Fast Computation of Dilation Integrals for Robustness Analysis

F. Dabbene* and P. Shcherbakov**

***Abstract*— It is shown how recent results on combinations of sequences of quadrature formulae with high polynomial exactness represent an efficient technique for the fast computation of the so-called dilation integrals. These integrals were introduced in the modern robustness literature as a tool for addressing approximate robustness in problems involving polynomial parameter dependence. As a result, the calculations are simplified and more precise conclusions can be made about approximate robustness. Illustrative examples of the proposed approach are presented.**

I. Introduction: Dilation Integral Approach

Many classical robustness problems with uncertain parameters entering nonlinearly in the problem data reduce to the following formulation. Let $f(q)$ be a problem-related multivariate polynomial in the vector of uncertain parameters $q \in \mathbb{R}^d$ confined to the unit hypercube $\Omega_d \doteq [-1,\, 1]^d$. The problem is said to be robust if $f(q) < 0$ for all $q \in \Omega_d$, see [1] for motivating examples.

Checking the robustness requirement $f(q) < 0$ on Ω_d may be prohibitive for large dimensions d of the uncertainty vector, due to the nonlinear dependence on the uncertainty. Indeed, many control problems that can be recast in this framework have been proved to be NP-hard, see for instance [2] and references therein. For this reason in [1], a relaxed formulation of the problem was proposed in the spirit of recent works in the area of probabilistic robustness, e.g. see [3]. Namely, the pair (f, Ω_d) is said to be *approximately robust with level* $\varepsilon > 0$ if

$$\frac{\mathbf{Vol}(\Omega_d^{bad})}{\mathbf{Vol}(\Omega_d)} < \varepsilon, \tag{1}$$

where

$$\Omega_d^{bad} \doteq \{q \in \Omega_d \colon\ f(q) \geq 0\}$$

represents the *violation set*. Checking (1) is accomplished via computing the so-called *dilation integrals* defined as

$$\Phi_k(\alpha) \doteq \int_{\Omega_d} \Big(1 + \alpha f(q)\Big)^k \, \mathrm{d}q \tag{2}$$

for even integer values of k, where α is a real parameter. The fundamental inequality

$$\mathbf{Vol}(\Omega_d^{bad}) \leq \Phi_k(\alpha) \quad \text{for all } \alpha \geq 0 \quad \text{and even integers } k$$

established in [1], together with closed form computability of the integrals and the fact that $\Phi_k(\alpha)$ are convex in the scalar variable α, leads to a simple procedure for evaluating the relative volume of violation $\mathbf{Vol}(\Omega_d^{bad})/\mathbf{Vol}(\Omega_d)$. Namely, compute $\Phi_k = \min_{\alpha \geq 0} \Phi_k(\alpha)$ for even integer k and check if $\Phi_k \leq \varepsilon \mathbf{Vol}(\Omega_d)$. Moreover, in [1] it was shown that the problem is robust (more precisely, $\mathbf{Vol}(\Omega_d^{bad}) = 0$) if and only if the sequence Φ_k converges to zero as k grows. Conversely, if the problem is not robust, there exists a limiting level $\Phi^* = \min_k \Phi_k > 0$ and no certification is possible for arbitrarily small volume of violation.

In [1], [4], [5], [6], this approach was shown to be quite an efficient tool for addressing approximate feasibility of robustness problems in various settings. At the same time, since computing the dilation integrals above (although being exact and straightforward) is performed symbolically, e.g. using Symbolic Math Toolbox in MATLAB, it is often the case that for high dimensions of q and large number of monomials in f, computations may be extremely cumbersome for high values of k required to obtain low volume certification for Ω_d^{bad}, and the amount of intermediate data may exceed MATLAB memory limitations. In other words, it may happen that even if the true relative volume of violation is indeed less than the tolerable user-specified ε, this desired level may not be recovered using the dilation integral estimation technique.

