0 \quad , \quad \tau > 0$$

$$\begin{bmatrix} A^*P + PA + \tau C^*C & PB + \tau C^*D \\ B^*P + \tau D^*C & -\tau\gamma^2 1 + \tau D^*D \end{bmatrix} < 0$$

which is the classical H_∞ result.

- No difficulty to generate LMIs for descriptor case

  & if there are more blocs in ∇ such as uncertainties ...

- Impulse to norm performance (H_2 in the LTI case if $D=0$)

$$E\dot{x} = Ax + Bv \ , \ \ g = Cx + Dv$$

- ▲ Prove that system is asymptotically stable
- ▲ and $\|g\| < \gamma$ if $v = \alpha\delta(t)\mathbf{1}_m$, $|\alpha| \leq 1$ and zero initial conditions $x(0) = 0$
- ! The Dirac delta function $\delta(t)$ is not in L_2
- Impulse inputs define jumps of the state

- Impulse to norm performance (H_2 in the LTI case if $D = 0$)

$$E\dot{x} = Ax + Bv \ , \ \ g = Cx + Dv$$

- Prove that system is asymptotically stable
- and $\|g\| < \gamma$ if $v = \alpha\delta(t)\mathbf{1}_m$, $|\alpha| \leq 1$ and zero initial conditions $x(0) = 0$

- Redefinition of the problem :

$$Ex(0) = \alpha B \qquad , \qquad g(0) = \alpha D$$
$$E\dot{x}(t > 0) = Ax(t > 0) \ , \ \ g(t > 0) = Cx(t > 0)$$

- Prove that system is asymptotically stable
- and $\|g\| < \gamma$ for all $\alpha \leq 1$

- Need to describe initial conditions as signals in L_2

■ Square-root of the shifted delta function φ_θ : $\begin{cases} L_2 \longrightarrow L_2 \\ x \longmapsto \varphi_\theta x \end{cases}$

with properties that φ_θ is linear, and whatever x, y in L_2 and whatever P:

$$[\varphi_\theta y]^*(t)P[\varphi_\theta x](t) = \delta(t-\theta)y^*(t)Px(t)$$

$$[\varphi_{\theta_1} y]^*(t)P[\varphi_{\theta_2} x](t) = 0 \quad \text{if } \theta_1 \neq \theta_2$$

● A formal definition:

$[\varphi_\theta x](t) = \varphi(t-\theta)x(t)$ where φ is the limit of complex valued functions

$$\varphi(t) = \lim_{\epsilon \to 0} \frac{\sqrt{\epsilon/\pi}}{t + j\epsilon} \qquad \left(\lim_{\epsilon \to 0} \frac{\epsilon/\pi}{(t-j\epsilon)(t+j\epsilon)} = \delta(t) \right)$$

● $\varphi_0 x$ is an L_2 signal that contains the information $x(0)$.

- Impulse to norm performance equivalent to well-poedness of

$$\underbrace{\left[\begin{array}{cc|cc} E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]}_{\mathcal{E}} \underbrace{\begin{pmatrix} \varphi_0 x \\ \dot{x} \\ \hline \varphi_0 g \\ g \end{pmatrix}}_{z} = \underbrace{\left[\begin{array}{c|c} 0 & B \\ A & 0 \\ 0 & D \\ C & 0 \end{array}\right]}_{\mathcal{A}} \underbrace{\begin{pmatrix} x \\ \hline v \end{pmatrix}}_{w} \quad , \quad \nabla = \begin{bmatrix} \mathcal{I}_2 & 0 \\ 0 & \nabla_{i2n} \end{bmatrix}$$

- $\mathcal{I}_2$ is the integrator with non-zero initial conditions

$$x(t) = \left[\mathcal{I}_2 \begin{pmatrix} \varphi_0 x \\ \dot{x} \end{pmatrix} \right] (t) = x(0) + \int_0^t \dot{x}(\tau) d\tau$$

$$v = \nabla_{i2n} \begin{pmatrix} \varphi_0 g \\ g \end{pmatrix} \quad : \quad v = \alpha \varphi_0 \mathbf{1}_m \ , \quad |\alpha| \leq \frac{1}{\gamma} \left\| \begin{matrix} \varphi_0 g \\ g \end{matrix} \right\|$$

