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Interval Parameter Estimation under Model Uncertainty

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This paper is devoted to the estimation of parameters of linear multi-output models with uncertain regressors and additive noise. The uncertainty is assumed to be described by intervals. Outer-bounding interval approximations of the non-convex feasible parameter set for uncertain systems are obtained. The method is based on the calculation of the interval solution for an interval system of linear algebraic equations and provides parameter estimators for models with a large number of measurements.

Keywords: Interval uncertainty, interval equations, uncertain model, parameter estimation, parameter bounding

1. Introduction

The set-membership estimation framework for uncertain systems has attracted much attention during the past few decades. It is an alternative to the stochastic approach where some prior information on the statistical distribution of errors is needed, because only bounds on uncertainty in system parameters and signals are required. This assumption is often much more acceptable in practice. Various types of compact sets (intervals, polytopes, ellipsoids, etc.) are usually used to characterize these bounds. They are called the membership set constraints on uncertain variables.

The parameter estimation problem for uncertain dynamic systems is one of the most natural in this context. The problem is to determine bounds or set constraints on system parameters based on output measurements, the model structure and bounds on uncertain variables. In this paper we focus on the so-called interval type of uncertainty. This means that each component of an uncertain vector or matrix is assumed to belong to a known finite interval. Although the description is natural and simple [1], combinatorial difficulties may become so severe as to make estimation intractable especially in the multidimensional case. The goal of the paper is to construct an

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effective interval parameter estimator for an uncertain multi-output static system that can be used with large sets of data.

Consider a linear regression model under interval uncertainty

$$y = Cx + w, \tag{1}$$

where $x \in \mathbb{R}^n$ is an unknown parameter vector, $y \in \mathbb{R}^m$ denotes a vector of results of measurements, $C \in \mathbb{R}^{m \times n}$ is a matrix of regressors and $w \in \mathbb{R}^m$ is an unknown vector of measurement errors. The classical parameter bounding approach is based on the assumption that the matrix C is known precisely, while the vector w is bounded and lies in the box $w \in [\underline{w}, \overline{w}]$ where $\underline{w}$ and $\overline{w}$ are known. This inclusion is understood componentwise. For instance, the set of all feasible parameter vectors x compatible with a single measurement $y_i \in \mathbb{R}$ is the strip between two hyperplanes

$$S_i = \left\{x \in \mathbb{R}^n : \ y_i - \overline{w_i} \le c_i^T x \le y_i - \underline{w_i}\right\}. \tag{2}$$

The sequence of measurements $y_1, \ldots, y_m$ then provides a convex polytope $\bigcap_{i=1}^m S_i$. A number of methods has been developed to characterize this polytope or to construct outer-bounding approximations of it (see [2,3,4,5,6]).

The present paper deals with the more general problem of where the matrix of regressors is also uncertain, i.e. $C \in \mathbf{C}$, and $\mathbf{C}$ is an interval matrix ($\mathbf{C} \in \mathbb{IR}^{m \times n}$). This situation arises in many real-life problems when we do not have complete information concerning the plant. Furthermore, weakly non-linear systems can be treated in the same manner if non-linearity is replaced by uncertainty. Particular cases of this problem have been considered in the literature [3,7,8]. Any matrix or vector is said to belong to the interval family if its elements are from some real intervals $[a;b]$, $a \leqslant b$. The standard notation $\mathbb{IR}^{m \times n}$ indicates the space of all interval $(m \times n)$ -matrices and $\mathbb{IR}^n$ is the space of all n-dimensional interval vectors. The presence of matrix uncertainty in the model leads to serious difficulties due to the non-convexity of the resulting set constraints. In [9,10] ellipsoidal techniques were applied to state and parameter estimation for linear models with matrix uncertainty where non-convexity of reachable and feasible parameter sets was pointed out. The main purpose of this article is to apply an interval technique to parameter estimation. The basic tool for this is the interval solution of linear interval systems of equations.

The paper is organized as follows. Section 2 states the problem in detail. The scalar-measurement case is considered in section 3. In the multi-dimensional case the idea is to split a given linear model into a number of interval systems of linear algebraic equations. A simple method for solving these systems is proposed, which easily computes an optimal interval solution for moderate-scale problems (section 4.2) whereas for large-scale problems effective interval over-bounding solutions are obtained (section 4.3). The resulting method for parameter bounding is finally described in section 5.