To address this issue, the notion of the underlying *conditioner* θ of the problem was introduced in [1], with robust pairs (f, Ω_d) being associated with $\theta < 1$, while in the non-robust case the value of θ is greater than one. In [6], a simple procedure was devised that yields a sequence of lower estimates θ_k for the unknown conditioner that converges monotonically to the true value of θ in the robust case and to one otherwise. Based on this information, the problem is said to be *ill-conditioned* if the θ_k values are close to unity, which is an indicator of the fact that the problem is inherently hard. In that case, the conclusion of its *practical nonrobustness* is made (for more details, see [4]).

In this paper, we propose the use of combinations of low-order quadrature formulae (QF) for the *exact* computation of the dilation integrals without recourse to symbolic manipulations. These formulae, originally prosed by Smolyak in [7], and recently exploited in [8] in the context of convex optimization, allow for the exact computation of multidimensional integrals by evaluating the integrand at a finite number of points (sparse grids).

The key message of this note is simple to formulate: As applied to the dilation integral framework, such formulae largely reduce the computational burden and the memory requirements. As a result, dilation integrals can be efficiently

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computed exactly for much higher values of k and d, as compared to those reachable with symbolic computations. In practice, this leads to a more definite answer on whether the problem is deemed to be approximately robust or not.

II. Exact solutions of integrals

In the case of one dimensional integration ($d = 1$), a well-known numerical computation method for approximating the definite integral

$$I[g] = \int_{-1}^{1} g(x)\mathrm{d}x$$

is given by so-called quadrature formulae, see for instance [9]. An N-point QF with nodes x_k and weights w_k, $k = 1, \ldots, N$, is defined as

$$Q_N[g] \doteq \sum_{k=1}^{N} w_k g(x_k) \tag{3}$$

and we have

$$I[g] = Q_N[g] + R_N[g],$$

where $R_N[g]$ represents the error, or residual, of the QF. Notice that (3) corresponds to evaluating the function g on the N-point grid

$$\mathbb{X}_N \doteq \{x_1, \ldots, x_N\}.$$

The *degree of exactness* $\deg(Q_N)$ of a quadrature formula is defined as the maximum integer s such that the QF is exact for all polynomials of degree less than or equal to s, and there exists a polynomial p of degree $s+1$ such that $R_N[p] \neq 0$. In particular, if the nodes are chosen as the zeros of the N-th order Legendre orthogonal polynomial $P_N(x)$, and the weights are computed by integrating the associated Lagrange polynomials, then the QF (3) has the maximum achievable degree of exactness

$$\deg(Q_N) = 2N - 1,$$

and is called a *Gauss formula*.

The approach proposed in this paper stems from the simple consideration that, if $f(q)$ in (2) is a polynomial in $q \in \mathbb{R}$ of degree ν, then the integrand is also a polynomial of degree $k\nu$. Hence, the application of a quadrature formula with degree of exactness $\deg(Q_N) = k\nu$ can be applied to solve the integral (2) without resorting to symbolic computations.

It is evident that the approach described above can be extended in a straightforward way to the general multidimensional problem of computing (2) for $d \geq 1$. In fact, assume that $f(q)$ has degree ν_i in the variable q_i, $i = 1, \ldots, d$. Then, it can be easily seen that the tensor product

$$\begin{aligned}\left(Q_{N_1} \otimes \cdots \otimes Q_{N_d}\right)[g] \doteq \\ \sum_{k_1=1}^{N_1} \cdots \sum_{k_d=1}^{N_d} (w_{k_1} \cdots w_{k_d}) g(x_{k_1} \cdots x_{k_d})\end{aligned} \tag{4}$$

with $\deg(Q_{N_i}) \geq k\nu_i$ can be used for computing exactly the dilation integral. Formula (4) corresponds to evaluating the function on the multidimensional grid

$$\mathbb{H}_d \doteq \left(\mathbb{X}_{N_1} \times \cdots \times \mathbb{X}_{N_d}\right) \subset \Omega_d.$$

The cardinality of this grid, i.e. the number of nodes in this formula, is given by $|\mathbb{H}_d| = \prod_{i=1}^{d} N_i$. It is obvious that such a trivial extension can easily become intractable in the number of function evaluations[1] due to the exponential growth with respect to the dimension d.

