Nonlinear output regulation theory based on center-stable manifold computation

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In collaboration with Branislav Rehak

June 26, 2012

Outline of the talk

- Introduction of $\begin{cases} \text{myself} \\ \text{stable manifold package} \end{cases}$
- Invariant manifolds (Stable, center-stable manifolds)
- Output regulation (servo system design)
  - Review (regulator equation)
  - CS-manifold algorithm computes optimal controller
  - No need to solve the regulator equation
- Application
  - Optimal servo system design under input norm constraint

Introduction of myself

- *Undergraduate:* Hokkaido University (mathematics)
- *PhD:* Nagoya University (aerospace, control theory)
- *Research fields:* Nonlinear control theory and its applications
- *Recent research topics:*
  - ✓ Hamilton-Jacobi equations for infinite horizon optimal control
  - ✓ Started as a collaboration with A.J. van der Schaft of Groningen (2006∼)
  - ✓ Matlab code ”stable manifold package”

Optimal control theory

$$(\Sigma_a) \quad \begin{cases} \dfrac{dx}{dt} = f(x(t)) + g(x(t))u(t) \\ y = h(x(t)) \end{cases}$$

$$J = \int_0^\infty y(t)^T y(t) + u(t)^T R u(t)\, dt$$

Optimal control theory

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$$J = \int_0^\infty y(t)^T y(t) + u(t)^T R u(t)\, dt$$

Fundamental theorem in optimal control (Bellman):
If there is a solution $V(x)$ with $V(0) = 0$ satisfying I), II), then the optimal controller is

$$u = -\tfrac{1}{2} R^{-1}(x) g(x)^T \frac{\partial V}{\partial x}^T$$

I) $\frac{\partial V}{\partial x} f(x) - \frac{1}{4}\frac{\partial V}{\partial x} g(x) R^{-1} g(x)^T \frac{\partial V}{\partial x}^T + h(x)^T h(x) = 0$ (HJ eq.)

II) $f(x) - \frac{1}{2} g(x) R^{-1} g(x)^T \frac{\partial V}{\partial x}^T$ is an asymptotically stable vector field at $x = 0$.

Hamilton-Jacobi equation

An HJ eq in optimal control:

$$\frac{\partial V}{\partial x}f(x) - \frac{1}{4}\frac{\partial V}{\partial x}g(x)R^{-1}g(x)^T\frac{\partial V}{\partial x}^T + h(x)^T h(x) = 0$$

In general,

$$\text{(HJ)} \qquad H(x,p) = p^T f(x) - \frac{1}{2}p^T R(x)p + q(x) = 0$$

$$\begin{cases} x_1, \cdots, x_n\text{: independent variables} \\ \qquad \cdots \text{ state space } X \\ p_j = \partial V/\partial x_j, \quad j = 1, \cdots, n \\ V\text{: unknown function} \end{cases}$$

Hamilton-Jacobi equation

An HJ eq in optimal control:

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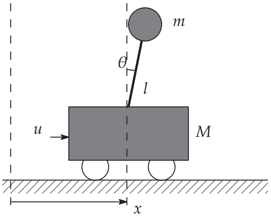
Find a solution $V(x)$ such that $f(x) - R(x)p(x)$ is asymptotically stable.

Stable manifold package—pendulum swing up

$$\begin{cases} (M+m)\ddot{x} + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + b\dot{x} = u \\ ml^2\ddot{\theta} + ml\ddot{x}\cos\theta - mgl\sin\theta = 0 \end{cases}$$

$$\dot{x} = f(x) + g(x)u \text{(state equations)}$$

$$f(x) = \begin{bmatrix} x_2 \\ \frac{mlx_4^2\sin x_3 - bx_2 - mg\cos x_3\sin x_3}{M+m\sin^2 x_3} \\ x_4 \\ \frac{(M+m)g\sin x_3 + bx_2\cos x_3 - mlx_4^2\sin x_3\cos x_3}{l(M+m\sin^2 x_3)} \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ \frac{1}{M+m\sin^2 x_3} \\ 0 \\ \frac{-\cos x_3}{l(M+m\sin^2 x_3)} \end{bmatrix}$$