2. Problem statement

Consider a multi-output model with measurement noise and uncertainty in the matrix of regressors

$$y = (C + \Delta C)\,x + w, \tag{3}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $C \in \mathbb{R}^{m\times n}$ and $w \in \mathbb{R}^m$. The number of measurements is usually much larger than the dimension of the parameter vector, so $m \gg n$. Assume

$$||\Delta C||_\infty \leq \epsilon, \quad ||w||_\infty \leq \delta. \tag{4}$$

The infinity norms of matrices and vectors are equal to the maximal absolute value of their elements, i.e.

$$||\Delta C||_\infty = \max_{1\leq i\leq m,\, 1\leq j\leq n} |(\Delta C)_{ij}|, \qquad ||w||_\infty = \max_{1\leq i\leq m} |w_i|. \tag{5}$$

Inequality (4) describes a particular case of interval uncertainty when all components of ΔC or w have the same bounds. The matrix C, vector y and scalars ϵ, δ are assumed to be known. All vectors $x \in \mathbb{R}^n$ satisfying (3) under the constraints (4) form the *feasible parameter set*

$$X = \left\{x \in \mathbb{R}^n : \quad y = (C + \Delta C)x + w, \ ||\Delta C||_\infty \leq \epsilon, \ ||w||_\infty \leq \delta\right\}. \tag{6}$$

Assume that $x \in \mathbf{X}_0$, where $\mathbf{X}_0 \in \mathbb{IR}^n$. The initial approximation $\mathbf{X}_0$ should be taken large enough to guarantee inclusion of all parameter vectors of interest. The problem is to construct a more accurate outer-bounding interval approximation for the vector x in accordance with the large number of measurements $\{y_1, \ldots, y_m\}$ and model structure (3)–(4). In other words, we look for an interval vector $\mathbf{X} \in \mathbb{IR}^n$ (preferably of minimal size) containing the intersection $\mathbf{X}_0 \cap X$.

3. Scalar observation

The single-measurement case ($m = 1$) is a good particular example of the parameter estimation. Set X for the scalar model

$$y = (c + \Delta c)^T x + w \tag{7}$$

where $x \in \mathbb{R}^n, y \in \mathbb{R}, ||\Delta c||_\infty \leq \epsilon$ and $|w| \leq \delta$ can be explicitly rewritten as

$$X_1 = \{x \in \mathbb{R}^n : \ |y - c^T x| \leq \epsilon ||x||_1 + \delta\}, \tag{8}$$

where $||x||_1 = \sum_{i=1}^n |x_i|$. Figure 1 depicts a typical shape of X_1, which is non-convex for any $\epsilon > 0$ (the region between two solid polygonal lines). This set reduces to a strip as $\epsilon \to 0$. However it is convex in each orthant of $\mathbb{R}^n$. Let E^k be the k-th orthant of the vector space, $k = 1, \ldots, 2^n$. If $x \in E^k$ for any fixed number k, the right-hand side of the inequality in (8) becomes a linear function and therefore $X_1 \cap E^k$ is a convex set.

Assume $x \in \mathbf{X}_0$. Then the smallest interval vector $\mathbf{X}$ containing $\mathbf{X}_0 \cap X_1$ can be found by solving a linear programming problem in each orthant of $\mathbb{R}^n$. Indeed, the vector $s^k = \text{sign}\, x$ for $x \in E^k$ is uniquely defined with elements s_i^k such that $|s_i^k| = 1$. Thus

$$X_1 \cap E^k = \left\{x \in \mathbb{R}^n : \ |y - c^T x| \leq \epsilon \sum_{i=1}^n x_i s_i^k + \delta\right\} \tag{9}$$

is a convex set given by linear constraints. Denote by e^j the j-th unit coordinate vector of $\mathbb{R}^n$, $j = 1, \ldots, n$. Then the j-th lower and upper bounds on the intersection $\mathbf{X}_0 \cap X_1 \cap E^k$ are calculated by linear programming as

$$\underline{x_j^k} = \arg \min_{x \in \mathbf{X}_0 \cap X_1 \cap E^k} (e^j, x), \qquad \overline{x_j^k} = \arg \max_{x \in \mathbf{X}_0 \cap X_1 \cap E^k} (e^j, x), \tag{10}$$

where $(\cdot,\cdot)$ denotes the scalar product. Hence $\mathbf{X^k} = ([\underline{x_1^k}, \overline{x_1^k}], \ldots, [\underline{x_n^k}, \overline{x_n^k}])^T$ gives an interval vector that is the minimal box containing $\mathbf{X}_0 \cap X_1 \cap E^k$.

Notice however that the intersection of the set $\mathbf{X}_0 \cap X_1$ with some orthants may be empty. The calculations in these orthants can obviously be omitted as long as the linear programming problem (10) turns out to become infeasible.

Further, let $K = \{k : \ \mathbf{X}_0 \cap X_1 \cap E^k \neq \emptyset\}$. Then $\{E^k : k \in K\}$ represents a family of orthants containing $\mathbf{X}_0 \cap X_1$. Checking all orthants E^k such that $k \in K$ we obtain the inclusion $\mathbf{X}_0 \cap X_1 \subseteq \bigcup_{k \in K} \mathbf{X^k}$. Finally, take

$$\underline{x_i} = \min_{k \in K} \left\{ \underline{x_i^k} \right\}, \quad \overline{x_i} = \max_{k \in K} \left\{ \overline{x_i^k} \right\}, \qquad i = 1, \ldots, n. \tag{11}$$

$\mathbf{X} = ([\underline{x_1}, \overline{x_1}], \ldots, [\underline{x_n}, \overline{x_n}])^T$ gives the optimal interval approximation of $\mathbf{X}_0 \cap X_1$.