To avoid this exponential growth, we propose here an approach based on a combination of low-order quadrature formulae. This method requires evaluating the function on a *sparse grid*, whose cardinality grows *polynomially* with respect to the dimension d. The approach was originally proposed in [7] and has been subsequently studied for instance in [10], [11], [12].

Let us first define particular sequences of quadrature formulae $Q^{(1)}, Q^{(2)}, \ldots$, with increasing index of precision i

$$Q^{(i)}[g] \doteq Q_{N_i}[g] = \sum_{k=1}^{N_i} w_k^{(i)} g(x_k^{(i)}), \quad i = 1, 2, \ldots$$

with nodes $\mathbb{X}^{(i)} \doteq \left\{x_1^{(i)}, \ldots, x_N^{(i)}\right\}$ and weights $w^{(i)} = [w_1^{(i)} \cdots w_N^{(i)}]^\top$. In order to apply our construction, these QF should satisfy the following properties:

(i) **Nested nodes.** The nodes of $Q^{(i)}$ should contain all the nodes used by $Q^{(i-1)}$, that is

$$\mathbb{X}^{(i-1)} \subset \mathbb{X}^{(i)} \quad i = 1, 2, \ldots;$$

(ii) **Precision.** The QF $Q^{(i)}$ should be exact for all polynomials of degree up to $2i - 1$, that is

$$\deg(Q^{(i)}) \geq 2i - 1 \quad i = 1, 2, \ldots;$$

(iii) **Initial condition.** The formula $Q^{(1)}$ is a one node formula, i.e., $N_1 = 1$ and

$$Q^{(1)}[g] = 2g(0).$$

Sequences of formulae sharing these properties are for instance Clenshaw-Curtis, Kronrod-Patterson and Petras delayed sequences, see [8], [10], [12] for definitions. These sequences are at the basis of the method proposed by Smolyak [7] for constructing particular cubature rules with low number of points. For a given integer ℓ, called *precision level*, the Smolyak formula is defined as

$$\begin{aligned}\mathcal{S}_{\ell,d}[g] = \sum_{\ell+1 \leq \|\mathbf{i}\|_1 \leq \ell+d} (-1)^{\ell+d-\|\mathbf{i}\|_1} \binom{d-1}{\|\mathbf{i}\|_1 - \ell - 1} \\ \cdot \left(Q^{(i_1)} \otimes \cdots \otimes Q^{(i_d)}\right)[g]\end{aligned} \tag{5}$$

where $\mathbf{i} \doteq [i_1 \cdots i_d]^\top$, $i_k > 0$, $k = 1, \ldots, d$, is the vector of precision indices for each dimension. It can be seen that Smolyak cubature rule is a linear combination of product formulae involving only relatively low-precision quadrature formulae, chosen in such a way that the interpolation property for $d = 1$ is maintained for $d > 1$. As a consequence, the number of nodes used by $\mathcal{S}_{\ell,d}$ is low. In particular, the

[1] Assume for instance that $\nu_i = \eta$, for $i = 1, \ldots, d$, then the number of evaluations becomes $|\mathbb{H}_d| = N^d$.

formula is based on the so-called sparse grid (see [10] and references therein)

$$\mathbb{H}_{\ell,d} \doteq \bigcup_{\|\mathbf{i}\|_1=\ell+d} \left(\mathbb{X}^{(i_1)} \times \cdots \times \mathbb{X}^{(i_d)}\right) \subset \Omega_d.$$

Two and three dimensional sparse grids for $\ell = 7$ are shown in Fig. 1.