Stable manifold package—pendulum swing up

$$\dot{x} = f(x) + g(x)u$$

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$$J = \int_0^\infty x^T Q x + u^T R u \, dt$$

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 40 \end{pmatrix}, R = 1.$$

Stable manifold package—pendulum swing up

However,

Input voltage limit: $|\mathrm{sat}(u)| \leqslant 18[\mathrm{V}]$

$$\dot{x} = f(x) + g(x)\mathrm{sat}(u)$$

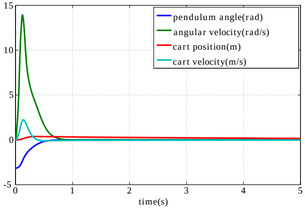
We solve the HJ eq. including saturation

$$\left(\frac{\partial V}{\partial x}\right)\left\{f(x) + g(x)\cdot \mathrm{sat}\left(\hat{u}\left(x, \left(\frac{\partial V}{\partial x}\right)\right)\right)\right\} + x^T Q x + \mathrm{sat}\left(\hat{u}\left(x, \left(\frac{\partial V}{\partial x}\right)\right)\right)^T \cdot R \cdot \mathrm{sat}\left(\hat{u}\left(x, \left(\frac{\partial V}{\partial x}\right)\right)\right) = 0,$$

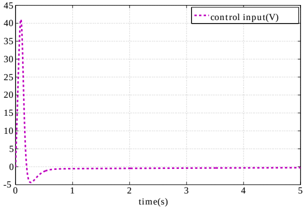
where $\hat{u}_i(x,p) = -\frac{1}{2}R^{-1}g_i(x)^T p$.

Stable manifold package—pendulum swing up

Optimal control without saturation



Responses(simulation)

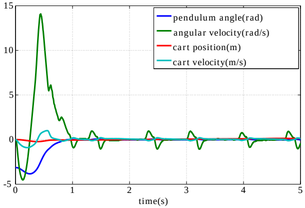


Input(simulation)

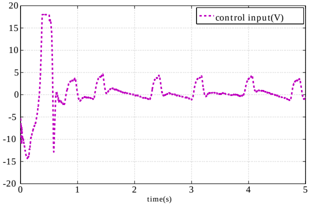
Swing up with 1 swing
Did not work in experiment...

Stable manifold package—pendulum swing up

Optimal control including saturation



Responses(experiment)

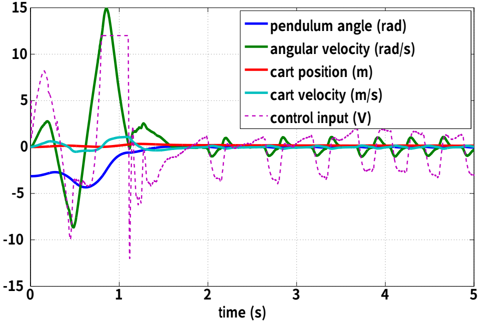


Input(experiment)

Input voltage satisfies the limitation $\leqslant 18$ [V]
Swing up with 2 swings

Stable manifold package—pendulum swing up

Optimal control including saturation $\leqslant$ 12[V]



Swing up with 3 swings

Stable manifold package—pendulum swing up

Optimal swing up with two swings

Stable manifold package—pendulum swing up

Optimal swing up with three swings

Invariant manifolds

Consider the following system of ordinary differential equations:

$$\dot{x} = Ax + X(x, y, z)$$
$$\dot{y} = By + Y(x, y, z)$$
$$\dot{z} = Cz + Z(x, y, z)$$

- $A \in R^{n_x \times n_x}$, $\mathrm{Re}\lambda(A) < 0$ (x: stable part)
- $B \in R^{n_y \times n_y}$, $\mathrm{Re}\lambda(B) = 0$ (y: center part)
- $C \in R^{n_z \times n_z}$, $\mathrm{Re}\lambda(C) > 0$ (z: unstable part)
- The functions X, Y, Z are continuously differentiable
- X, Y, Z together with all of their first derivatives vanish at the origin.