Example 1. Let $\mathbf{X}_0 = ([-1,4],[-0.5,5])^T$ and $y = 0$, $c = (1,1)^T$, $\epsilon = \delta = 0.5$. The set $X_1 = \{ x \in \mathbb{R}^2 : \ 2|x_1 + x_2| \leq |x_1| + |x_2| + 1\}$ is shown in figure 2 (shaded region). The auxiliary interval vectors $\mathbf{X^k}$ are found via linear programming according to (10) in each orthant E^k, $k = 1, \ldots, 4$ (bold boxes). Then $\mathbf{X} = ([-1,2.5],[-0.5,4])^T$.

In the multi-output case, one can consider the scalar observations recursively and apply the above linear programming procedure. However this technique becomes unsuitable for models with a large number of measurements ($m \gg n$). The main idea of the present paper is to consider blocks of n measurement equations in (3) and to treat each of them as a system of linear algebraic equations under interval uncertainties. A

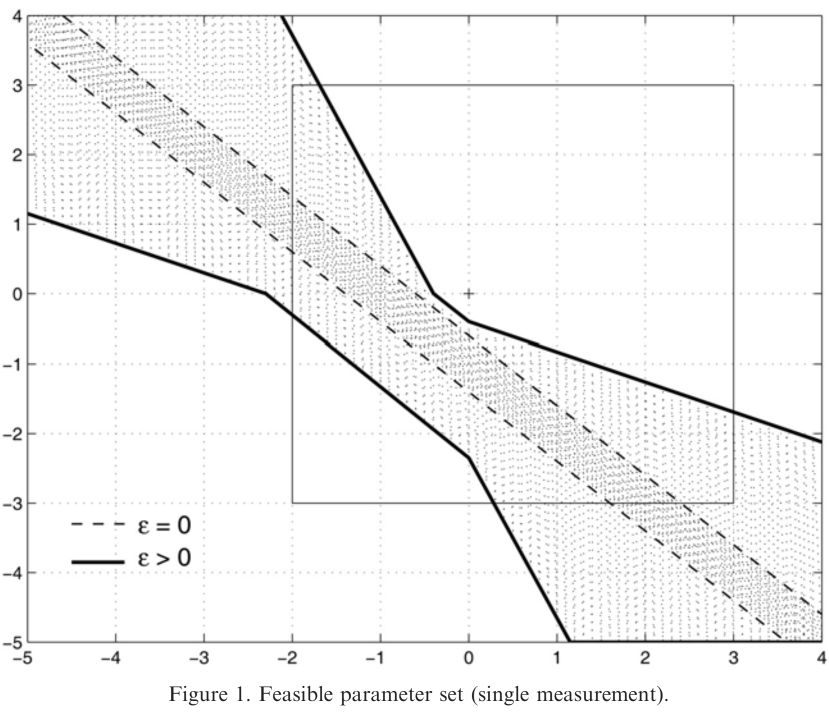

Figure 1. Feasible parameter set (single measurement).

simple algorithm to obtain an interval solution for this system is described in the following section.

4. Interval system of linear algebraic equations

Let $C \in \mathbb{R}^{n\times n}$, i.e. the number of parameters is equal to the number of observations. Then rewrite (3) in the form

$$(A + \Delta A)\, x = b + \Delta b \tag{12}$$

with $A = C,\ b = y,\ \Delta A = \Delta C$ and $\Delta b = -w$ such that $||\Delta A||_\infty \leqslant \epsilon,\ ||\Delta b||_\infty \leqslant \delta$.

The calculation of the interval solution for the interval system of equations (12) is a challenging problem in numerical analysis and robust linear algebra. This problem was first considered in the 1960s by Oettli and Prager [11]. Since then, the problem has attracted much attention and was developed in the context of the modelling of uncertain systems.

Assume that the matrix family $\mathbf{A} = \{A + \Delta A :\ ||\Delta A||_\infty \leq \epsilon\} \in \mathbb{IR}^{n\times n}$ is non-singular (it contains no singular matrix) and that the interval vector is $\mathbf{b} = \{b + \Delta b :\ ||\Delta b||_\infty \leq \delta\} \in \mathbb{IR}^n$. Then for any $A \in \mathbf{A}$ and any $b \in \mathbf{b}$ the ordinary linear system $Ax = b$ has a unique solution. We are interested in a set $\hat{X}$ of all these solutions of the interval system:

$$\hat{X} = \{x \in \mathbb{R}^n :\quad Ax = b,\ A \in \mathbf{A},\ b \in \mathbf{b}\}. \tag{13}$$

Our main objective is to find an interval solution of the linear interval system (12), that is, to determine the smallest interval vector $\mathbf{X}^*$ containing all possible solutions