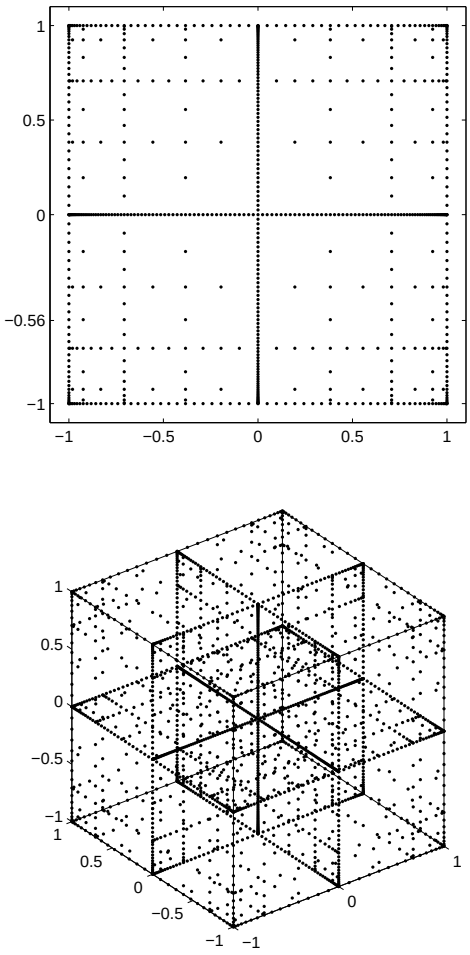

Fig. 1. Two and three dimensional sparse grids for $\ell = 7$. The number of grid points is respectively 705 and 2561.

The key properties of the cubature formula (5) are reported in the next proposition, see [12], [13].

Proposition 1: Assume that the sequence of QF used for constructing the Smolyak formula in (5) satisfies properties (i)–(iii). Then, the formula is exact for all multivariate polynomials of total degree[2] less than or equal to $2\ell + 1$, that is

$$\deg(\mathcal{S}_{\ell,d}) \geq 2\ell + 1.$$

[2]The total degree of a monomial $m(x) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$ is the sum of the exponents $deg(m(x)) = \alpha_1 + \alpha_2 + \cdots + \alpha_d$. The total degree of a multivariate polynomial is defined as the maximum total degree of its monomials.

Moreover, for fixed ℓ, the number of points grows polynomially in d as

$$N_{\ell,d} \approx \frac{2^\ell}{\ell!} d^\ell.$$

⋆

Remark 1: An important point which should be noted is that, for given ℓ and d, the $N_{\ell,d}$ nodes and weights of the Smolyak cubature formula $\mathcal{S}_{\ell,d}$ can be computed *once for all* and stored for successive computations. Indeed, these quantities do not depend on the integrand, but only on ℓ, d. The procedure for actually computing the nodes and weights is the most time consuming part of the proposed computational scheme. For this reason, a repository of nodes and weights for different values of ℓ, d has been created and is available on request.

Once these nodes x_j and weights w_j, $j = 1, \ldots, N_{\ell,d}$, are obtained, we end up with the following $N_{\ell,d}$-point cubature formula

$$\mathcal{S}_{\ell,d}[g] = \sum_{j=1}^{N_{\ell,d}} \mathsf{w}_j g(\mathsf{x}_j) \tag{6}$$

that only requires $N_{\ell,d}$ evaluations of the integrand.

This approach can be directly applied to the exact computation of dilation integrals. In particular, the following result holds.

