ISSN 0005-1179, Automation and Remote Control, 2007, Vol. 68, No. 3, pp. 467–486. © Pleiades Publishing, Ltd., 2007.
Original Russian Text © S.A. Nazin, B.T. Polyak, and M.V. Topunov, 2007, published in Avtomatika i Telemekhanika, 2007, No. 3, pp. 106–125.

Rejection of Bounded Exogenous Disturbances by the Method of Invariant Ellipsoids[1]

S. A. Nazin, B. T. Polyak, and M. V. Topunov

Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia

Received September 12, 2006

Abstract—Rejection of the bounded exogenous disturbances was first studied by the l_1-optimization theory. A new approach to this problem was proposed in the present paper on the basis of the method of invariant ellipsoids where the technique of linear matrix inequalities was the main tool. Consideration was given to the continuous and discrete variants of the problem. Control of the "double pendulum" was studied by way of example.

PACS number: 02.30.Yy

DOI: 10.1134/S0005117907030083

1. INTRODUCTION

Rejection of exogenous disturbances is one of the main problems in the control theory. It is studied both by the linear quadratic Gaussian optimization where the disturbance is assumed to be random and the H_∞-optimization where the noise is regarded as random or belonging to the class L_2, that is, decreasing with time.

The interest in the problem of rejection of the arbitrary bounded exogenous disturbances attracted interest of the researchers as early as in the middle of the last century. B.V. Bulgakov considered the so-called problem of disturbance accumulation already in the 1940's [1]. Yet then the main attention was focused on analysis, that is, on what is the maximum deviation caused by the arbitrary bounded exogenous disturbances, which in essence is the problem of the optimal open-loop control because the exogenous disturbances were considered as controls. The works on compensation of the bounded disturbances (see [2]) appeared much later, though they did not propose any methods for design of optimal controllers.

The problem proper of optimal rejection of the arbitrary bounded disturbances was formulated by E.D. Yakubovich [3], and for some special cases it was solved in [3–5]. The full solution was constructed in the works of A.E. Barabanov and O.N. Granichin [6] and later on, of M.A. Dahleh and J.B. Pearson [7]. This problem was later on christened the l_1-optimization. Nevertheless, the methods of l_1-optimization have some fundamental disadvantages of which we mention only a rather high order of the resulting optimal controllers and the asymptotic nature of the estimates. Along with the l_1-optimal control, methods of dynamic programming for such problems are known as well [8–10].

The aforementioned results concern the discrete systems. Their generalization to the continuous case (L_1-optimization) creates additional problems. On the whole, rejection of the arbitrary bounded disturbances is traditionally regarded as a difficult problem of the control theory [11, 12].

[1] This work was supported in part by the Russian Foundation for Basic Research, projects nos. 05-01-00114 and 05-08-01177, and the Complex Program no. 22 for Basic Research of the Presidium of the Russian Academy of Sciences. The work of S.A. Nazin was supported by the grant of the President of the Russian Federation, project no. MK-1294.2005.8.

We propose another approach to this problem which relies on the method of invariant sets, on the invariant ellipsoids in particular. The invariant sets find rather wide use in various problems of the theory of guaranteed estimation, filtering, and minimax control in the dynamic systems under uncertainties. In this field, the works of F.C. Schweppe [13], D.P. Bertsekas [9, 14], A.B. Kurzhanskii [15], and F.L. Chernous'ko [16] may be regarded as fundamental. We note that the invariant sets often prove to be useful approximations, for example, of the reachability domains of the dynamic systems, which enables their wide use in analysis. Yet, the concept of invariance also is actively used in other divisions of the theory of systems and automatic control (see the review of F. Blanchini [17]). The ellipsoidal approximation of the reachability set of the linear discrete system was discussed in [18].

The present work formulates the problem of rejection of the arbitrary bounded exogenous disturbances in terms of the invariant ellipsoids. Consideration is given to the design of the static state feedback minimizing the size of the invariant ellipsoids of the dynamic system. At that, one succeeds in reducing the original problems of control analysis and design to the equivalent conditions in the form of linear matrix inequalities and the problem of semidefinite programming which readily yield to numerical treatment.

It deserves noting that the popular technique of linear matrix inequalities (LMI) [19] was already used for the similar purposes of disturbance rejection [19–21]. The paper [20] where the authors solve the problem of analysis and design under bounded disturbances for continuous systems provides an example. The present paper used the method of invariant ellipsoids to establish similar results for the discrete systems as well. A more detailed comparison with [20] is made in what follows. The proposed approach enables one to establish simple optimal controllers and, in our view, has great potentiality for generalizations. The paper also considers the continuous and discrete variants of the problem, as well as the numerical example of control of the "double pendulum." Preliminary results were reported at conferences [22, 23].

2. INVARIANT ELLIPSOIDS. ANALYSIS

2.1. Continuous Linear System

Let us consider a continuous-time stationary dynamic linear system

$$\begin{aligned} \dot{x} &= Ax + Dw, \\ y &= Cx, \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system phase state, $y(t) \in \mathbb{R}^l$ is the system output, $w(t) \in \mathbb{R}^m$ are the exogenous disturbances bounded at each time instant, and

$$\|w(t)\| \leqslant 1 \quad \forall\, t \geqslant 0 \tag{2}$$

with $\|\cdot\|$ for the Euclidean norm of the vector. Therefore, we consider the L_∞-bounded exogenous disturbances. We note that no other constraints are imposed on the disturbances $w(t)$—for example, they are not assumed to be random or harmonic.

We assume that system (1) is stable, that is, A is a Hurwitzian matrix with negative real parts, the pair (A, D) is controllable, and C is the maximum-rank matrix. We determine the family of invariant ellipsoids of this system.

Definition 1. The ellipsoid with the center at the origin

$$\mathcal{E}_x = \left\{ x \in \mathbb{R}^n : \quad x^{\mathrm{T}} P^{-1} x \leqslant 1 \right\}, \quad P > 0, \tag{3}$$

is referred to as invariant to the variable x (in state) for the dynamic system (1), (2) if it follows from the condition $x(0) \in \mathcal{E}_x$ that $x(t) \in \mathcal{E}_x$ for all time instants $t \geqslant 0$. P will be called the matrix of the ellipsoid $\mathcal{E}_x$.

Stated differently, any system trajectory $x(t)$ going out of a point lying within the ellipsoid $\mathcal{E}_x$ belongs to this ellipsoid at any time. The ellipsoid invariant to the variable y, that is, system output, is determined in a similar way. It obeys the expression

$$\mathcal{E}_y = \left\{ y \in \mathbb{R}^m : \quad y^{\mathrm{T}} \left(CPC^{\mathrm{T}}\right)^{-1} y \leqslant 1 \right\},$$

where $P > 0$ is the matrix of the invariant ellipsoid $\mathcal{E}_x$.