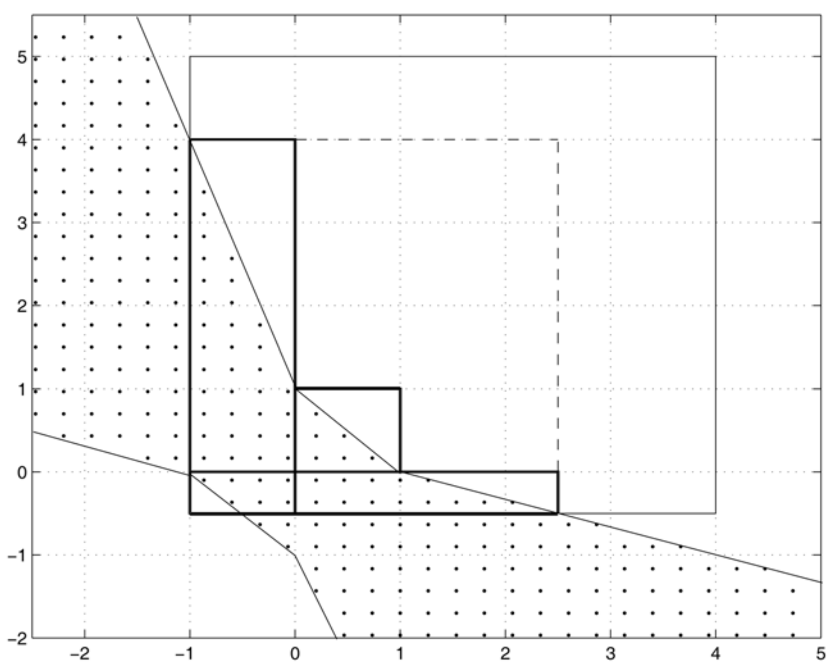

Figure 2. Single-output interval parameter bounding.

(13). In other words, we want to embed the solution set $\hat{X}$ into the minimal box in $\mathbb{R}^n$. This problem is known to be NP-hard [12] and complicated from a computational viewpoint for large-scale systems. Paper [11] shows how multiple linear programming can be used to obtain $\mathbf{X}^*$; this line of research was continued in [13,14]. Iterative approaches have been established at this context as well as direct numerical methods that provide an over-bounding of $\mathbf{X}^*$; see the monographs [15,16] and papers [17,18].

In this section we briefly describe a simple approach proposed in [19] for interval approximation of the solution set. Instead of employing linear programming in each orthant it is suggested to deal with a scalar equation. This method is based on Rohn's result [17] and simplifies his algorithm. In order to find the optimal interval estimate $\mathbf{X}^*$ of the solution set $\hat{X}$, all vertices of its convex hull Conv $\hat{X}$ should be obtained. The search of each vertex is via the solution of a scalar equation. In the case of large-scale systems we also provide a simple and fast procedure for over-bounding of the optimal interval solution.

4.1. The solution set

A detailed description of the solution set for the linear interval systems was given in the pioneering work by Oettli and Prager [11] for a general situation of interval uncertainty. In our case their result is reduced as follows.

Lemma 1 (Oettli & Prager [11]) The set of all admissible solutions of the system (12) is a non-convex polytope:

$$\hat{X} = \{x \in \mathbb{R}^n : ||Ax - b||_\infty \leq \epsilon ||x||_1 + \delta\}. \tag{14}$$

This result also follows from (8). The set $\hat{X}$ remains bounded as long as the interval matrix $\mathbf{A}$ is regular. This regularity is characterized by a non-singularity radius. For the interval family $\mathbf{A}$ this radius is equal to

$$\rho(A) = \frac{1}{||A^{-1}||_{\infty,1}}, \tag{15}$$

see [20] for details. Recall that for any matrix G its $(\infty,1)$ -norm is defined as

$$||G||_{\infty,1} = \max_{||x||_\infty \leq 1} ||Gx||_1. \tag{16}$$

Note also that the calculation of this norm is NP-hard.

While $\epsilon < \rho(A)$, $\mathbf{A}$ remains regular and $\hat{X}$ is bounded. If the solution set $\hat{X}$ lies in a given orthant of $\mathbb{R}^n$, then it becomes convex, and the search for its interval approximation reduces to convex optimization. However this is no longer the case in most situations, and the problem then meets combinatorial difficulties.

4.2. Optimal interval estimates of the solution set

The problem is to determine exact lower $\underline{x}_i$ and upper $\overline{x}_i$ bounds on each component x_i of the vector $x \in \mathbb{R}^n$ under the assumption that $x \in \hat{X}$. The approach is focused on searching for vertices of the convex hull $\mathrm{Conv}\,\hat{X}$ of the solution set $\hat{X}$ instead of employing linear programming in each orthant. The main base for this technique is the paper by Rohn [17], where a key result defining $\mathrm{Conv}\,\hat{X}$ was proved.

Let S be the set of vertices of the unit cube $S = \{s \in \mathbb{R}^n : |s_i| = 1, i = 1, \ldots, n\}$. Consider a system of equations

$$(a_i^T x - b_i)\, s_i = \epsilon ||x||_1 + \delta, \quad i = 1, \ldots, n, \tag{17}$$

for some $s \in S$, where a_i is the i-th row of the matrix A.