Theorem 1: Assume that $f(q)$ is a multivariate polynomial of total degree ν. Let $\ell = \frac{k\nu}{2}$, and let x_j and w_j, $j = 1, \ldots, N_{\ell,d}$ be the nodes and weights of the (ℓ, d) Smolyak formula (6). Define

$$\begin{aligned} \mathsf{w} &\doteq [\mathsf{w}_1 \cdots \mathsf{w}_{N_{\ell,d}}]^\top \\ \mathsf{f} &\doteq [f(\mathsf{x}_1) \cdots f(\mathsf{x}_{N_{\ell,d}})]^\top. \end{aligned}$$

Then, for the dilation integral (2) we have

$$\Phi_k(\alpha) = \sum_{i=0}^{k} \eta_i \alpha^i,$$

where $\eta_i \doteq \begin{pmatrix} k \\ i \end{pmatrix} \mathsf{w}^\top \mathsf{f}^i$. ⋆

Proof. First notice that the integrand in (2) may be written as

$$\Big(1 + \alpha f(q)\Big)^k = \sum_{i=0}^{k} \begin{pmatrix} k \\ i \end{pmatrix} f^i(q) \alpha^i.$$

Then, since for the particular choice of ℓ, the Smolyak formula is exact for every d-polynomial of total degree less than or equal to $k\nu$, we can write

$$\int_{\Omega_d} f^i(q) \alpha^i \, \mathrm{d}q = \sum_{j=1}^{N_{\ell,d}} \mathsf{w}_j f^i(\mathsf{x}_j) \alpha^i.$$

The statement then follows from simple algebraic manipulations and from the definition of w and f. □

It is also worth noting that the whole set of $N_{l,d}$ nodes is used for evaluation of $f^k(\cdot)$ only, while to compute the integrals of $f^i(\cdot)$ for $i < k$, the respective subsets of $N_{l,d}$,

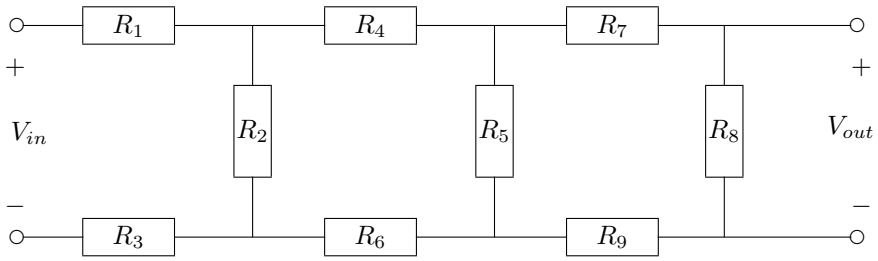

Fig. 2. Three-loop resistive ladder network.

would suffice — according to the lower total degree of $f^i(\cdot)$. This reduces the overall computational burden of the proposed scheme as applied to the specific dilation integral structure.

The result of Theorem 1 can be directly applied to the computation of the minimum of $\Phi_k(\alpha)$. In fact, it was proved in [1] that the functions $\Phi_k(\alpha)$ are convex and attain the minimum at a finite unique point. Hence, the minimizer can be directly computed as the unique real zero of the derivative. This fact is summarized in the following corollary.

Corollary 1: Under the assumptions and definitions of Theorem 1, the (unique) minimizer

$$\alpha_k^* \doteq \arg\min_{\alpha\geq 0} \Phi_k(\alpha)$$

can be computed as the only real zero of the polynomial

$$\Phi_k'(\alpha) = \sum_{i=1}^{k} i\,\eta_i \alpha^{i-1}.$$

$\star$

III. Examples

In this section, we exemplify the use of Theorem 1 in computing dilation integrals by showing its superiority over straightforward symbolic computations.

Below, we denote $\varepsilon_k(\alpha) = \Phi_k(\alpha)/\mathbf{Vol}(\Omega_d)$ and $\varepsilon_k = \min_{\alpha\geq 0} \varepsilon_k(\alpha)$ and specify $\varepsilon = 0.01$ as a tolerable level of approximate robustness.

Example 1: (Resistive ladder network).

We consider the three-loop ladder network depicted in Fig. 2, which contains nine resistors. This configuration has been subject of research in [14] within a probabilistic distributionally-robust approach.