- Elementary IQS for bloc $\mathcal{I}_2$ is

$$\Theta_{\mathcal{I}_2} = \left[\begin{array}{cc|c} -P & 0 & 0 \\ 0 & 0 & -P \\ \hline 0 & -P & 0 \end{array}\right] \quad : \quad P > 0$$

Indeed (recall $x(t) = [\mathcal{I}_2 \begin{pmatrix} \varphi_0 x \\ \dot{x} \end{pmatrix}](t) = x(0) + \int_0^t \dot{x}(\tau) d\tau$)

$$\left\langle \begin{pmatrix} \varphi_0 x \\ \dot{x} \\ \hline x \end{pmatrix} \middle| \Theta_{\mathcal{I}_2} \begin{pmatrix} \varphi_0 x \\ \dot{x} \\ \hline x \end{pmatrix} \right\rangle_T = -\text{Trace}(x^*(T) P x(T)) \leq 0$$

- Elementary IQS for bloc ∇_{i2n} is

$$\Theta_{\nabla_{i2n}} = \left[\begin{array}{cc|c} -\tau 1 & 0 & \\ 0 & -\tau 1 & 0 \\ \hline 0 & 0 & Q \end{array}\right] \quad : \quad \text{Trace}(Q) < \tau\gamma^2$$

Indeed (recall $v = \nabla_{i2n} \begin{pmatrix} \varphi_0 g \\ g \end{pmatrix} \quad : \quad v = \alpha\varphi_0 \mathbf{1}_m \ , \ |\alpha| \leq \frac{1}{\gamma} \left\| \begin{matrix} \varphi_0 g \\ g \end{matrix} \right\|$)

$$\left\langle \begin{pmatrix} \varphi_0 g \\ g \\ \hline v \end{pmatrix} \middle| \Theta_{\nabla_{i2n}} \begin{pmatrix} \varphi_0 g \\ g \\ \hline v \end{pmatrix} \right\rangle = -\tau \left\| \begin{matrix} \varphi_0 g \\ g \end{matrix} \right\|^2 + \alpha^2 \text{Trace}(Q) \leq 0$$

- Apply IQS and get (for non-descriptor case $E = 1$)

$$P > 0 \quad , \quad \tau > 0 \quad , \quad \text{Trace}(Q) \leq \tau\gamma^2$$

$$A^* P + PA + \tau C^* C < 0 \quad , \quad Q > B^* PB + \tau D^* D$$

which is the classical H_2 result (when $D = 0$) as expected.

- No difficulty to generate LMIs for descriptor case

  & if there are more blocs in ∇ such as uncertainties ...

■ Impulse to peak performance

$$E\dot{x} = Ax + Bv \;\; , \;\; g = Cx + Dv$$

▲ Prove that system is asymptotically stable

▲ and $\max_{t\geq 0} \|g(t)\| < \gamma$ if $v = \delta(t)\alpha$, $\|\alpha\| \leq 1$ and $x(0) = 0$

● Redefinition of the problem :

▲ Let $\theta = \arg\max_{t\geq 0} \|g(t)\|$ (unknown positive or zero)

$$Ex(0) = B\alpha \qquad , \quad g(0) = D\alpha$$

$$E\dot{x}(\theta > t > 0) = Ax(\theta > t > 0) \;\; , \;\; g(\theta) = Cx(\theta)$$

▲ Prove that system is asymptotically stable

▲ and $\|g(0)\| < \gamma$, $\|g(\theta)\| < \gamma$ for all $\|\alpha\| \leq 1$

■ Need to describe final conditions.

■ Truncation operator $\mathbb{T}_\theta$: $\begin{cases} L_2 \longrightarrow L_2 \\ x \longmapsto \mathbb{T}_\theta x \end{cases}$

with properties $\begin{cases} [\mathbb{T}_\theta x](t) = x(t) & \forall t \in [\,0 \;\; \theta\,] \\ [\mathbb{T}_\theta x](t) = 0 & \forall t > \theta \end{cases}$