Invariant manifolds

$$\begin{aligned}\dot{x} &= Ax + X(x,y,z) & &(\textit{stable})\\ \dot{y} &= By + Y(x,y,z) & &(\textit{center})\\ \dot{z} &= Cz + Z(x,y,z) & &(\textit{unstable})\end{aligned}$$

Kelly (1967) shows that there exist invariant manifolds

Stable manifold	$y = v^{+}(x),\ z = w^{+}(x)$
Center manifold	$x = u^{*}(y),\ z = w^{*}(y)$
Unstable manifold	$x = u^{-}(z),\ y = v^{-}(z)$
Center stable manifold	$z = w^{*+}(x,y)$
Center unstable manifold	$x = u^{*-}(y,z)$

Roughly speaking, on these manifolds, one variable synchronizes the other

Invariant manifolds and control system design

- Solutions of a pde $=$ an integrable invariant manifold of an associated Hamiltonian system
- In control system, stability needs to be considered

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    Output regulation (servo system design)
    Hamilton-Jacobi equation with center part

Invariant manifolds and control system design

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  - Stable manifold:
    Optimal stabilization, Hamilton-Jacobi equation
  - Center-stable manifold:
    Output regulation (servo system design)
    Hamilton-Jacobi equation with center part
  - Approximate computation of the invariant manifolds
    Iterative algorithms
    Suitable for computer calculation

Iterative algorithms

- Stable manifold algorithm:

$$x_1(t) = e^{At}x_0, y_1(t) = 0, z_1(t) = 0$$

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{pmatrix}(t, x_0) = \begin{pmatrix} e^{At}x_0 + \int_0^t e^{A(t-s)} X(x_k(s), y_k(s), z_k(s))\, ds \\ -\int_t^\infty e^{B(t-s)} Y(x_k(s), y_k(s), z_k(s))\, ds \\ -\int_t^\infty e^{C(t-s)} Z(x_k(s), y_k(s), z_k(s))\, ds \end{pmatrix}$$

- Center-stable manifold algorithm:

$$x_1(t) = e^{At}x_0, y_1(t) = e^{Bt}y_0, z_1(t) = 0$$

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{pmatrix}(t, x_0, y_0) = \begin{pmatrix} e^{At}x_0 + \int_0^t e^{A(t-s)} X(x_k(s), y_k(s), z_k(s))\, ds \\ e^{Bt}y_0 + \int_0^t e^{B(t-s)} Y(x_k(s), y_k(s), z_k(s))\, ds \\ -\int_t^\infty e^{C(t-s)} Z(x_k(s), y_k(s), z_k(s))\, ds \end{pmatrix}$$

Existence of limit

Theorem

i) There exists $\delta_s > 0$ such that for all $x_0,\ |x_0| < \delta_s$ the sequence converges point-wise to the solutions on $y = v^{+}(x)$, $z = w^{+}(x)$ (convergence to the stable manifold).

ii) There exists $\delta_{cs} > 0$ such that for all $(x_0, y_0),\ |(x_0, y_0)| < \delta_{cs}$ the sequence converges point-wise to the solutions on $z = w^{*+}(x, y)$ (convergence to the center-stable manifold).