Lemma 2 (Rohn [17]). For a given nominal matrix A, let the interval family

$$\mathbf{A} = \{A + \Delta A : \ ||\Delta A||_\infty \leq \epsilon\} \tag{18}$$

be regular, i.e. all matrices in $\mathbf{A}$ are non-singular. Then the non-linear system of equations (17) has exactly one solution $x_s \in \hat{X}$ for every fixed vector $s \in S$, and $\mathrm{Conv}\,\hat{X} = \mathrm{Conv}\,\{x_s : \ s \in S\}$.

To simplify the search for vertices x_s, introduce $\hat{y} = Ax - b$. After change of the variables equalities (17) are converted to

$$\hat{y}_i\, s_i = (\epsilon ||A^{-1}(\hat{y} + b)||_1 + \delta), \quad i = 1, \ldots, n. \tag{19}$$

Recall that $s_i = \pm 1; \forall i$. The transformed solution set $\hat{Y} = \{\hat{y} : \ ||\hat{y}||_\infty \leq \epsilon ||A^{-1}(\hat{y} + b)||_1 + \delta\}$ is the affine image of $\hat{X}$ that is $\hat{Y} = A\hat{X} - b$. Note that $\mathrm{Conv}\,\hat{Y} = A\, \mathrm{Conv}\,\hat{X} - b$. For any positive value ϵ the intersection of $\hat{Y}$ with each orthant of $\mathbb{R}^n$ is non-empty. Following Lemma 2 each orthant contains only one vertex of $\mathrm{Conv}\,\hat{Y}$ that gives the solution of the system of equations (19) while the vector $s = (s_1, \ldots, s_n)^T = \mathrm{sign}\,\hat{y}$ specifies the choice of the orthant under consideration. Taking all vectors s from S we find all vertices for $\mathrm{Conv}\,\hat{Y}$. Moreover, (19) is equivalent to one scalar equation

$$\tau = \varphi(\tau), \tag{20}$$

where $\tau = \hat{y}_i\, s_i$, $\hat{y}_i = \tau / s_i = \tau s_i$ and $\varphi(\tau) = \epsilon ||A^{-1}(\tau s + b)||_1 + \delta$. The function $\phi(\tau)$ is defined for all $\tau \geqslant 0$ and it is a convex piecewise linear function of τ.

Lemma 3 (see [19]) For any regular interval family $\mathbf{A} \in \mathbb{IR}^{n \times n}$ and any fixed vector $s \in S$ the scalar equation (20) has a unique solution over $[0, \infty)$.

The solution τ^* of (20) can be obtained using a simple iterative scheme, for example, Newton iterations

$$\tau_{k+1} = \left[\tau_k + \frac{\varphi(\tau_k) - \tau_k}{1 - \varphi'(\tau_k)}\right]_+, \tag{21}$$

where we use the notation $[\alpha]_+ = \max\,\{0, \alpha\}$. Procedure (21) converges to τ^* for any initial $\tau_0 \geqslant 0$ in a finite (no more than n) iterations. Finally, we can formulate the following theorem.

Theorem 1 The set $\mathrm{Conv}\,\hat{X}$ has 2^n vertices. Each vertex x_s can be found by solving the scalar equation (20) for a given vector $s \in S$ by algorithm (21). Then $x_s = A^{-1}(\hat{y}(\tau^*) + b)$, where $\hat{y}(\tau) = \tau s$ and τ^* is the solution of (20).

With these vertices we find the optimal lower and upper bounds for each component of x in the solution set $\hat{X}$

$$\underline{x}_i = \min_{s\in S}\{x_{s_i}\}, \quad \overline{x}_i = \max_{s\in S}\{x_{s_i}\}, \qquad i = 1,\dots,n, \tag{22}$$

and finally $\mathbf{X}^* = ([\underline{x}_1, \overline{x}_1], \dots, [\underline{x}_n, \overline{x}_n])^T$.

Example 2. For $A = \binom{11}{01}$, $b = \binom{1}{0.5}$, and $\epsilon = \delta = 0.25$ the solution set $\hat{X}$ is a bounded and non-convex polytope depicted in figure 3. Its image after the affine transformation $\hat{y} = Ax - b$ is shown in figure 4. All vertices of the convex hull Conv $\hat{Y}$ of the solution set with the variables $\hat{y}$ are represented by the vector s^k with elements $s_i^k = \pm 1$ and the value of τ from (20): $y^1 = (2/3,\ 2/3)^T$, $y^2 = (-1,1)^T$, $y^3 = (1,-1)^T$ and $y^4 = (-0.4,\ -0.4)^T$. By inverse transformation $x = A^{-1}(\hat{y} + b)$ the vertices of Conv $\hat{X}$ are obtained, and then it is trivial to find the interval bounds on $\hat{X}$ using (22). Finally $\mathbf{X}^* = ([-1.5,\ 2.5],\ [-0.5,\ 1.5])^T$.

4.3. Interval over-bounding technique

As already mentioned, the calculation of the optimal interval solution X* may be hard for large-dimensional problems. Hence, its simple interval over-bounding is of interest. This over-bounding is often said to be an interval solution of the interval system of equations as well. We provide below two such estimates.