The invariant ellipsoids may be regarded as the characteristic of the impact of the exogenous disturbances on the trajectories of the dynamic system. In the case under study, we have to estimate the degree of influence of the exogenous disturbances $w(t)$ on the system vector $y(t)$. In this connection, we are interested in the minimal-in-a-sense invariant ellipsoids $\mathcal{E}_y$.

Here we consider the trace criterion

$$f(P) = \operatorname{tr}\left[CPC^{\mathrm{T}}\right] \tag{4}$$

which corresponds to the sum of the quadrates of the semiaxes of the ellipsoid invariant to the output of the original system as the objective function. Some other functions—for example, $g(P) = \det\left[CPC^{\mathrm{T}}\right]$ which is proportional to the volume of the ellipsoid $\mathcal{E}_y$ or the operator norm $h(P) = \|CPC^{\mathrm{T}}\|$ of the matrix CPC^{T} which corresponds to the value of the greatest semiaxis of the ellipsoid $\mathcal{E}_y$—can be regarded as criteria. In virtue of its linearity, the trace criterion (4), however, is the simplest one. Consequently, the degree of influence of the L_∞-bounded exogenous disturbances $w(t)$ on the system output $y(t)$ comes to determining the invariant ellipsoid which is minimal in the criterion $f(P)$. Since the system is assumed to be stable, there exists a finite unique invariant ellipsoid minimizing any of the aforementioned functions.

The invariant ellipsoids were considered before as approximations of the reachable set

$$\mathcal{R} = \left\{x \in \mathbb{R}^n : \quad x = x(t),\ t \geqslant 0 \text{ is the solution of (1), (2) for } x(0) = 0\right\}.$$

It is clear that, generally speaking, $\mathcal{R}$ is not an ellipsoid but rather some closed bounded convex set and at that $\mathcal{R} \subset \mathcal{E}_x$. However, $\mathcal{E}_x$ (even it is is minimal in terms of some criterion) may be a very bad approximation of $\mathcal{R}$ (see an example in [24]). From this point of view, the approach based on the invariant ellipsoids was criticized [24] as overconservative, that is, providing only suboptimal solutions. By the opinion of the present authors, the notion of invariant ellipsoid is more useful and robust as compared with the reachability set where it is assumed that the initial conditions are zero. Yet a small deviation in the initial condition may drive the trajectory outside the boundaries of $\mathcal{R}$. For the invariant ellipsoid, it is possible to take into account the initial uncertainty

$$x(0) \in \mathcal{E}_0 = \left\{x : \quad x^{\mathrm{T}} P_0^{-1} x \leqslant 1\right\}, \quad P_0 > 0,$$

and require that $\mathcal{E}_0 \subset \mathcal{E}_x$, that is,

$$P \geqslant P_0. \tag{5}$$

In what follows, we as a rule include the initial uncertainty, that is, condition (5), in the definition of $\mathcal{E}_x$.

Note 1. If the initial condition $x(0) \neq 0$ is defined directly, then the condition

$$x^{\mathrm{T}}(0)P^{-1}x(0) \leqslant 1$$

which is obviously representable as the matrix linear inequality

$$\begin{pmatrix} I & x^{\mathrm{T}}(0) \\ x(0) & P, \end{pmatrix} \geqslant 0$$

will be used instead of the constraint (5) on the matrix P.

Theorem 1. *The ellipsoid $\mathcal{E}_x$ of the form* (3) *is state-invariant for the dynamic system* (1) *with L_∞-bounded exogenous disturbances if and only if for some $\alpha > 0$ the matrix P satisfies the matrix linear inequality*

$$AP + PA^{\mathrm{T}} + \alpha P + \frac{1}{\alpha}DD^{\mathrm{T}} \leqslant 0, \quad P \geqslant P_0. \tag{6}$$

Theorem 1 is proved in the Appendix.

Corollary 1. *The invariant ellipsoid of system* (1), (2) *which is minimal relative to the criterion $f(P)$ belongs for $\mathcal{E}_0 = \{0\}$ to the one-parameter family of ellipsoids generated by the matrices $P(\alpha)$ satisfying the Lyapunov equation*

$$AP + PA^{\mathrm{T}} + \alpha P + \frac{1}{\alpha}DD^{\mathrm{T}} = 0 \tag{7}$$

over the interval $0 < \alpha < -2\max\limits_i \operatorname{Re}\lambda_i(A)$, where $\lambda_i(A)$ are the eigenvalues of the matrix A. At that, the function $\varphi(\alpha) = \operatorname{tr}\left[CP(\alpha)C^{\mathrm{T}}\right]$ is strictly convex over the mentioned interval.

Corollary 1 is proved in the Appendix.

One can readily see that all minimal invariant ellipsoids satisfy Eq. (7) independently of the particular criterion. This corollary enables one to confine the search for the minimum invariant ellipsoid to the one-parameter family (7), which reduces the problem to the one-dimensional convex minimization over a finite interval.

2.2. Discrete Linear System

Similar definitions will be introduced for the discrete dynamic linear system

$$\begin{aligned} x_{k+1} &= Ax_k + Dw_k, \\ y_k &= Cx_k, \end{aligned} \tag{8}$$

where $x_k \in \mathbb{R}^n$ is the system phase state, $y_k \in \mathbb{R}^l$ is the system output, and $w_k \in \mathbb{R}^m$ are the exogenous disturbances that are bounded at all time instants,

$$\|w_k\| \leqslant 1, \quad k = 0, 1, 2, \ldots . \tag{9}$$

Therefore, we consider the l_∞-bounded exogenous disturbances.

We assume that system (8) is stable, that is, A is a Schur matrix whose eigenvalues lie within the unit circle, the pair (A, D) is controllable, and C is the maximum-rank matrix. We define the family of invariant ellipsoids of the given system.

Definition 2. The ellipsoid centered at the origin

$$\mathcal{E}_x = \left\{ x_k \in \mathbb{R}^n : \quad x_k^{\mathrm{T}} P^{-1} x_k \leqslant 1 \right\}, \quad P > 0, \tag{10}$$

is referred to as invariant to the variable x_k (in state) for the discrete dynamic system (8), (9) if it follows from the condition $x_0 \in \mathcal{E}_x$ that the condition $x_k \in \mathcal{E}_x$ is satisfied at all time instants $k = 1, 2, \dots$. By the matrix of the ellipsoid $\mathcal{E}_x$ is meant the matrix P.

