Anomalous Measurements in the Satellite Navigation. Detection and Isolation

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Approaches to Anomalies Detection

Navigation equations

$$P_{f,s,r}(t) = \rho_{s,r}(t) + c\tau_s(t) + c\tau_r(t) + T_{s,r}(t) + \left(\lambda_f / \lambda_1\right)^2 I_{1,s,r}(t) + d_{f,s,r} + \xi_{f,s,r}(t)$$

$$\Phi_{f,s,r}(t) = \rho_{s,r}(t) + c\tau_s(t) + c\tau_r(t) + \lambda_f N_{f,s,r}(t) + T_{s,r}(t) - \left(\lambda_f / \lambda_1\right)^2 I_{1,s,r}(t) + d_{f,s,r} + \varepsilon_{f,s,r}(t)$$

Error budget

- White (thermal) noise
- Ephemeris errors; vanish when constructing across-receiver differences
- Ionospheric delays, inverse proportional to frequency; vanish when constructing across-receiver differences
- Tropospheric delays; can be modeled and partially compensated
- Multipath

Approaches to Anomalies Detection

Errors that do not fit into classification are ‘anomalies’:

- Pseudorange outliers
- Cycle slips in carrier phase

They badly affect positioning quality if are not isolated.

Approaches to detection and isolation

- Triple difference approach applied to each frequency and each satellite independently
- Multiple frequencies: geometry-free combinations
- Multiple frequencies: wide-lane combinations
- Using signal redundancy

Using Signal Redundancy

Cycle slips and pseudorange outliers detection explores the same linear model

$$Jx + \delta + \xi = b$$

where J is design matrix, $x \in R^4$

δ is a vector of anomalies (0 if no anomalies)

ξ is noise, $C = E\left(\varepsilon\varepsilon^T\right) = \sigma^2 I_n$

b pseudorange residuals or carrier phase between epoch increments,

$NDF = n - 4 > 0$ is redundancy.

Right hand side form n - dimensional space, while x spans 4 - dimensional subspace if no anomalies happened => χ^2 test can be applied to check 0-hypothesis (absence of outliers).

Challenges

- Unpredictable kinematics
- Single frequency (L1 only) measurements for each satellite
- Low redundancy and nested (multiple) cycle slips

Assumptions:

The vector of anomalies δ is assumed to have small number of non-zero entries i.e. small number of signals is affected:

$$\delta^T = [0 \quad 0 \quad 0 \quad \times \quad 0 \quad 0 \quad 0 \quad \times]$$

The symbol $\times$ stays for non-zero value.

Compressive Sensing Theory. Basic Facts (1/2)

See http://dsp.rice.edu/cs for Compressive Sensing Resources

Let $\delta \in R^n$ be the signal, $c \in R^m$ be observables, $m < n$ => underdetermined system.

$$F\delta = c + \varepsilon$$

$$\begin{bmatrix} \times & \times & \times & \times & \times & \vdots & \times & \times \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \times & \times & \times & \times & \times & \vdots & \times & \times \end{bmatrix} \begin{bmatrix} \times \\ \times \\ \times \\ \times \\ \vdots \\ \vdots \\ \times \\ \times \end{bmatrix} = \begin{bmatrix} \times \\ \vdots \\ \times \end{bmatrix}$$

We need to recover a signal from significantly underdetermined observables. Additional information needed is that δ is sparse: zero entries dominate.

$$\delta^T = [0 \quad 0 \quad 0 \quad \times \quad 0 \quad 0 \quad 0 \quad \times]$$

Sparsity allows us to recover uniquely the high-dimensional vector from low-dimensional data, given in advance that at most $k < m$ are non-zero. Straightforward approach consists in using exhaustive search, inspecting all subsets $I \subseteq \{1, \cdots, n\}$, $\mathrm{Card}(I) \le k$.