▲ Integration $\mathcal{I}_3$ maps $\begin{pmatrix} \varphi_0 x \\ \mathbb{T}_\theta \dot{x} \end{pmatrix}$ to $\begin{pmatrix} \mathbb{T}_\theta x \\ \varphi_\theta x \end{pmatrix}$

$$\left[\mathcal{I}_3 \begin{pmatrix} \varphi_0 x \\ \mathbb{T}_\theta \dot{x} \end{pmatrix}\right](t) = x(0) + \int_0^t \dot{x} d\tau = x(t) = \mathbb{T}_\theta x(t) \;\; , \;\; \forall t \in [0\;\theta]$$

$$\left[\mathcal{I}_3 \begin{pmatrix} \varphi_0 x \\ \mathbb{T}_\theta \dot{x} \end{pmatrix}\right](t) = x(0) + \int_0^\theta \dot{x} d\tau = x(\theta) \;\; , \;\; \forall t > \theta \,.$$

- Impulse to peak performance equivalent to well-poedness of

$$\underbrace{\left[\begin{array}{cc|c|c} E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]}_{\mathcal{E}} \underbrace{\left(\begin{array}{c} \varphi_0 x \\ \mathbb{T}_\theta \dot{x} \\ \hline \varphi_0 g \\ \hline \varphi_\theta g \end{array}\right)}_{z} = \underbrace{\left[\begin{array}{cc|c|c} 0 & 0 & 0 & B \\ A & 0 & 0 & 0 \\ 0 & 0 & D & 0 \\ 0 & C & 0 & 0 \end{array}\right]}_{\mathcal{A}} \underbrace{\left(\begin{array}{c} \mathbb{T}_\theta x \\ \varphi_\theta x \\ \hline v_0 \\ \hline v_\theta \end{array}\right)}_{w}$$

$$\nabla = \begin{bmatrix} \mathcal{I}_3 & 0 & 0 \\ 0 & \nabla_{i2p,0} & 0 \\ 0 & 0 & \nabla_{i2p,\theta} \end{bmatrix}$$

where $v_\theta = \nabla_{i2p,\theta} \varphi_\theta g \;:\; v = \varphi_0 \bar{v} \;,\; \bar{v}^* \bar{v} \leq \frac{1}{\gamma^2} < \varphi_\theta g | \varphi_\theta g >$

... LMIs can be produced in the same way as for other performances...

Outline

1. Topological separation [Safonov 1980]
2. Integral Quadratic Separation (IQS) for the descriptor case
3. Performances in the IQS framework
4. System augmentation : a way to conservatism reduction
5. The Romuald toolbox

❹ System augmentation and conservatism reduction

■ General formulation of robust performance analysis

● Well-posedness of

$$\underbrace{\mathcal{E}z(t) = \mathcal{A}w(t)}_{F} \;,\; \underbrace{w(t) = [\nabla z](t)}_{G} \quad \nabla \in \mathbb{W}$$

where ∇ contains

▲ integrator $\mathcal{I}_1$, $\mathcal{I}_2$ or $\mathcal{I}_3$ (or delay operators for discrete-time systems)

▲ performance operator ∇_{n2n}, ∇_{i2n} or ∇_{i2p}

▲ delay operators $\begin{cases} x(t-d) = [\mathcal{D}_0 x](t) \\ x(t) - x(t-d) = [\mathcal{D}_1 \dot{x}](t) \\ \dots \end{cases}$ see [Gouaisbaut]

▲ uncertainties Δ of norm-bounded type (and others : polytopes...)

● LMI results based on DG-scaling type separators

■ May be conservative as soon as more than 2 blocs !

■ Towards less-conservative conditions: System augmentation

▲ Example of stability of uncertain system with parametric uncertainty ($\dot{\delta} = 0$)

$$\dot{x} = (A + \delta B_\Delta (1 - \delta D_\Delta)^{-1} C_\Delta) x$$

▲ Corresponds to well-posedness of

$$\left(\begin{array}{c} \dot{x} \\ z_\Delta \end{array} \right) = \left[\begin{array}{cc} A & B_\Delta \\ C_\Delta & D_\Delta \end{array} \right] \left(\begin{array}{c} x \\ w_\Delta \end{array} \right) , \quad \nabla = \left[\begin{array}{cc} \mathcal{I}_1 1_n & 0 \\ 0 & \delta 1_m \end{array} \right]$$