According to the inequality $||\hat{y}||_\infty \le \epsilon ||A^{-1}(\hat{y} + b)||_1 + \delta$ for the set $\hat{Y}$ we write

$$||A^{-1}(\hat{y} + b)||_1 \le ||A^{-1}\hat{y}||_1 + ||x^*||_1 \le ||A^{-1}||_{\infty,1}\, ||\hat{y}||_\infty + ||x^*||_1, \tag{23}$$

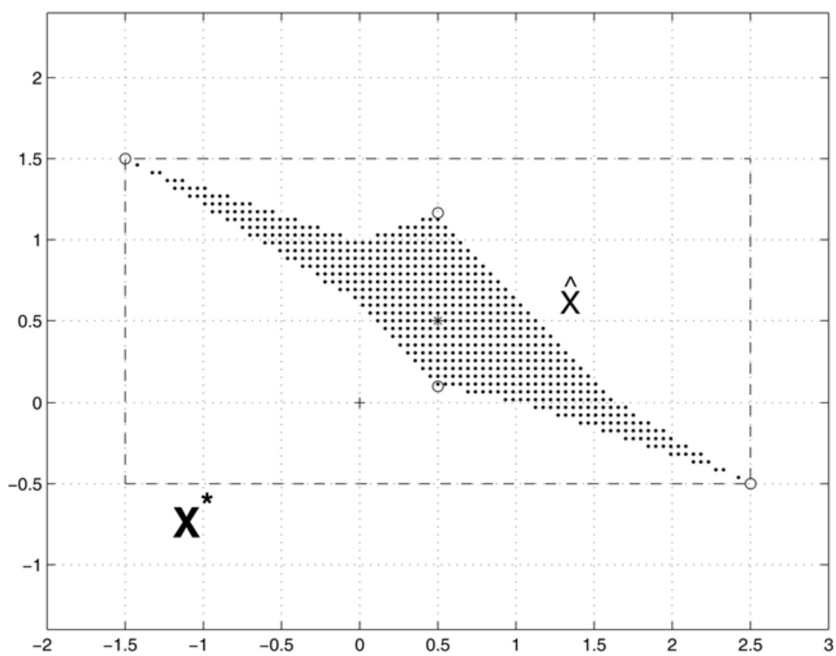

Figure 3. The original solution set $\hat{X}$.

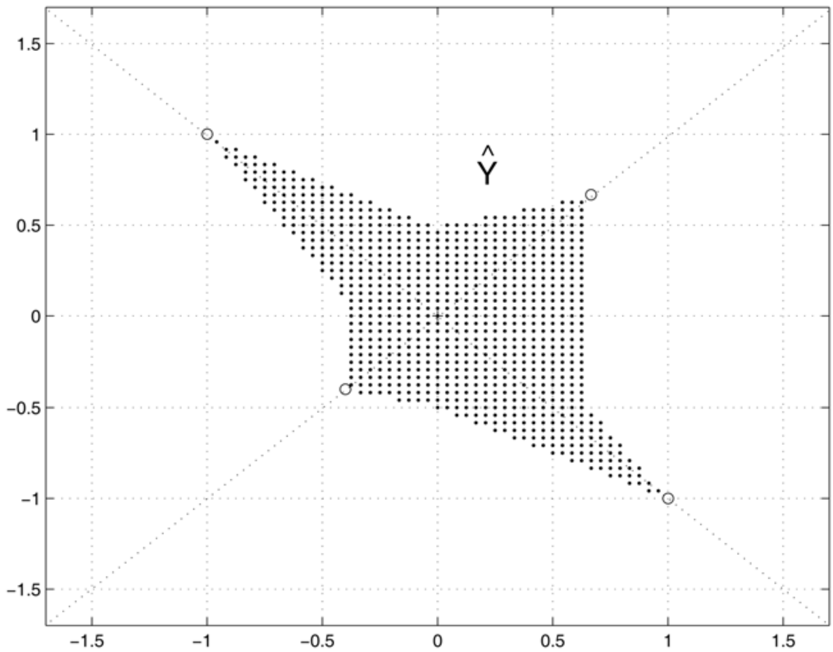

Figure 4. The transformed solution set $\hat{Y}$.

where $x^* = A^{-1}b$. Therefore

$$||\hat{y}||_\infty \leq \gamma = \frac{\epsilon||x^*||_1 + \delta}{1 - \epsilon||A^{-1}||_{\infty,1}}. \tag{24}$$

All vectors $\hat{y}$ that belong to $\hat{Y}$ thus also lie inside the ball in ∞-norm of radius γ. This ball is the first over-bounding interval estimate. In most cases (24) is the minimal cube centred at the origin containing $\hat{Y}$. The main difficulty here is to calculate the $(\infty,1)$ norm of A^{-1}; this is again an NP-hard problem. There exist tractable upper bounds for this norm; we use the simplest one: for any given matrix G with entries g_{ij} $(i, j = 1, \ldots, n)$ the value of $||G||_{\infty,1}$ can always be approximated by a 1-norm:

$$||G||_{\infty,1} = \max_{||x||_\infty \leq 1} ||Gx||_1 = \max_{||x||_\infty \leq 1} \sum_{i=1}^{n} \left| \sum_{j=1}^{n} g_{ij} x_j \right| \leq \sum_{i,j=1}^{n} |g_{ij}|| = ||G||_1. \tag{25}$$

Hence, the inequality (24) is replaced by

$$||\hat{y}||_\infty \leq \frac{\epsilon||x^*||_1 + \delta}{1 - \epsilon||A^{-1}||_1}, \tag{26}$$

where ϵ should be less than $1/||A^{-1}||_1$. An interval estimate for $\hat{Y}$ implies an interval estimate for $\hat{X}$. Indeed, x is an affine function of $\hat{y} : x = x^* + A^{-1}\hat{y}$ and component-wise optimization for x_i on a cube can be performed explicitly. Then we arrive at the following result.