Like in the continuous case, if $\mathcal{E}_x$ defines for system (8) the invariant ellipsoid (in state x_k) with the matrix P, then the ellipsoid

$$\mathcal{E}_y = \left\{ y_k \in \mathbb{R}^m : \quad y_k^{\mathrm{T}} \left(CPC^{\mathrm{T}}\right)^{-1} y_k \leqslant 1 \right\}$$

with the matrix CPC^{T} will be invariant to the output of system y_k. At that, function (4) defines the size of the invariant ellipsoid $\mathcal{E}_y$. We also assume that x_0 lies within the ellipsoid $\mathcal{E}_0$.

Theorem 2. *For the dynamic system* (8) *with the* l_∞*-bounded exogenous disturbances, the ellipsoid* $\mathcal{E}_x$ *of the form* (10) *is invariant if and only if the matrix* P *satisfies the linear matrix inequality*

$$\frac{1}{\alpha} APA^{\mathrm{T}} - P + \frac{1}{1-\alpha} DD^{\mathrm{T}} \leqslant 0, \quad P \geqslant P_0, \tag{11}$$

for some $\alpha \in (0, 1)$.

Theorem 2 is proved in the Appendix.

Corollary 2. *The invariant ellipsoid of system* (8), (9) *which is minimal with respect to the criterion* $f(P)$ *for* $\mathcal{E}_0 = \{0\}$ *belongs to the one-parameter family of the ellipsoids generated by the matrices* $P(\alpha)$ *satisfying the discrete Lyapunov equation*

$$\frac{1}{\alpha} APA^{\mathrm{T}} - P + \frac{1}{1-\alpha} DD^{\mathrm{T}} = 0 \tag{12}$$

over the interval $\rho^2(A) < \alpha < 1$, *where* $\rho(A) = \max_i |\lambda_i(A)|$ *is the spectral radius of the matrix* A. *At that, the function* $\varphi(\alpha) = \operatorname{tr}\left[CP(\alpha)C^{\mathrm{T}}\right]$ *is strictly convex over the mentioned interval.*

Corollary 2 is proved in the Appendix.

Therefore, the search for the minimum invariant ellipsoid comes to the one-dimensional convex problem of minimization in the family generated by Eq. (12).

3. INVARIANT ELLIPSOIDS. DESIGN

To compensate the impact of the arbitrary bounded exogenous disturbance on the output of the stationary dynamic system, we consider a static controller in the form of the state feedback. The proposed approach to the design of control lies in defining the desired optimal controller minimizing the effect of exogenous disturbances by the least invariant ellipsoid of the closed-loop system. This section considers successively the continuous and discrete cases.

3.1. Continuous Controllable System

Let us consider the controllable linear system

$$\begin{aligned} \dot{x} &= Ax + B_1 u + Dw, \quad x(0) \in \mathcal{E}_0, \\ y &= Cx + B_2 u, \end{aligned} \tag{13}$$

where $x \in \mathbb{R}^n$ is the system phase state, $y \in \mathbb{R}^l$ is the system output, $u \in \mathbb{R}^p$ is the control, and $w \in \mathbb{R}^m$ is the exogenous disturbance satisfying the constraint (2). At that, the system matrix A is not assumed to be Hurwitzian, but the matrix pair (A, B_1) is controllable and also $B_2^{\mathrm{T}} C = 0$.

We aim at finding a controller K in the form of a static linear state feedback

$$u = Kx, \tag{14}$$

which stabilizes the closed-loop system and optimally rejects (in the sense of minimality of the trace of the invariant output ellipsoid) the effect of the exogenous disturbances $w(t)$. We note that the presence of the nonzero component $B_2 u$ in (13) is only natural and allows one to avoid great values of control.

With regard for (14), system (13) takes the closed-loop form

$$\begin{aligned} \dot{x} &= (A + B_1 K)x + Dw, \\ y &= (C + B_2 K)x. \end{aligned} \tag{15}$$

The results of Section 2.1 on determination of the minimal invariant ellipsoid give rise to the following theorem where the search for the optimal controller is reduced to the problem of semidefinite programming and one-dimensional convex minimization.

Theorem 3. *For the controllable system* (13), *let the exogenous disturbances be L_∞-bounded and the pair* (A, B_1), *controllable. Then, the problem of designing a static controller by state* (14) *which rejects optimally* (*in the sense of the trace that is output-invariant to the ellipsoid*) *the exogenous disturbances is equivalent to that of minimization of*

$$\operatorname{tr}\left[CPC^{\mathrm{T}} + B_2 Z B_2^{\mathrm{T}}\right] \longrightarrow \min \tag{16}$$

under the constraints

$$AP + PA^{\mathrm{T}} + \alpha P + B_1 Y + Y^{\mathrm{T}} B_1^{\mathrm{T}} + \frac{1}{\alpha} DD^{\mathrm{T}} \leqslant 0, \quad \alpha > 0, \tag{17}$$

$$\begin{pmatrix} Z & Y \\ Y^{\mathrm{T}} & P \end{pmatrix} \geqslant 0, \quad P \geqslant P_0, \tag{18}$$

where $Y = KP$, and minimization is carried out with respect to the variables $\alpha \in \mathbb{R}$, $P = P^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{p\times n}$ and $Z = Z^{\mathrm{T}} \in \mathbb{R}^{p\times p}$.

Theorem 3 is proved in the Appendix.

Note 2. It would be of interest to prove an assertion about the strict convexity of the objective function and the interval boundaries for the parameter α which is similar to Corollary 1. We just note that there exists an $\alpha^* > 0$ such that for $\alpha \geqslant \alpha^*$ the Lyapunov inequality has no positive definite solution and the system of linear matrix inequalities from the condition of Theorem 3 becomes contradictory.

Let $\widehat{\alpha}$, $\widehat{P}$, $\widehat{Y}$, and $\widehat{Z}$ minimize (16) under constraints (17) and (18). Then, the optimal controller is established from the expression $\widehat{K} = \widehat{Y}\widehat{P}^{-1}$. At that,

$$u_{\max} = \max_{x^{\mathrm{T}}\widehat{P}^{-1}x \leqslant 1} \widehat{K}x = \sqrt{\widehat{K}\widehat{P}\widehat{K}^{\mathrm{T}}}.$$

We note that under a fixed α this problem comes to that of minimization of the linear function (16) under constraints (17) and (18) representing the linear matrix inequalities, that is, to the semidefinite programming problem which belongs to the class of convex optimization problems. There exist numerous packages for its numerical solution such as SeDuMi Toolbox, YALMIP Toolbox, and also LMI Toolbox of the MATLAB system.