Parametrization and expression of the manifolds

i) Parameterized stable manifold:

$$M_S = \{(x_k(t,x_0), y_k(t,x_0), z(t,x_0)) \mid |x_0| : \text{ sufficiently small, } t \geqslant 0\}$$

Polynomial expression:

$$y = v^+(x),\ z = w^+(x)$$

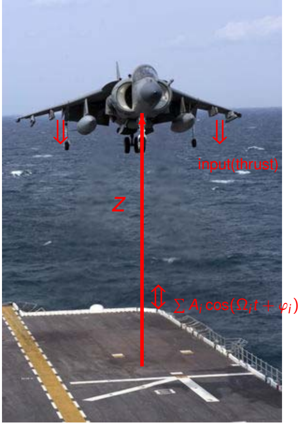
ii) Parameterized center-stable manifold:

$$M_{CS} = \{(x_k(t,x_0,y_0), y_k(t,x_0,y_0), z(t,x_0,y_0)) \mid |x_0|, |y_0| : \text{ sufficiently small, } t \geqslant 0\}$$

Polynomial expression:

$$z = w^{*+}(x,y)$$

Output regulation



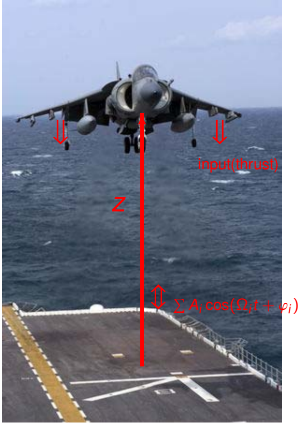
Achieve

$z(t)$ (controlled output) $\sim \sum A_i \cos(\Omega_i t + \varphi_i)$

for sufficiently large t.

Fundamental technique in robotics, factory automation etc

Output regulation



$$\dot{x} = f(x, u)$$
$$e = h(x, u, w) \qquad \cdots \ (z - w)$$
$$\dot{w} = s(w) \qquad \cdots \ (\cos \Omega_i t)$$

Design a control u such that after some time,

$$e(t) = 0.$$

$\Longrightarrow$

Invariant manifolds $x = \pi(w)$, $u = c(w)$

Output regulation

$$\begin{aligned} \dot{x} &= f(x,u) \\ e &= h(x,u,w) \qquad \cdots\ (z-w) \\ \dot{w} &= s(w) \qquad \cdots\ (\cos \Omega_i t) \end{aligned}$$

Design a control u such that after some time,

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$\Longrightarrow$

Invariant manifolds $x = \pi(w)$, $u = c(w)$

Invariance condition
(regulator equation)
$$: \begin{cases} \dfrac{\partial \pi}{\partial w} s(w) = f(\pi(w), c(w)) \\ 0 = h(\pi(w), c(w), w) \end{cases}$$

Optimal output regulation - equations

System: $\dot{x} = f(x) + g(x)u, \quad x(t) \in \mathbb{R}^n, \; f(0) = 0$

Exosystem: $\dot{w} = s(w), \quad w(t) \in \mathbb{R}^p, \; s(0) = 0$

Error (output) equation: $e = h(x, w)$

Regulator equation:

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w)) + g(\pi(w))\sigma(w), \quad h(\pi(w), w) = 0$$

(Nececary and sufficient cond. Isidori & Byrnes '90 *IEEE TAC*)

Denote

$$A = \frac{\partial f}{\partial x}(0), \quad B = g(0), \quad C = \frac{\partial h}{\partial x}(0,0),$$

$$S = \frac{\partial s}{\partial w}(0), \quad Q = \frac{\partial h}{\partial w}(0,0).$$

Optimal output regulation - assumptions

- The exosystem is Lyapunov stable at $w = 0$, Poisson stable around $w = 0$ and all eigenvalues of S are purely imaginary.
- $(A,\ B)$ is stabilizable, (C, A) is detectable
- The number of inputs $\geqslant$ the number of outputs (square)
- The system has well-defined relative degree (rel.deg. =1, $\det L_g h(0,0) \neq 0$)

Optimal output regulation

$$\text{Cost function:} \quad J = \frac{1}{2}\int_0^\infty |e|^2 + |\dot{e}|^2 \, dt$$