$\mathrm{supp}(\delta) = \{i : \delta_i \ne 0\}$ - a 'support set', $\|\delta\|_0 = \mathrm{Card}(\mathrm{supp}(\delta))$

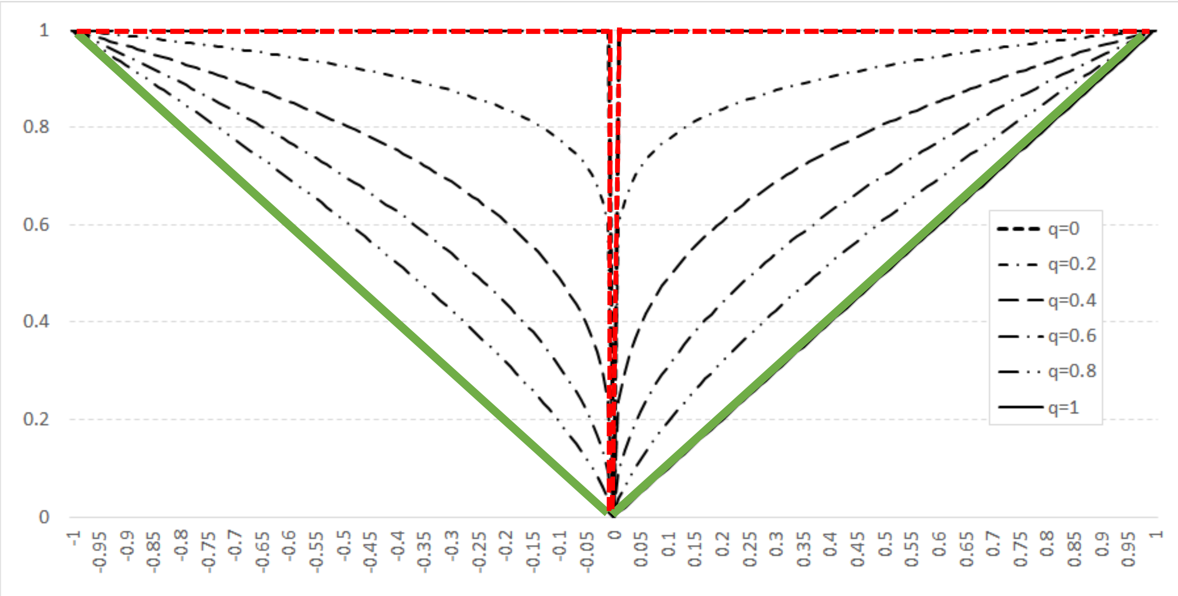
=> l_0-minimization problem, minimizing cardinality of the support set

$$\min_{\delta} \left\{ \|\delta\|_0 : F\delta = c \right\}$$

Compressive Sensing Theory. Basic Facts (2/2)

Note that $\|\delta\|_0 = \sum_{i=1}^{n} \xi(\delta_i)$ with $\xi(\delta_i)$ being an indicator function $\xi(\delta_i) = \begin{cases} 0 \text{ if } \delta_i = 0 \\ 1 \text{ if } \delta_i \neq 0 \end{cases}$

$$\xi(\delta_i) = \lim_{q \to 0} |\delta_i|^q$$



Consider $\xi_q(\delta_i) = |\delta_i|^q$.

Among them only $q = 1$ defines a convex function $\xi_1(\delta_i) = |\delta_i|$. Substitution of $\xi(\delta_i)$ by its best convex approximation gives the l_1 -minimization problem

$$\min_{\delta} \left\{ \|\delta\|_1 = \sum_{i=1}^{n} |\delta_i| : \; F\delta = c \right\}$$

which is Linear Programming (LP).

Simplex Method and Interior Point Method are efficient tools, good for real time implementation, provided dimension is not too large, not exceeding several tens.