The *gain* of the network is defined as the ratio of the output and input voltages

$$g = \frac{V_{out}}{V_{in}},$$

and can be computed using Kirchhoff rule as

$$g = \frac{R_2 R_5 R_8}{\det(M_R)},$$

where

$$M_R = \begin{bmatrix} R_1+R_2+R_3 & -R_2 & 0 \\ -R_2 & R_2+R_4+R_5+R_6 & -R_5 \\ 0 & -R_5 & R_5+R_7+R_8+R_9 \end{bmatrix}$$

is the resistance matrix of the network.

We assume now that the values of the resistors R_i are uncertain, with nominal values $R_i^0 = 1$ and uncertainty radius $\mu > 0$. That is, for $i = 1, \ldots, 9$, we write

$$R_i = R_i(q) = R_i^0 + \mu q_i, \quad |q_i| \leq 1. \tag{7}$$

Then, due to parameter variations, the actual gain $g(q)$ differs from the nominal $g_0 = g(0)$, and the robustness requirement is that $g(q)$ does not exceed a certain level. We formulate this requirement in the form

$$g(q) < \gamma\, g_0 \qquad \text{for all} \quad q \in \Omega_9 \tag{8}$$

for some *looseness* level $\gamma > 1$. Inequality (8) writes

$$R_2(q)R_5(q)R_8(q) - \gamma g_0 \det(M_R(q)) < 0 \quad \text{for all} \quad q \in \Omega_9.$$

To check this robustness specification, we resort to the dilation integral method and introduce the function

$$f(q) = R_2(q)R_5(q)R_8(q) - \gamma g_0 \det(M_R(q)),$$

which is seen to be a polynomial in nine variables of total degree $\nu = 3$. The theories above can now be used for checking sign-definiteness of $f(q)$; we see that the integrand in (2) is a polynomial of total degree $3k$ in q.

For this network, we have $g_0 = 0.0244$, and in the experiments, looseness $\gamma = 1.5$ was chosen. The radius of robustness, that is the largest value of μ in (7) guaranteeing (8), in this problem can be found by checking the vertices of the cube Ω_d (since $f(q)$ is multilinear, it attains its maximum value at a vertex[3]). Calculations resulted in $\mu_{\max} = 0.1128$.

Then, we specify $\mu = 1.75\mu_{\max}$, so that in the classical sense the problem is not robust, and compute the dilation integrals symbolically. For $k = 2$ this gives $\varepsilon_2 = 0.09133$ (with a cpu time of about 20 sec). Already for $k = 4$, the MATLAB memory capacity is exceeded, and we cannot make definite conclusions about approximate robustness of the problem.

[3]It is straightforward to see that the maximal gain for the network over all variations of q_i is attained by putting q_2, q_5, q_8 at their maximum admissible values and the rest of the uncertainties at their minimum values.

By way of comparison, computing the requisite integrals using the representation given by Theorem 1, we obtain the same value of ε_2 with cpu time 0.1 sec, and then $\varepsilon_4 = 0.026011$, $\varepsilon_6 = 0.012582$ and $\varepsilon_8 = 0.008514$ thus arriving at the conclusion that the problem is approximately robust with the desired level ε. Succeeding Monte Carlo simulation with $N = 152,000$ samples gives $\varepsilon_{MC} = 0.001432$ as a statistical estimate of this quantity. To interpret this probabilistic result, we apply the Chernoff bound [15] which guarantees that $\varepsilon \in (0,\, 0.006432)$ with probability at least 99.9%. This confirms the fact established by the dilation theory that ε_k are upper estimates for the relative volume of violation.

Example 2: (Ackermann's track guided bus [16]).