Within the framework of the present approach to the rejection of the exogenous disturbances, it is only natural to introduce constraints on control. Let $u \in \mathbb{R}^p$, $\mu > 0$, and

$$\|u\| \leqslant \mu. \tag{19}$$

The following lemma reduces the last constraint to an equivalent linear matrix inequality.

Lemma 1. *Let a controllable system* (13) *with L_∞-bounded exogenous disturbances and control like $u = Kx$ be given, P define the matrix of the invariant ellipsoid $\mathcal{E}_x$ of the system, and $Y = KP$. Then, constraint* (19) *amounts to satisfying the linear matrix inequality*

$$\begin{pmatrix} P & Y^{\mathrm{T}} \\ Y & \mu^2 I \end{pmatrix} \geqslant 0 \tag{20}$$

for the matrices P and Y,

Lemma 1 is proved in the Appendix.

Note 3. Lemma 1 fully retains its validity in the discrete case.

For the closed-loop system (15), while proving Theorem 3 we construct a Lyapunov function $V(x)$ such that $\dot{V}(x) \leqslant 0$ for $V(x) \geqslant 1$ and $w^{\mathrm{T}}w \leqslant 1$. It is only natural to seek an L_∞-bounded exogenous disturbance $w^*(t)$ maximizing $\dot{V}(x)$.

Lemma 2. *For the linear continuous controllable system* (13)*, the exogenous disturbance $w^*(t)$ obeys the formula*

$$w^*(t) = \frac{D^{\mathrm{T}}\widehat{P}^{-1}x(t)}{\|D^{\mathrm{T}}\widehat{P}^{-1}x(t)\|}.$$

In particular, for $m = 1$,

$$w^*(t) = \operatorname{sgn}\left(D^{\mathrm{T}}\widehat{P}^{-1}x(t)\right).$$

Lemma 2 is proved in the Appendix.

3.2. Discrete Controllable System

We present a discrete counterpart of the above discussion. Let us consider a discrete linear controllable system

$$\begin{aligned} x_{k+1} &= Ax_k + B_1 u_k + Dw_k, \quad x_0 \in \mathcal{E}_0, \\ y_k &= Cx_k + B_2 u_k, \end{aligned} \tag{21}$$

where $x_k \in \mathbb{R}^n$ is the system phase state, $y_k \in \mathbb{R}^l$ is the system output, $u \in \mathbb{R}^p$ is the control, and $w_k \in \mathbb{R}^m$ is the exogenous disturbance satisfying constraint (9). At that, A is not assumed to be a Schur matrix, but the pair (A, B_1) is controllable, and also $B_2^{\mathrm{T}} C = 0$.

Needed is to determine a controller K in the form of the static linear state feedback

$$u_k = Kx_k, \tag{22}$$

providing the output-invariant ellipsoid that is minimal by the trace criterion (4).

With regard for (22), system (21) takes the closed-loop form

$$\begin{aligned} x_{k+1} &= (A + B_1 K)x_k + Dw_k, \\ y_k &= (C + B_2 K)x_k. \end{aligned} \tag{23}$$

The following theorem is the discrete counterpart of Theorem 3.

Theorem 4. *Let for the discrete controllable system* (21) *the exogenous disturbances be* l_∞*-bounded and the pair* (A, B_1) *be controllable. Then, the problem of designing a controller which is static by the state* (22) *and rejects optimally* (*in the sense of the output trace of the invariant ellipsoid*) *the exogenous disturbances is equivalent to the problem of minimizing the linear function*

$$\operatorname{tr}\left[CPC^{\mathrm{T}} + B_2 Z B_2^{\mathrm{T}}\right] \longrightarrow \min \tag{24}$$

under the constraints

$$\frac{1}{\alpha}\left(APA^{\mathrm{T}} + B_1 Y A^{\mathrm{T}} + AY^{\mathrm{T}} B_1^{\mathrm{T}} + B_1 Z B_1^{\mathrm{T}}\right) - P + \frac{DD^{\mathrm{T}}}{1-\alpha} \leqslant 0, \quad \alpha < 1, \tag{25}$$

$$\begin{pmatrix} Z & Y \\ Y^{\mathrm{T}} & P \end{pmatrix} \geqslant 0, \quad P \geqslant P_0, \tag{26}$$

where $Y = KP$, *minimization is carried out with respect to the variables* $\alpha \in \mathbb{R}$, $P = P^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{p\times n}$ *and* $Z = Z^{\mathrm{T}} \in \mathbb{R}^{p\times p}$.

Theorem 4 is proved in the Appendix.

Let $\widehat{\alpha}$, $\widehat{P}$, $\widehat{Y}$, and $\widehat{Z}$ minimize (24) under constraints (25) and (26). Then, the optimal controller is established from the expression $\widehat{K} = \widehat{Y}\widehat{P}^{-1}$. At that,

$$u_{k\,\max} = \max_{x_k^{\mathrm{T}} \widehat{P}^{-1} x_k \leqslant 1} \widehat{K} x_k = \sqrt{\widehat{K}\widehat{P}\widehat{K}^{\mathrm{T}}}.$$

We note that under constraints (25) and (26) and for fixed α, the minimization problem (24) is that of semidefinite programming.

Note 4. One can readily see that $B_2^{\mathrm{T}}C = 0$ is not a restrictive requirement. If it is not satisfied, then all the above results (in particular, Theorems 3 and 4) retain their validity, and only the relations for the objective functions (16) and (24) will undergo obvious changes:

$$\mathrm{tr}\left[CPC^{\mathrm{T}} + B_2YC^{\mathrm{T}} + CY^{\mathrm{T}}B_2^{\mathrm{T}} + B_2ZB_2^{\mathrm{T}}\right] \longrightarrow \min.$$

The Lyapunov function $V(x_k)$ for closed-loop system (23) such that $V(x_{k+1}) \leqslant 1$ for $V(x_k) \leqslant 1$ and $w^{\mathrm{T}}w \leqslant 1$ is constructed in the course of proving Theorem 4. It is only natural to determine an l_∞-bounded exogenous disturbance w_k^* maximizing $V(x_{k+1})$.

Lemma 3. *For the linear discrete controllable system* (21), *the disturbance* w_k^* *for* $m = 1$ *obeys*

$$w_k^* = \mathrm{sgn}\left(D^{\mathrm{T}}\widehat{P}^{-1}(A + B_1\widehat{K})x_k\right).$$

Lemma 3 is proved in the Appendix.