Considering noise leads to Quadratic Programming $\min_{\delta} \left\{ \|\delta\|_1 : \; (F\delta - c)^T C^{-1} (F\delta - c) \leq \sigma^2 \right\}$

Linear Programming Decoding and Error-Correction (1/3)

For the over-determined system $Jx+\delta+\xi=b$ consider the problem of decoding of the 'message' x from the code ('ciphertext') Jx corrupted by error consisting of two parts:

- small zero centered noise ξ with known covariance matrix
- sparse vector of outliers δ consisting mostly of zeros, though non-zero entries can be arbitrary large

$$\delta^T=\begin{bmatrix}0 & 0 & \times & 0 & \cdots & 0 & 0 & \times\end{bmatrix}$$

Outliers affect only few symbols of the code Jx leaving a hope for correct decoding if the code length is greater than the message length $J\in R^{n\times m},\ n \gg m$

The problem has connection to the error-correction, however there is a distinction between the finite-alphabet setting and the real-valued setting.

Given the coding matrix J and the ciphertext $Jx+\delta$ can we recover the sparse corruption vector δ and, therefore, correctly decode the message x ?

Linear Programming Decoding (2/3)

With natural assumption that the number of corrupted symbols is not too large

$$|\delta|_0 \le \rho m \qquad \rho \ll 1$$

reduce the problem to the sparse signal recovery problem

First, construct the matrix $F \in R^{(n-m)\times n}$ annulating the matrix $J \in R^{n\times m}$: $FJ = 0$. This step is accomplished calculating the $QR = J$ decomposition.

Then, apply F to both sides of $Jx + \delta = b$ resulting in $F\delta = Fb \equiv c$

The LP problem

$$\min_{\delta}\left\{\|\delta\|_1 :\ F\delta = c\right\}$$

and QP problem

$$\min_{\delta}\left\{\|\delta\|_1 :\ \left(F\delta - c\right)^T C^{-1}\left(F\delta - c\right) \le \sigma^2\right\}$$

can be used for sparse outliers recovery for noiseless and noisy cases respectively

Linear Programming Decoding (3/3)

[1] Candez, E., Tao, T. (2005a) Decoding by Linear Programming *IEEE Transactions on Information Theory*, **57**(12): 4203-4215

[2] Candez, E., Rudelson, M., Tao, T. (2005b) Error Correction via Linear Programming *Proceedings of the 46th Annual IEEE Symposium on Foundations of Computer Science, 2005. FOCS 2005*, 668-681

[3] Cai, T., Wang L. (2011). Orthogonal Matching Pursuit for Sparse Signal Recovery With Noise. *IEEE Trans. Information Theory*, **57**(7):4680-4688

Orthogonal Matching Pursuit (OMP) (1/2)

Orthogonal Matching Pursuit for sparse vector recovery, a 'greedy' algorithm, neither exhaustive search, nor restricted exhaustive search are involved. Just picks up indices one by one.

Step 0. Initialize the algorithm calculating the residual and the support set

$$r^{(0)} = c, \quad S^{(0)} = \varnothing, \quad k = 0$$

Step 1. Check if the termination criterion using χ^2 criterion with confidence level α

$$\frac{1}{\sigma^2} \left| r^{(k)} \right|_2^2 < T(\alpha, m - k - n)$$

Step 2. Find the index $j^{(k)}$ (find the column of F most correlated with residuals $r^{(k-1)}$)

$$\max_{j=1,\cdots,n} \frac{\left| f_j^T r^{(k-1)} \right|}{\sqrt{f_j^T f_j}}$$

with f_j being a column of the matrix F.

Orthogonal Matching Pursuit (OMP) (2/2)

Step 3. Update the support set

$$S^{(k)} = S^{(k-1)} \cup \{ j^{(k)} \}$$

Step 4. Solve the least squares problem for the matrix $F(S^{(k)})$ consisting

of columns $f_j, \ j \in S^{(k)}$

$$\delta^{(k)} = \left(F^T(S^{(k)}) F(S^{(k)}) \right)^{-1} F^T(S^{(k)}) c, \qquad r^{(k)} = c - F(S^{(k)}) \delta^{(k)}$$

Step 5. go to step 1).

Usually, reconstruction results obtained by the OMP algorithm are less precise than results, obtained by the LP or convex programming algorithms, but for sure OMP is faster than LP

CoSaMP, StOMP, BaOMP

The vast amount of literature is devoted to enhancement of OMP.