▲ [Meinsma] rule indicates results may be conservative

4 System augmentation and conservatism reduction

- Well-posedness of

$$\begin{pmatrix} \dot{x} \\ z_\Delta \end{pmatrix} = \begin{bmatrix} A & B_\Delta \\ C_\Delta & D_\Delta \end{bmatrix} \begin{pmatrix} x \\ w_\Delta \end{pmatrix} \ , \ \nabla = \begin{bmatrix} \mathcal{I}_1 1_n & 0 \\ 0 & \delta 1_m \end{bmatrix}$$

- adding the fact that $\dot{w}_\Delta = \delta \dot{z}_\Delta$, is also equivalent to well-posedness of

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -C_\Delta & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \left(\begin{array}{c} \dot{z}_\Delta \\ \dot{x} \\ \hline z_\Delta \\ \dot{z}_\Delta \end{array}\right) = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & A & B_\Delta & 0 \\ 0 & C_\Delta & D_\Delta & 0 \\ 0 & 0 & 0 & D_\Delta \\ 1 & 0 & 0 & 0 \end{array}\right] \left(\begin{array}{c} z_\Delta \\ x \\ \hline w_\Delta \\ \dot{w}_\Delta \end{array}\right)$$

$$\nabla = \begin{bmatrix} \mathcal{I}_1 1_{n+m} & 0 \\ 0 & \delta 1_{2m} \end{bmatrix}$$

❹ System augmentation and conservatism reduction

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -C_\Delta & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{pmatrix} \dot{z}_\Delta \\ \dot{x} \\ \hline z_\Delta \\ \dot{z}_\Delta \end{pmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & A & B_\Delta & 0 \\ 0 & C_\Delta & D_\Delta & 0 \\ 0 & 0 & 0 & D_\Delta \\ 1 & 0 & 0 & 0 \end{array}\right] \begin{pmatrix} z_\Delta \\ x \\ \hline w_\Delta \\ \dot{w}_\Delta \end{pmatrix}$$

$$\nabla = \begin{bmatrix} \mathcal{I}_1 1_{n+m} & 0 \\ 0 & \delta 1_{2m} \end{bmatrix}$$

- It is descriptor model.
- More decisions variables in the separator (increased dimensions of ∇)

- Lyapunov function is with respect to the augmented state

(vector involved in the integrator operator)

$$\begin{pmatrix} z_\Delta^* & x^* \end{pmatrix} P \begin{pmatrix} z_\Delta \\ x \end{pmatrix}$$

▲ Recalling that

$$z_\Delta = \delta(1 - \delta D_\Delta)^{-1} C_\Delta x$$

the result corresponds to looking for a parameter dependent Lyapunov function

$$x^* \begin{bmatrix} \delta(1 - \delta D_\Delta)^{-1} C_\Delta \\ 1 \end{bmatrix}^* P \begin{bmatrix} \delta(1 - \delta D_\Delta)^{-1} C_\Delta \\ 1 \end{bmatrix} x$$

- Proves to be less conservative than for LMIs obtained on original system.

④ System augmentation and conservatism reduction

■ Towards less-conservative conditions: System augmentation

- Adding more equations for higher derivatives of the state:

  less conservative LMI conditions

- Same technique works for time varying uncertainties

  (if known bounds on derivatives)

- Has been applied successfully to time-delay systems [Gouaisbaut]:

  gives sequences of LMI conditions with decreasing conservatism

▲ Related to SOS representations of positive polynomials [Sato 2009]:

  conservatism decreases as the order of the representation is augmented

- No need to manipulate by hand LMIs (Schur complements etc.), polynomials...

▲ Does conservatism vanishes? Exactly? Asymptotically?

▲ Is it possible to cope with non-linearities in the same way?

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❺ The Romuald toolbox

■ Freely distributed software to test the theoretical results

● Existing software : RoMulOC

`www.laas.fr/OLOCEP/romuloc`

▲ Contains some of the analysis results plus some state-feedback features

● Currently developed software : Romuald

▲ Dedicated to analysis of descriptor systems

▲ Fully coded using the quadratic separation theory

▲ Allows systematic system augmentation

▲ First preliminary tests currently done for satellite and plane applications

```
>> quiz = ctrpb( OrderOfAugmentation ) + i2n (usys);
>> result = solvesdp( quiz )
```

Conclusions