This is a widely known benchmark problem with nonlinear parameter dependence, for which the classical robustness analysis is difficult to perform. Considered is the robust stability problem for an uncertain system with characteristic polynomial given by

$$p(s,\vartheta) = \sum_{k=0}^{8} a_k(\vartheta) s^k \tag{9}$$

with uncertain parameter $\vartheta(q) \in \mathbb{R}^2$ of the form

$$\vartheta(q) = \vartheta^0 + \mu\delta_\vartheta q, \quad q \in \Omega_2,$$

where $\vartheta^0 \doteq [11.5\ 21]^\top$, $\delta_\vartheta \doteq [8.5\ 11]^\top$, $0 \le \mu \le 1$ defines the radius of uncertainty, and the coefficients $a_k(\vartheta)$, which depend polynomially on ϑ, are given by (see [16])

$$\begin{aligned}
a_0 &= 4.53\cdot 10^8\vartheta_1^2,\\
a_1 &= 5.28\cdot 10^8\vartheta_1^2 + 3.64\cdot 10^9\vartheta_1,\\
a_2 &= 5.72\cdot 10^6\vartheta_1^2\vartheta_2 + 1.13\cdot 10^8\vartheta_1^2 + 4.25\cdot 10^9\vartheta_1,\\
a_3 &= 6.93\cdot 10^6\vartheta_1^2\vartheta_2 + 9.11\cdot 10^8\vartheta_1 + 4.22\cdot 10^9,\\
a_4 &= 1.45\cdot 10^6\vartheta_1^2\vartheta_2 + 16.8\cdot 10^6\vartheta_1\vartheta_2 + 3.38\cdot 10^8,\\
a_5 &= 15.6\cdot 10^3\vartheta_1^2\vartheta_2^2 + 840\vartheta_1^2\vartheta_2 + 1.35\cdot 10^6\vartheta_1\vartheta_2 + 13.5\cdot 10^6,\\
a_6 &= 1.25\cdot 10^3\vartheta_1^2\vartheta_2^2 + 16.8\vartheta_1^2\vartheta_2 + 5.39\cdot 10^4\vartheta_1\vartheta_2 + 270\cdot 10^3,\\
a_7 &= 50\vartheta_1^2\vartheta_2^2 + 1080\vartheta_1\vartheta_2,\\
a_8 &= \vartheta_1^2\vartheta_2^2.
\end{aligned}$$

To apply the dilation integral method, we follow [4] and take

$$f(q) = -\det \mathcal{H}(q)$$

in (2), $\mathcal{H}$ being the Hurwitz matrix for $p(s,q)$. Notice that $f(q)$ is a multivariate polynomial in q.

The analysis in [16] shows that for $\mu = 1$ polynomial (9) is robustly stable. Hence, convergence of ε_k to zero should be observed. However, the uncertainty set Ω_d nearly touches the boundary of the stability domain in the q parameter space, which makes the problem extremely poorly conditioned, with conditioner $\theta > 0.999999$ (see [4] for details). Using Symbolic Math Toolbox in MATLAB for computing the dilation integrals, we obtain $\varepsilon_2 = 0.94793$ and $\varepsilon_4 = 0.92923$, and respectively, the estimate $\theta_4 = 0.98184$ for the conditioner. Based on this information, in [4] a conclusion was made on *practical instability* of polynomial (9). Further increase in k yields integrals which are intractable with symbolics due to MATLAB memory limitations. In contrast, using cubature formulae as above[4], we can go much higher in k. Figure 3 presents the values of ε_k for k ranging 2 to 30; these indeed testify to extremely slow convergence of the estimates and it is reasonable to conclude that the uncertain system is *practically unstable*.