Theorem 2 The interval vector $\mathbf{X} = ([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n])^T$ with

$$\underline{x}_i = x_i^* - \gamma||g_i||_1, \quad \overline{x}_i = x_i^* + \gamma||g_i||_1, \quad i = 1, \ldots, n \tag{27}$$

contains the solution set $\hat{X}$, where g_i is the i-th row of $G = A^{-1}$ while γ is the right-hand side of (24) or (26).

Thus the calculation of $\mathbf{X} \supseteq \mathbf{X}^*$ given by (26), (27) is not involved, it does not lead to any combinatorial difficulties and does not require the solution of linear programming problems. Numerous examples confirm that this over-bounding solution is close to optimal. One such example is considered in [19] for the linear system $Hx = b$ with H being a Hilbert matrix. Hilbert matrices are poorly conditioned even for small dimensions and for this reason it is a good test example in the framework of interval uncertainty. It was demonstrated in [19] that the over-bounding estimates (26), (27) and (24), (27) coincide in this case and give a very precise approximation of the smallest interval solution.

5. Large-scale interval parameter bounding

Assume now that $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $m \gg n$. The interval vector $\mathbf{X}_0$ is taken to be a prior approximation containing the parameter vector x. Let c_i be the i-th row of C. Below, we describe two recursive algorithms for an outer-bounding interval approximation of the intersection $X \cap \mathbf{X}_0$.

In Algorithm 1, for simplicity, we assume $m = Kn$, where K is an integer.

Algorithm 1 *Let* $k = 1$. Assume $\mathbf{X} = \mathbf{X}_0$ as an initial interval approximation.

- **Step 1:** Consider the interval system of linear equations from (3) that corresponds to the regressors $c_{(kn-n+1)}, \ldots, c_{kn}$. Compute the non-singularity radius ρ_k for the nominal matrix A of this system. If $\epsilon < \rho_k$, then find its interval solution $\mathrm{X_k}$, else (in particular, if A is singular) $\mathrm{X_k}$ is assumed to be infinitely large and go to step 3.
- **Step 2:** Find the smallest interval vector $\tilde{\mathbf{X}}$ containing $\mathbf{X} \cap \mathbf{X_k}$. Put $\mathbf{X} = \tilde{\mathbf{X}}$.
- **Step 3:** If $k = K$, then terminate, else set $k = k + 1$ and go to step 1.

The interval solution $\mathbf{X_k}$ in step 1 can be calculated as described in section 2. For large-scale systems (e.g., $n > 15$) it can be obtained as a simple interval over-bounding, see section 3. The interval vector $\mathbf{X}$ computed by Algorithm 1 contains $\bigcap_{k=0}^{K} \mathbf{X_k}$. The main benefit of Algorithm 1 is its relatively low complexity. It requires the solution of $K = m/n$ interval systems of equations.

Algorithm 2 Let $k = 1$. Assume $\mathbf{X} = \mathbf{X}_0$ as an initial interval approximation.

- **Step 1:** Consider the interval system of linear equations from (3) corresponding to the regressors $c_k, \ldots c_{k+n-1}$. Compute the non-singularity radius ρ_k for the nominal matrix A of this system. If $\epsilon < \rho_k$, then find its interval solution $\mathbf{X_k}$, else go to step 3.
- **Step 2:** Find the smallest interval vector $\tilde{\mathbf{X}}$ containing $\mathbf{X} \cap \mathbf{X_k}$. Put $\mathbf{X} = \tilde{\mathbf{X}}$.
- **Step 3:** If $k = m-n + 1$, then terminate, else set $k = k + 1$ and go to step 1.

Algorithm 2 requires the solution of $m-n+1$ interval systems of equations instead of m/n for Algorithm 1, but it provides a more accurate interval estimate.