4. DISCUSSION

We compare the above theoretical findings with the previous results, in particular, with those of [20]:

—the continuous and discrete variants of the problem were considered equally, whereas in [20] only the continuous case was studied;

—in the present paper, instead of the operator norm of the matrix $\|P\|$ [20], the trace criterion $\mathrm{tr}\left[CPC^{\mathrm{T}}\right]$ was used, which enabled us to reduce the problem to the standard semidefinite programming problem and thereby substantially simplify the results;

—constraint (19) on control in the form of the linear matrix inequality (20) was introduced;

—uncertainty (5) of the initial system state was included in the definition of the invariant ellipsoid;

—a single parameter α is contained in the results obtained instead of two parameters as in [20];

—a basically new technique for proving the assertions was used: the S-procedure with two constraints was used instead of the standard S-procedure with one constraint;

—effectiveness of the results obtained was demonstrated by the example of a system of sufficiently high order, the double pendulum problem which is discussed in detail below.

5. CONTROL OF THE DOUBLE PENDULUM

We demonstrate the proposed invariant ellipsoid-based approach to rejection of the exogenous disturbances by the example of control of the double pendulum, that is, a system of two solid bodies of unit masses connected by a spring of unit elastic stiffness and sliding without friction along a fixed horizontal rod (Fig. 1). The name of the model is due to the fact that its equations coincide with the linearized equations describing behavior of the double pendulum.

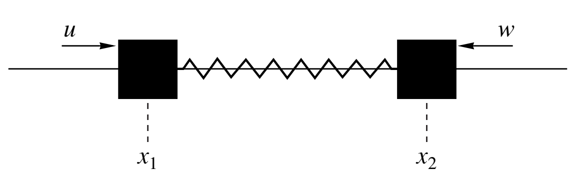

Fig. 1. Double pendulum.

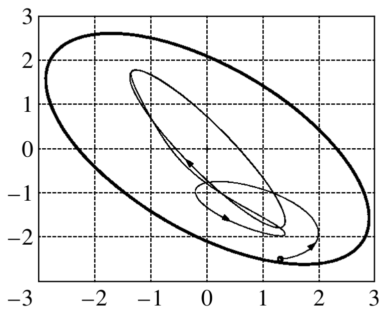

Fig. 2. Invariant output ellipse.

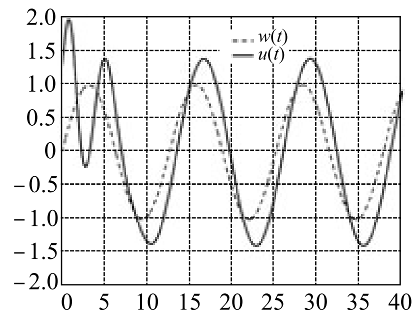

Fig. 3. Disturbance $w(t)$ and control $u(t)$.

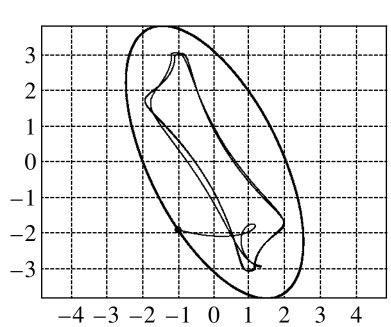

Fig. 4. Invariant output ellipse.

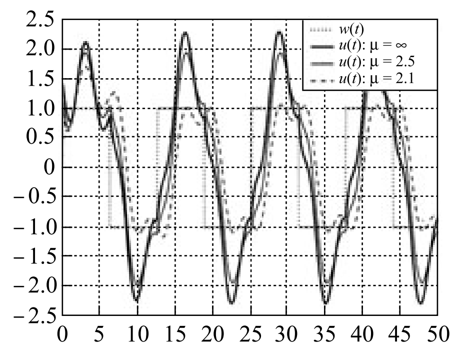

Fig. 5. Disturbance $w(t)$ and control $u(t)$.

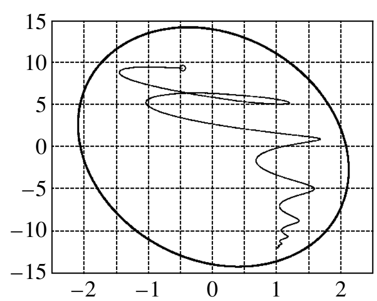

Fig. 6. Invariant output ellipse.

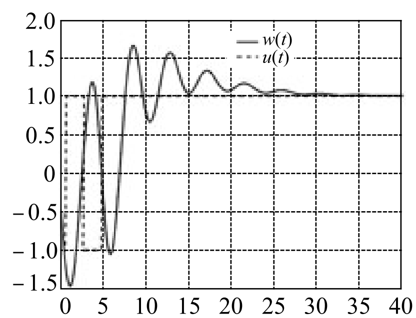

Fig. 7. Disturbance $w(t)$ and control $u(t)$.

The control action $u \in \mathbb{R}$ is applied to the left body with the aim of compensating the effect of the exogenous disturbances $w \in \mathbb{R}$ acting on the right body. The disturbances are assumed to be arbitrary but bounded at any time instant: $|w(t)| \leqslant 1$. We denote by x_1 and v_1, respectively, the coordinate and velocity of the left body, and by x_2 and v_2, of the right body. Then, $x = \begin{pmatrix} x_1 & v_1 & x_2 & v_2 \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^4$ is the vector of the phase state of the given dynamic system which describes completely the motion of the double pendulum. The part of the output is played by the vector $y = \begin{pmatrix} u & x_2 \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^2$ which is characterized by the value of control and the coordinate of the second body subjected to the exogenous disturbances.

We consider the continuous model of disturbed oscillations of the double pendulum

$$\begin{cases} \dot{x}_1 = v_1 \\ \dot{v}_1 = -x_1 + x_2 + u \\ \dot{x}_2 = v_2 \\ \dot{v}_2 = x_1 - x_2 - w, \end{cases}$$

or in the matrix form,

$$\dot{x} = Ax + B_1 u + Dw,$$
$$y = Cx + B_2 u,$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix},$$
$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We note that the matrix A is not stable, but the pair of matrices (A, B_1) is controllable, and also $B_2^{\mathrm{T}} C = 0$.

The optimal controller $\widehat{K}$ minimizing (by the trace criterion) the invariant output ellipse was determined using Theorem 3. At that, the SeDuMi Toolbox and YALMIP Toolbox based on the MATLAB system were used for numerical solution of the problem of semidefinite programming (16) with constraints (17) and (18). As the result, for the system under consideration (no constraint (5) on the initial state) we obtained $\widehat{K} \approx \begin{pmatrix} -2.2724 & -2.3341 & 0.6420 & -1.6564 \end{pmatrix}$ and $u_{\max} \leqslant 2.8913$.