Its ‘greedy’ nature obeys high computational efficiency.
Variations of OMP introduce ‘back-stepping’ allowing to release indices wrongly included into $S^{(k)}$. To accomplish this step, multiple indices are included at the Step 3 of OMP and some indices are excluded after solving for

$$\delta^{(k)} = \left(F^T(S^{(k)})F(S^{(k)})\right)^{-1} F^T(S^{(k)})c,$$

at the Step 4.

Experimental Results and Comparison with Exhaustive Search (1/5)

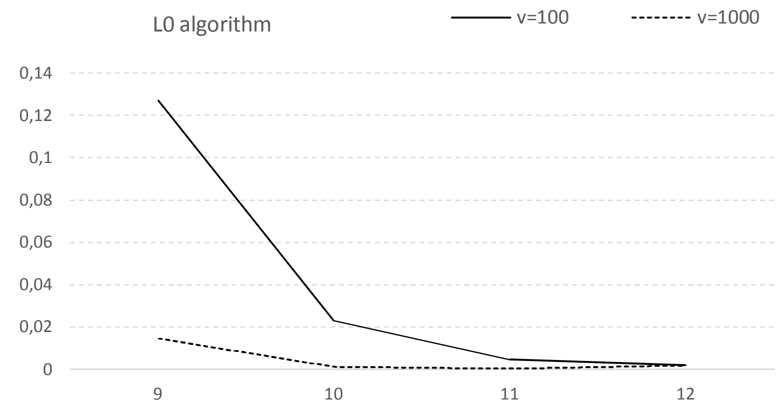
- The probability of wrong detection

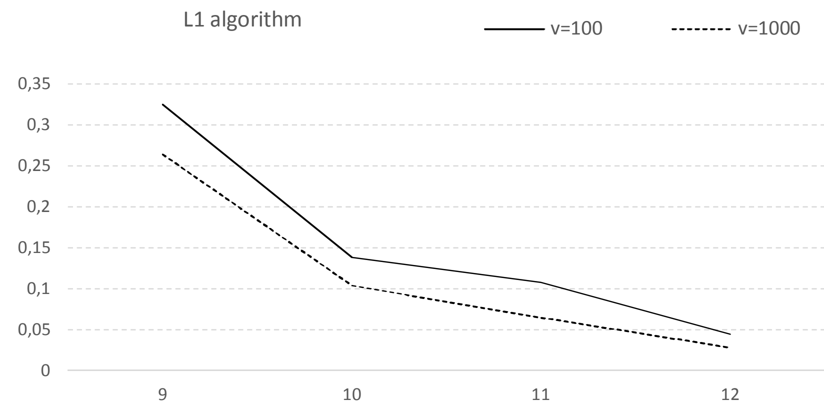
$$p_{err} = \frac{\text{number of not detected cycle slips}}{\text{total number of generated cycle slips}}$$

- Two data sets collected at different conditions, approximately one hour long were used for experiment.
- For the first data set the antenna connected to Topcon Net-G3 receiver was placed near the building wall and thus, half of the sky was blocked. The number of GPS and GLONASS satellites observed at different epochs varied from 8 to 12. For the second data set the Triumph-1 receiver from Javad GNSS was placed at the open sky place for another one hour. From 13 to 15 GPS and GLONASS satellites were observed at different epochs.
- All satellites provided dual frequency measurements. The raw data was affected by noticeable multipath for the first data set. The data was relatively clear for the second data set. Anomalous measurements were artificially generated.