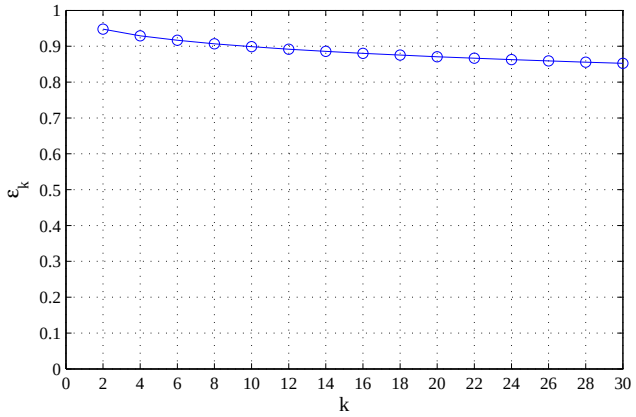

Fig. 3. Slowly converging sequence of ε_k in Example 2 with $\mu = 1$.

In the second experiment, we took $\mu = 0.1$ still keeping the problem poorly conditioned, with $\theta = 0.8622$. However in that case, computing dilation integrals for higher k using Theorem 1 rather quickly leads to certification of approximate robustness, see Fig. 4. Thus, for the index as low as $k = 12$, we obtain $\varepsilon_{12} = 0.0098057$, which is below the desired level ε. By increasing k we observe convergence of the estimates ε_k to zero; e.g., $\varepsilon_{30} = 0.000257$.

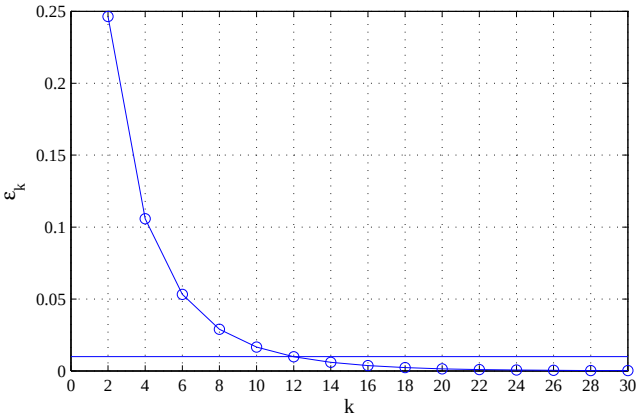

Fig. 4. Convergence of estimates $\varepsilon_k \to 0$ in Example 2 with $\mu = 0.1$.

IV. Conclusion

We have presented a new technique for the exact computation of dilation integrals. The proposed solution, based on the application of special cubature formulae with high degree of polynomial exactness, allows to overcome

[4]In this case, due to lower dimension of d, a direct application of the tensor formula (4) is preferable. In this case, we had $\nu_1 = 16$ and $\nu_2 = 10$.

the main drawback of the dilation approach, that is the need to resort to symbolic computation. The method is numerically stable and has polynomial complexity in the problem dimension d. From a robustness analysis point of view, the method provides a more definite certification of low volume of violation, thus allowing for a better assessment of practical robustness for problems which may be computationally untractable within a deterministic framework.

The approach still suffers some limitations. This is due to the non-polynomial dependence on the index k. This fact is not surprising, since the problem is just a reformulation of the originally NP hard prototype. In the light of the above considerations, by way of future research it would be interesting to analyze the accuracy of computations when leaning on a subset of nodes (that is when using cubature formulae with degree of exactness less than the required one). For instance, in Example 1, we used the sparse grid required for exact computation of ε_4 for computing ε_8, and the result was exact up to the eighth digit. Proceeding in a similar way, the quantities $\varepsilon_{10} = 0.0072534$, $\varepsilon_{12} = 0.0071538$, $\varepsilon_{14} = 0.0073456$ were obtained, which are in agreement with the expected behavior of the function.

Finally, the resolution of the following issue seems very important in reducing the amount of computations. Given two polynomials $g(x)$ and $h(x)$ together with the associated sets of nodes and weights required for exact computation of $\int g(x)\mathrm{d}x$ and $\int h(x)\mathrm{d}x$, compose the set of nodes and weights required for computing of $\int g(x)h(x)\mathrm{d}x$.

Acknowledgments

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