Figure 2 depicts the reslting minimal invariant output ellipse for the closed-loop system with the controller $\widehat{K}$. The same figure shows the trajectory $y(t)$ for some choice of the initial position inside this ellipse and action of the exogenous disturbance $w(t) = \sin(t/2)$ on the system. Figure 3 shows the graph of disturbances $w(t)$ and control $u(t)$.

Figure 4 shows the minimal invariant output ellipse under constraints (5) in the form $P \geqslant I$ and constraint (19) in the form of $\mu = 2.5$. $w(t) = \operatorname{sgn} \sin(t/2)$ was used as the exogenous disturbance. Figure 5 shows the graphs of disturbances $w(t)$ and control $u(t)$ for different values of μ.

Figure 6 shows the minimal invariant output ellipse under the constraint on control $\mu = 2.1$. According to Lemma 2, $w^*(t)$ was used as the exogenous disturbance. Figure 7 shows the graph of disturbance $w^*(t)$ and control $u(t)$.

Within the framework of this example, the discrete case of the disturbed oscillations of the double pendulum also can be demonstrated by approximating its motion by a discrete-time model. Therefore, the proposed method enables effective solution of the given problem.

6. CONCLUSIONS

The paper proposed a simple universal approach to rejection of the arbitrary bounded exogenous disturbances by means of the static linear state feedback. It relies on the method of invariant ellipsoids, which reduces design of the optimal controller to the search of the least invariant ellipsoid of the closed-loop dynamic system. The concept of invariant ellipsoids allows one to restate the problem in terms of the linear matrix inequalities and reduce the controller design straight to the problems of semidefinite programming and one-dimensional convex minimization which readily yield to numerical solution. Effectiveness of the method was demonstrated by way of the double pendulum problem.

We note the possible generalizations of the proposed method and the paths of its further development. The authors used the controller in the form of a static linear state feedback. It seems possible to approach the problem by constructing a dynamic observer-based output feedback. In contrast to the *proper systems* in whose output there are exogenous disturbances, the considered systems, both continuous and discrete, belong to the class of the so-called *strictly proper systems* [20] whose output has no component of the exogenous disturbances. The authors plan to investigate these questions in future.

ACKNOWLEDGMENTS

The authors should like to thank A.S. Nemirovskii who drew their attention to the problem of double pendulum and also A.V. Nazin and P.S. Shcherbakov for valuable discussions, remarks, and suggestions concerning the present paper.

APPENDIX

Assertion A.1 (S-procedure). *Let the uniform quadratic forms* $f_i(x) = x^{\mathrm{T}} A_i x$, $i = 0, 1, \ldots, m$, *in* $\mathbb{R}^n$ *and the numbers* $\alpha_0, \alpha_1, \ldots, \alpha_m \in \mathbb{R}$ *be given. If there exist real numbers* $\tau_i \geqslant 0$, $i = 1, \ldots, m$ *such that*

$$A_0 \leqslant \sum_{i=1}^{m} \tau_i A_i, \quad \alpha_0 \geqslant \sum_{i=1}^{m} \tau_i \alpha_i, \tag{A.1}$$

then it follows from

$$f_i(x) \leqslant \alpha_i, \quad i = 1, \ldots, m, \tag{A.2}$$

that

$$f_0(x) \leqslant \alpha_0. \tag{A.3}$$

Inversely, if (A.3) *follows from* (A.2), *any of the conditions*

(a) $m = 1$ *or*

(b) $m = 2$, $n \geqslant 3$

is satisfied, and there exist numbers $\mu_1, \mu_2 \in \mathbb{R}$ *and vector* $x^0 \in \mathbb{R}^n$ *such that*

$$\mu_1 A_1 + \mu_2 A_2 > 0, \quad f_1(x^0) < \alpha_1, \quad f_2(x^0) < \alpha_2,$$

then there exist numbers $\tau_i \geqslant 0$, $i = 1, \ldots, m$, *for which inequalities* (A.1) *are valid.*

A full proof of this assertion can be found in [25]. More detailed information about the S-procedure—history, theory, applications to control—is given in [26].

Proof of Theorem 1. We consider the quadratic Lyapunov function

$$V(x) = x^{\mathrm{T}} Q x, \quad Q > 0,$$

built on the solutions of the system (1). Then,

$$\dot{V}(x) = \dot{x}^{\mathrm{T}} Q x + x^{\mathrm{T}} Q \dot{x} = (Ax + Dw)^{\mathrm{T}} Q x + x^{\mathrm{T}} Q (Ax + Dw) = x^{\mathrm{T}} (A^{\mathrm{T}} Q + QA) x + 2 w^{\mathrm{T}} D^{\mathrm{T}} Q x.$$

For the trajectories $x(t)$ of system (1) to remain within the boundaries of the ellipsoid

$$\mathcal{E}_x = \{x : \quad V(x) \leqslant 1\},$$

we require that $\dot{V}(x) \leqslant 0$ be satisfied for $V(x) \geqslant 1$, that is,

$$x^{\mathrm{T}} (A^{\mathrm{T}} Q + QA) x + 2 w^{\mathrm{T}} D^{\mathrm{T}} Q x \leqslant 0 \qquad \forall\, (x, w) : \quad x^{\mathrm{T}} Q x \geqslant 1, \quad w^{\mathrm{T}} w \leqslant 1. \tag{A.4}$$

Let $s = \begin{pmatrix} x & w \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^{n+m}$ and

$$M_0 = \begin{pmatrix} A^{\mathrm{T}}Q + QA & QD \\ D^{\mathrm{T}}Q & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} -Q & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix},$$

and also $\widetilde{f}_i(s) = s^{\mathrm{T}} M_i s$, $i = 0, 1, 2$. Then, (A.4) will be rearranged in

$$\widetilde{f}_0(s) \leqslant 0 \quad \forall\, s: \quad \widetilde{f}_1(s) \leqslant -1, \quad \widetilde{f}_2(s) \leqslant 1.$$

Since conditions b. in Assertion A.1 are met, (A.4) is equivalent to the linear matrix inequality

$$M_0 \leqslant \tau_1 M_1 + \tau_2 M_2$$

for some values of τ_1, τ_2 such that $\tau_1 \geqslant \tau_2 \geqslant 0$, or

$$\begin{pmatrix} A^{\mathrm{T}}Q + QA + \tau_1 Q & QD \\ D^{\mathrm{T}}Q & -\tau_2 I \end{pmatrix} \leqslant 0. \tag{A.5}$$