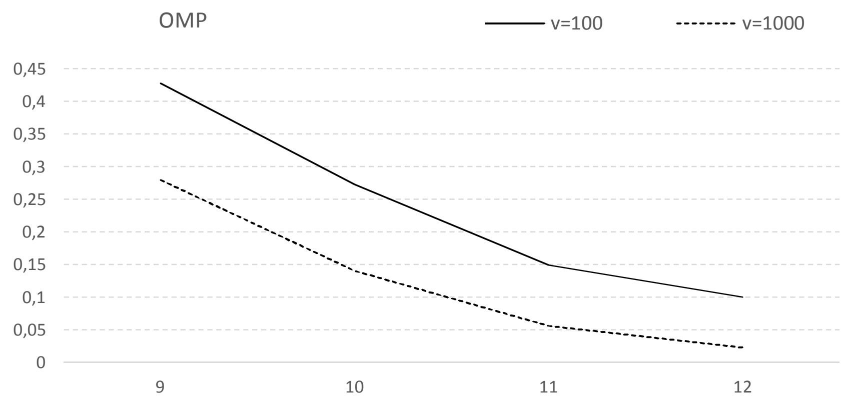
Experimental Results and Comparison with Exhaustive Search (2/5)

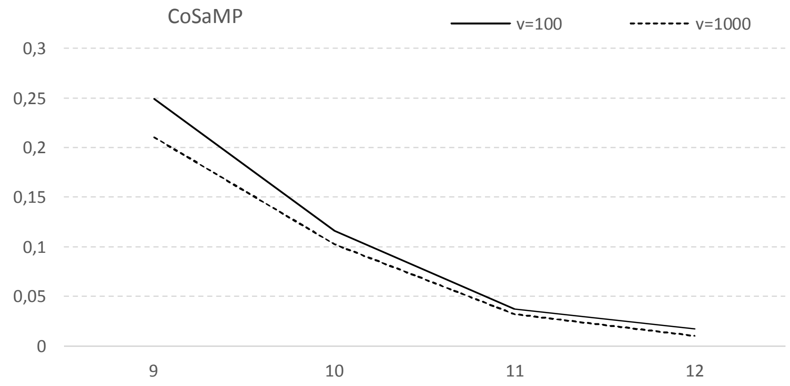
- At each epoch a set of randomly generated outliers were added to the pseudorange measurements. The number of artificially generated outliers k varied from 1 to 4. Indices were generated without repetition. Magnitude of the outlier was randomly chosen from the range $[-v_{\max}, v_{\max}]$. Two cases, $v_{\max} = 100$ and $v_{\max} = 1000$ were considered.
- Four algorithms, l_0 (exhaustive search), l_1 (linear programming), OMP, and CoSaMP were checked for sensitivity to the value of the outlier.





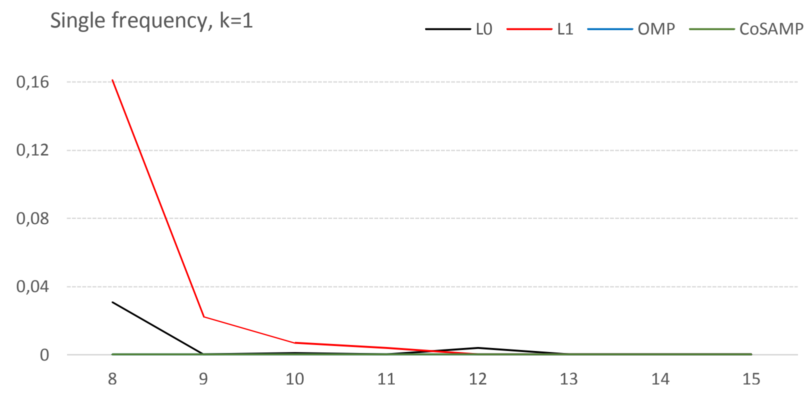
Experimental Results and Comparison with Exhaustive Search (3/5)