Example 3 Let $n=2$, $m=40$ and $x=(1,1)^T$ be the parameter vector to be estimated, i.e. there are two parameters and 40 measurements in the model. The data are generated as follows. Take C be a $(m \times n)$ -matrix with rows c_i, which are samples of uniformly distributed vectors on the unit sphere. Interval uncertainty is defined by $\epsilon = 0.2$ and $\delta = 0.5$, and then $\Delta C = 2\varepsilon(\texttt{rand (m,n)} - 0.5)$ and $w = 2\delta(\texttt{rand (m,1)} - 0.5)$. The measurement vector $y \in \mathbb{R}^n$ is taken to be $y = (C + \Delta C)x + w$. These measurements are compatible with model (3) and given parameter vector x. Our purpose is to estimate x under given C, y. Algorithm 1 considers $K = m/n = 20$ linear interval systems. Let the initial interval approximation be $\mathbf{X_0} = ([-5,5],[-5,5])^T$. The interval estimator is constructed as an intersection of the optimal interval solutions for the linear interval systems; see figure 5. Algorithm 1 provides $\mathbf{X_1}$ (dashed line box in figure 6) while Algorithm 2 computes a more precise interval approximation $\mathbf{X_2}$ of the non-convex feasible parameter set X as the intersection of $m-n+1=39$ optimal interval solutions for linear interval systems (solid line box in figure 6). Notice that the decrease in size of interval approximations in both recursive algorithms slows down after an initial rapid decrease. This effect often appears in parameter estimation.

6. Conclusions

In this paper we considered the parameter estimation problem for linear multi-output models under interval uncertainty. The model uncertainty involves additive measurement noise vector and a bounded uncertain regressor matrix. Outer-bounding interval

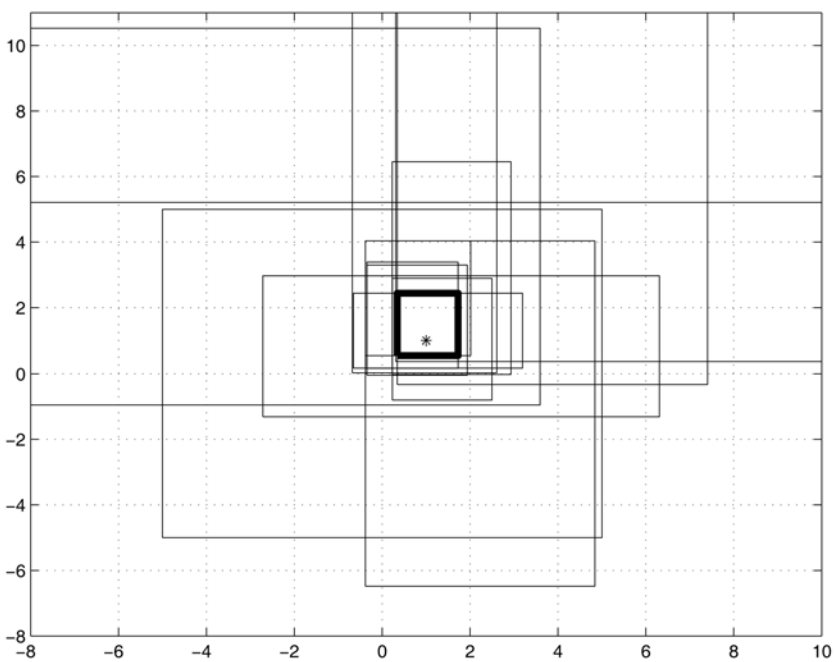

Figure 5. Interval parameter estimator.

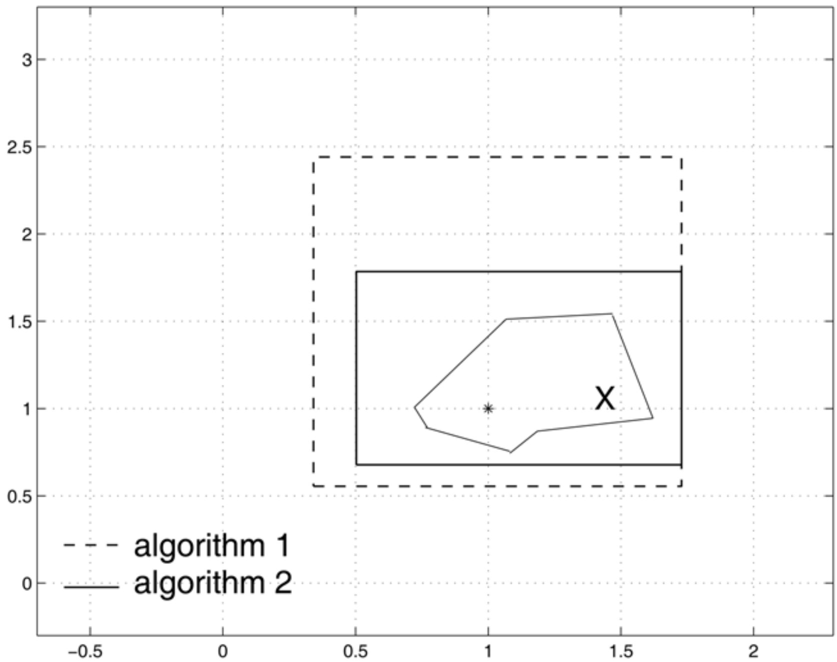

Figure 6. Interval approximations of *X*.

approximations of the non-convex feasible parameter set for this uncertain model are obtained. The algorithms described are based on the computation of interval solutions for square interval systems of linear equations. This approach allows computational difficulties to be avoided and provides a parameter estimator for models with a large number of measurements.

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