It suffices to consider the case of $\tau_2 > 0$. In the case of equality, we assume that $\tau_2 = \varepsilon > 0$, carry out the following calculations, and then make ε tend to zero. Then, using the Schur formula, inequality (A.5) is rearranged in

$$A^{\mathrm{T}}Q + QA + \tau_1 Q + \frac{1}{\tau_2} QDD^{\mathrm{T}}Q \leqslant 0.$$

By denoting $P = Q^{-1}$ and pre-multiplying and post-multiplying the resulting inequality by P, we get

$$PA^{\mathrm{T}} + AP + \tau_1 P + \frac{1}{\tau_2} DD^{\mathrm{T}} \leqslant 0.$$

Therefore, the condition for invariance of the ellipsoid with the matrix $P > 0$ amounts to satisfying the last linear matrix inequality for some $\tau_1 \geqslant \tau_2 > 0$. Since it is the minimal ellipsoids that are of interest to us,

$$\tau_2 = \tau_{2\max} = \tau_1.$$

By redefining $\tau_1 = \alpha$, we get the desired inequality (6), which proves the theorem. □

Proof of Corollary 1. The first statement of the corollary follows from Lemma A.16 [11, Appendix]: *let A be a Hurwitzian matrix and the pair (A, B) be controllable. Then, for any matrix C, the solution of the Lyapunov equation*

$$AP + PA^{\mathrm{T}} + BB^{\mathrm{T}} = 0$$

under the constraint

$$AP + PA^{\mathrm{T}} + BB^{\mathrm{T}} \leqslant 0$$

is that of the problem

$$\operatorname{tr}\left[CPC^{\mathrm{T}}\right] \longrightarrow \min.$$

Equation (7) is representable as

$$\left(A + \frac{\alpha}{2} I\right) P + P \left(A + \frac{\alpha}{2} I\right)^{\mathrm{T}} = -\frac{1}{\alpha} DD^{\mathrm{T}}$$

and according to [11] has a unique positive definite solution if the matrix $A + \frac{\alpha}{2}I$ is stable (Hurwitzian):

$$\mathrm{Re}\,\lambda_i\left(A + \frac{\alpha}{2}I\right) < 0,$$

that is, $0 < \alpha < -2\max\limits_i \mathrm{Re}\,\lambda_i(A)$.

It remains to prove that the function $\varphi(\alpha) = \mathrm{tr}\left[CP(\alpha)C^{\mathrm{T}}\right]$ is strictly convex over the interval $(0, -2\max\limits_i \mathrm{Re}\,\lambda_i(A))$. According to Lemma A.13 [11, Appendix], solution of Eq. (7) is explicitly representable as

$$P(\alpha) = \int\limits_0^{+\infty} e^{\left(A+\frac{\alpha}{2}I\right)t}\frac{1}{\alpha}DD^{\mathrm{T}}e^{\left(A+\frac{\alpha}{2}I\right)t}dt = \int\limits_0^{+\infty}\frac{e^{\alpha t}}{\alpha}e^{At}DD^{\mathrm{T}}e^{A^{\mathrm{T}}t}dt > 0.$$

Consequently,

$$\varphi(\alpha) = \mathrm{tr}\left[CP(\alpha)C^{\mathrm{T}}\right] = \int\limits_0^{+\infty}\frac{e^{\alpha t}}{\alpha}\mathrm{tr}\left[Ce^{At}DD^{\mathrm{T}}e^{A^{\mathrm{T}}t}C^{\mathrm{T}}\right]dt > 0$$

because C is the maximum-rank matrix and, therefore, $CP(\alpha)C^{\mathrm{T}} > 0$.

We note that the function

$$\alpha \to \frac{e^{\alpha t}}{\alpha}$$

is strictly convex over the interval $\left(0, -2\max\limits_i \mathrm{Re}\,\lambda_i(A)\right)$ for all $t \geqslant 0$, and

$$\mathrm{tr}\left[Ce^{At}DD^{\mathrm{T}}e^{A^{\mathrm{T}}t}C^{\mathrm{T}}\right] \geqslant 0,$$

this inequality being strict for some $t \geqslant 0$. In virtue of the continuous time-dependence of the function $\mathrm{tr}[Ce^{At}DD^{\mathrm{T}}e^{A^{\mathrm{T}}t}C^{\mathrm{T}}]$, the function $\varphi(\alpha)$ is strictly convex over the interval $\left(0,-2\max\limits_i \mathrm{Re}\lambda_i(A)\right)$, which proves the corollary. □

Proof of Theorem 2. We consider the quadratic Lyapunov function

$$V(x_k) = x_k^{\mathrm{T}}Qx_k, \quad Q > 0,$$

constructed on the solutions of system (8). For the trajectories x_k of system (8) to remain the boundaries of the ellipsoid

$$\mathcal{E}_x = \{x_k : \quad V(x_k) \leqslant 1\},$$

we require that $V(x_{k+1}) \leqslant 1$ be satisfied for $V(x_k) \leqslant 1$, that is,

$$(Ax + Dw)^{\mathrm{T}}Q(Ax + Dw) \leqslant 1 \quad \forall\,(x, w): \quad x^{\mathrm{T}}Qx \leqslant 1, \quad w^{\mathrm{T}}w \leqslant 1. \tag{A.6}$$

Let $s = \begin{pmatrix} x & w \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^{n+m}$,

$$M_0 = \begin{pmatrix} A^{\mathrm{T}}QA & A^{\mathrm{T}}QD \\ D^{\mathrm{T}}QA & D^{\mathrm{T}}QD \end{pmatrix}, \quad M_1 = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

and $\widetilde{f}_i(s) = s^{\mathrm{T}}M_is$, $i = 0, 1, 2$. Then, (A.6) is rearranged in

$$\widetilde{f}_0(s) \leqslant 1 \quad \forall\,s: \quad \widetilde{f}_1(s) \leqslant 1, \quad \widetilde{f}_2(s) \leqslant 1.$$

According to Assertion A.1, for some values of $\tau_1, \tau_2 \geqslant 0$ such that $\tau_1 + \tau_2 \leqslant 1$, condition (A.6) is equivalent to the linear matrix inequality

$$\begin{pmatrix} A^{\mathrm{T}}QA - \tau_1 Q & A^{\mathrm{T}}QD \\ D^{\mathrm{T}}QA & D^{\mathrm{T}}QD - \tau_2 I \end{pmatrix} \leqslant 0. \tag{A.7}$$

We note that

$$\tau_1 > 0, \quad \tau_2 > 0$$

follows from the matrix inequality (A.7) because $Q > 0$, $A^{\mathrm{T}}QA > 0$, and $D^{\mathrm{T}}QD > 0$; moreover, $\tau_2 \geqslant \lambda_{\max}(D^{\mathrm{T}}QD) > 0$. It suffices to consider the case of $\tau_2 > \lambda_{\max}(D^{\mathrm{T}}QD)$. In the case of equality, we assume that $\tau_2 = \lambda_{\max}(D^{\mathrm{T}}QD) + \varepsilon$, $\varepsilon > 0$, carry out the following calculations, and make ε tend to zero.