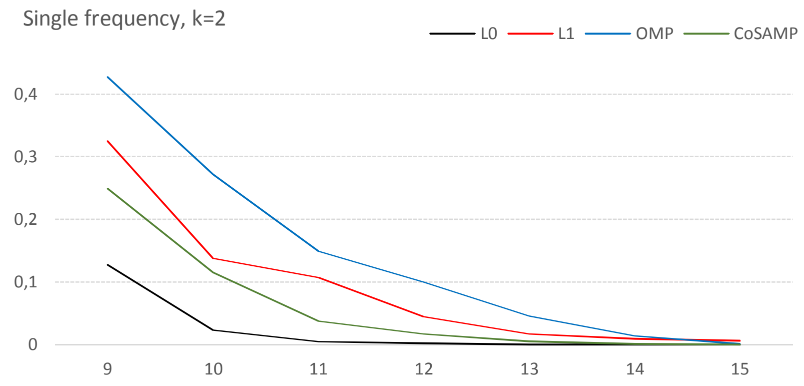


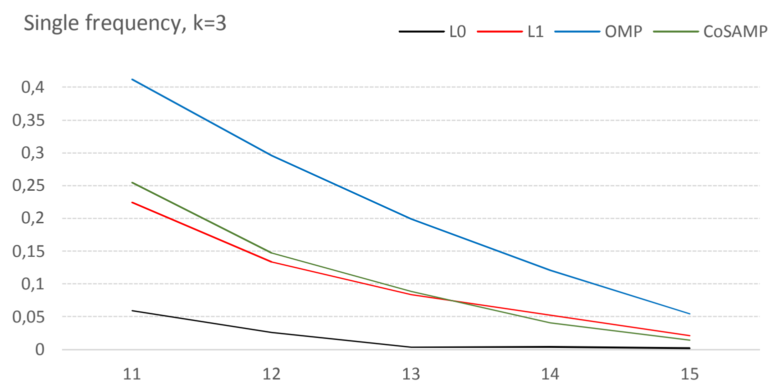


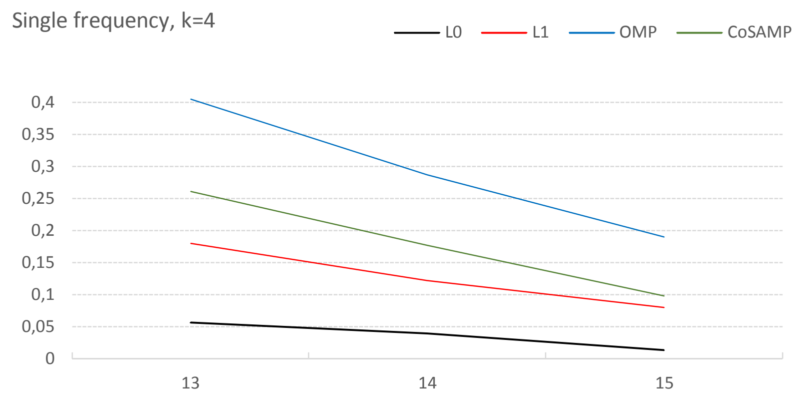
As it follows from plots, the l_0 optimization algorithm looks sensitive to the magnitude of the outlier. The larger the outlier is, the higher probability of its correct detection. Three other algorithms look less sensitive to the value of the outlier, while being less precise showing less performance. The CoSaMP algorithm shows superior behavior among three algorithms based on the compressive sensing approach.

Experimental Results and Comparison with Exhaustive Search (4/5). Single Frequency Case