The Hamiltonian H_D:

$$H_D = p_x^T(f + gu) + p_w^T s(w) + \frac{1}{2}\left|h(x,w)\right|^2 + \frac{1}{2}\left|L_f h(x,w) + (L_g h(x,w))u + L_s h(x,w)\right|^2,$$

The control vector $\bar{u}$ minimizing H_D:

$$\bar{u} = -(L_g h)^{-1}\left\{(L_g h)^{-T} g(x)^T p_x + L_f h + L_s h\right\}.$$

The *Hamilton-Jacobi equation* is then

$$p_x^T\left\{f - g(L_g h)^{-1}(L_f h + L_s h)\right\} + p_w^T s(w) - \frac{1}{2}p_x^T g(L_g h)^{-1}(L_g h)^{-T} g^T p_x + \frac{1}{2}|h(x,w)|^2 = 0$$

Associated Hamiltonian System

The Hamiltonian system is

$$
\begin{aligned}
\dot{x} =& (A - B(CB)^{-1}CA)x - B(CB)^{-1}QSw \\
& -B(B^TC^TCB)^{-1}B^Tp_x + N_1(x, w, p_x) \\
\dot{w} =& Sw + N_2(w) \\
\dot{p}_x =& C^TCx - C^TQw \\
& -(A - B(CB)^{-1}CA)^Tp_x + N_3(x, w, p_x) \\
\dot{p}_w =& -Q^TCx - Q^TQw + S^TQ^T(B^TC^T)^{-1}B^Tp_x \\
& -S^Tp_w + N_4(x, w, p_x, p_w).
\end{aligned}
$$

Define the Hamiltonian matrix H as

$$
\frac{d}{dt}[x, w, p_x, p_w]^T = H\,[x, w, p_x, p_w]^T + [N_1, N_2, N_3, N_4]^T
$$

Block diagonalization

The linear regulator equation

$$\Pi S = A\Pi + B\Sigma, \quad C\Pi + Q = 0$$

A Riccati equation

$$P\bar{A} + \bar{A}^T P - PR_B P + C^T C = 0;$$
$$\bar{A} = A - B(CB)^{-1}CA, \quad R_B = B(B^T C^T CB)^{-1}B^T$$

Lyapunov equation

$$VA_c + A_c^T V = B(B^T C^T CB)^{-1}B^T$$
$$\text{where} \quad A_c = A - B(CB)^{-1}CA - B(B^T C^T CB)^{-1}B^T P$$

Linear symplectic coordinate transformations

$$T_1 = \begin{pmatrix} I & \Pi & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & -\Pi^T & I \end{pmatrix}, \quad T_2 = \begin{pmatrix} I & 0 & V & 0 \\ 0 & I & 0 & 0 \\ P & 0 & PV + I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

New Hamiltonian system

$$[x''^T, w''^T, p_x''^T, p_w''^T]^T = T_2^{-1} T_1^{-1} [x^T, w^T, p_x{}^T, p_w{}^T]^T$$

Hamiltonian system with block-diagonalized linear part:

$$\begin{cases} \dot{x}'' = A_c x'' + \bar{N}_1(x'', w'', p_x'') & \textit{stable} \\ \dot{w}'' = S w'' + \bar{N}_2(w'') & \textit{center1} \\ \dot{p}_x'' = -A_c^T p_x'' + \bar{N}_3(x'', w'', p_x'') & \textit{unstable} \\ \dot{p}_w'' = -S^T p_w'' + \bar{N}_4(x'', w'', p_x'', p_w'') & \textit{center2} \end{cases}$$

✓ Center manifolds: $x'' = \bar{\pi}_1(w')$, $p_x'' = \bar{\pi}_2(w')$
represents synchronization of states ⇒ Regulator equation

✓ Center-stable manifold $p_x'' = \pi_3(x'', w'')$
represents transient behavior until synchronization
⇒ Feedback controller $u = u(w, x)$