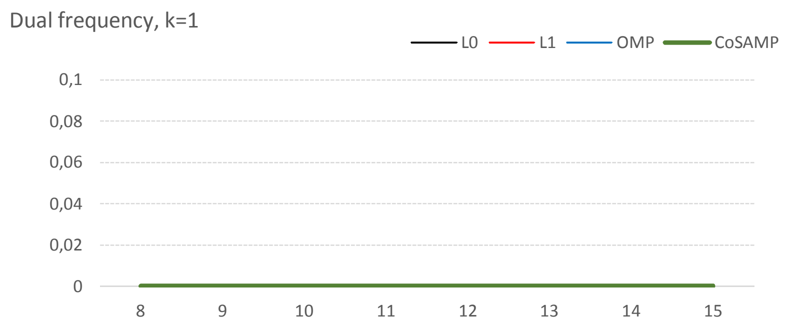


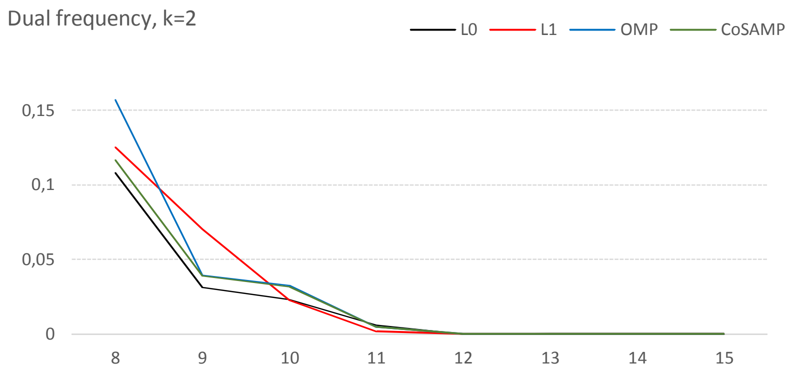


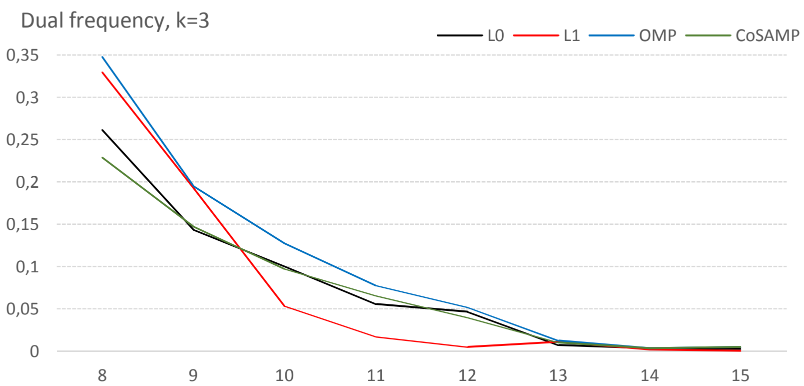


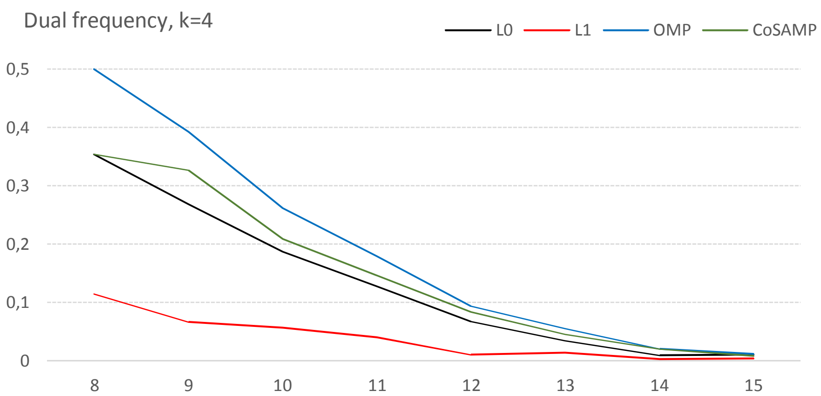


Experimental Results and Comparison with Exhaustive Search (5/5). Dual Frequency Case









Conclusion

1. The l_0 optimization (exhaustive search) algorithm is sensitive to the outlier magnitude. The greater the outlier, the easier it is detected. The algorithms based on the compressive sensing approach look not sensitive to the outlier magnitude.
2. The exhaustive search giving the best performance can be recommended for relatively small number of measurements. If the number of measurements is large (10 and more satellites, dual frequency) the CoSaMP algorithm can be recommended without significant degradation of the detection performance.
3. The linear programming algorithm having the polynomial complexity does not show significant benefits over the CoSaMP algorithm. Therefore, the linear programming algorithm can be considered as having rather theoretical value. On the other hand, for large number of measurements it is preferable to the l_0 optimization exhaustive search having the exponential complexity.
4. As the number of satellite systems grows, the CoSaMP algorithm becomes the reasonable choice comparing with the exhaustive search. It is true especially for the real time processing.
5. For the single frequency case and the large number of satellites the CoSaMP algorithm is preferable because other approaches based on the multiple frequencies data for each satellite are not applicable.