A design example

Optimal OR with input norm constraint

$$\begin{cases} \dot{x} = Ax + Bu \\ e = Cx + Qw \\ \dot{w} = Sw = 0 \end{cases} \qquad u^T u \leqslant \delta$$

(input norm constraint)

$$J = \int_0^\infty e^T e + \dot{e}^T \dot{e} \, dt \quad \text{(cost function)}$$

Minimize

$$H_D = p_x^T (Ax + Bu) + |Cx + Qw|^2 + |C(Ax + Bu)|^2$$

w.r.t. u under $u^T u \leqslant \delta$

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w.r.t. u under $u^T u \leqslant \delta$

Lagrange multiplier method yields the minimizing vector as

$$\bar{u} = \bar{u}(x, w, p_x, \lambda(x, w, p_x)) \qquad \text{(nonlinear)}$$

A design example

The HJ eq:

$$p_x^T(Ax + B\bar{u}) + |Cx + Qw|^2 + |C(Ax + B\bar{u})|^2 = 0$$

The associated Hamiltonian system:

$$\begin{cases} \dot{x} = \bar{A}x - R_B p_x + N_1 \\ \dot{w} = 0 & \longleftarrow \text{ center part} \\ \dot{p}_x = -C^T C - C^T Q w - \bar{A}^T p_x + N_3 \\ \dot{p}_w = -Q^T C x - Q^T Q w + N_4 & \longleftarrow \text{ center part} \end{cases}$$

✓ The center-stable manifold is the family of flows that converge to the final states $(x^f, p_x{}^f)$ corresponding to each constant w

✓ The representation $p_x = p_x(x, w)$ on CS manifold gives the desired controller
$u^*(x, w) = \bar{u}(x, w, p_x(x, w), \lambda(x, w, p_x(x, w)))$

Numerical example

Motivation: torque control of a synchronous motor
under inverter voltage limitation

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \end{cases}$$

with input norm constraint: $|u|^2 - 1 \leq 0$.

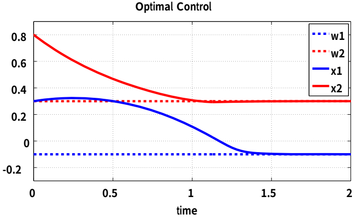
Constant reference signals: $\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Torque control is equivalent to:

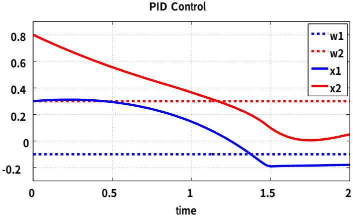
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \longrightarrow \begin{bmatrix} w_1(0) \\ w_2(0) \end{bmatrix} \text{ (current control)}$$

Numerical example—Similation results

Tracking performance



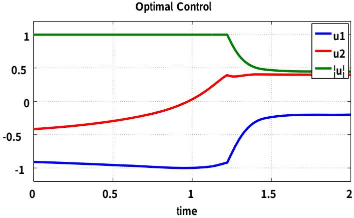
Optimal OR



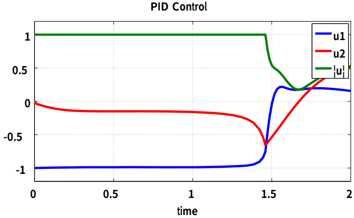
PID control

Numerical example—Simulation results

Input behavior



Optimal OR



PID control

The inputs of the PID controller, when its norm exceeds δ, are held until the norm is under δ

Concluding remarks

- Output regulation (servo system design)
- The role of invariant manifolds in control system design

$$\begin{cases} \text{Stable manifold:} & \text{optimal control} \\ \text{Center-stable manifold:} & \text{output regulation} \end{cases}$$

- Iterative computation methods for center-stable manifolds
- No need to solve the regulator equation, center-stable manifold algorithm computes the controller
- Application for AC motor torque control under voltage limitation

Matlab codes are available upon request