Using the Schur formula, inequality (A.7) is rearranged in

$$A^{\mathrm{T}}QA - \tau_1 Q \leqslant A^{\mathrm{T}}QD(D^{\mathrm{T}}QD - \tau_2 I)^{-1}D^{\mathrm{T}}QA. \tag{A.8}$$

Since we are interested in the minimal, that is, having the greatest matrix Q, ellipsoids, and on the other hand, $D^{\mathrm{T}}QD - \tau_2 I < 0$ must be satisfied, we get

$$\tau_2 = \tau_{2\max} = 1 - \tau_1.$$

According to the lemma of matrix inversion (see [27]),

$$\left(Q^{-1} - (1-\tau_1)^{-1}DD^{\mathrm{T}}\right)^{-1} = Q + QD\left((1-\tau_1)I - D^{\mathrm{T}}QD\right)^{-1}D^{\mathrm{T}}Q,$$

and (A.8) may be rearranged in

$$\tau_1 Q \geqslant A^{\mathrm{T}}\left(Q^{-1} - (1-\tau_1)^{-1}DD^{\mathrm{T}}\right)^{-1}A,$$

or

$$P \geqslant \tau_1^{-1}APA^{\mathrm{T}} + (1-\tau_1)^{-1}DD^{\mathrm{T}}, \quad P = Q^{-1}. \tag{A.9}$$

On the other hand, it follows from inequality (A.9) that

$$I = Q^{1/2}PQ^{1/2} \geqslant (1-\tau_1)^{-1}Q^{1/2}DD^{\mathrm{T}}Q^{1/2}$$

and

$$1 - \tau_1 \geqslant \lambda_{\max}(Q^{1/2}DD^{\mathrm{T}}Q^{1/2}) = \lambda_{\max}(D^{\mathrm{T}}QD).$$

Therefore, conditions (A.7) and (A.9) are equivalent. By redefining $\tau_1 = \alpha$, we obtain (11) from (A.9), which proves the theorem. □

Proof of Corollary 2. The first statement of the corollary follows from a statement similar to Lemma A.16 [11, Appendix]: *let A be a Schur matrix and the pair (A, B) be controllable. Then, for any matrix C, the solution of the discrete Lyapunov equation*

$$APA^{\mathrm{T}} - P + BB^{\mathrm{T}} = 0$$

under the constraint

$$APA^{\mathrm{T}} - P + BB^{\mathrm{T}} \leqslant 0$$

is that of the problem

$$\operatorname{tr}\left[CPC^{\mathrm{T}}\right] \longrightarrow \min.$$

Equation (12) is representable as

$$\left(\frac{A}{\sqrt{\alpha}}\right) P \left(\frac{A}{\sqrt{\alpha}}\right)^{\mathrm{T}} - P = -\frac{1}{1-\alpha} DD^{\mathrm{T}}$$

and according to [11] has a unique positive definite solution if $A/\sqrt{\alpha}$ is a stable (Schur) matrix:

$$\rho\left(\frac{A}{\sqrt{\alpha}}\right) < 1,$$

that is, $\rho^2(A) < \alpha < 1$.

It remains to prove that the function $\varphi(\alpha) = \mathrm{tr}\left[CP(\alpha)C^{\mathrm{T}}\right]$ is strictly convex over the interval $(\rho^2(A), 1)$. According to Lemma A.19 [11, Appendix], the solution of Eq. (12) is explicitly representable as

$$P(\alpha) = \sum_{k=0}^{\infty} \left(\frac{A}{\sqrt{\alpha}}\right)^k \frac{1}{1-\alpha} DD^{\mathrm{T}} \left(\frac{A^{\mathrm{T}}}{\sqrt{\alpha}}\right)^k = \sum_{k=0}^{\infty} \frac{1}{(1-\alpha)\alpha^k} A^k DD^{\mathrm{T}} (A^{\mathrm{T}})^k > 0.$$

Consequently,

$$\varphi(\alpha) = \mathrm{tr}\left[CP(\alpha)C^{\mathrm{T}}\right] = \sum_{k=0}^{\infty} \frac{1}{(1-\alpha)\alpha^k} \mathrm{tr}\left[CA^k DD^{\mathrm{T}} (A^{\mathrm{T}})^k C^{\mathrm{T}}\right] > 0$$

because C is the maximum-rank matrix and, therefore, $CP(\alpha)C^{\mathrm{T}} > 0$.

We note that the function

$$\alpha \to \frac{1}{(1-\alpha)\alpha^k}$$

is strictly convex over the interval $(\rho^2(A), 1)$ for all nonnegative k and

$$\mathrm{tr}\left[CA^k DD^{\mathrm{T}} (A^{\mathrm{T}})^k C^{\mathrm{T}}\right] \geqslant 0,$$

this inequality being strict for some $k \geqslant 0$. As the sum of convex and strictly convex functions, the function $\varphi(\alpha)$ is therefore strictly convex over the interval $(\rho^2(A), 1)$, which proves the corollary. □

Proof of Theorem 3. We use Theorem 1 to determine the minimal output-invariant ellipsoid for the closed-loop system (15) with bounded exogenous disturbances $\|w(t)\| \leqslant 1$. Then, the problem is represented as the minimization

$$\mathrm{tr}\left[(C + B_2K)P(C + B_2K)^{\mathrm{T}}\right] \longrightarrow \min \tag{A.10}$$

under the constraints (5) and

$$(A + B_1K)P + P(A + B_1K)^{\mathrm{T}} + \alpha P + \frac{1}{\alpha} DD^{\mathrm{T}} \leqslant 0, \quad \alpha > 0. \tag{A.11}$$

The variables P and K appear nonlinearly in the matrix inequality (A.11), but after the change $Y = KP$ it takes the form (17). With regard for the introduced variable Y, the objective function in (A.10) can be rearranged in

$$f(P, Y) = \mathrm{tr}\left[CPC^{\mathrm{T}} + B_2YP^{-1}Y^{\mathrm{T}}B_2^{\mathrm{T}}\right].$$

To reduce problem (A.10) to minimization of a linear function, we consider the matrix

$$H = \begin{pmatrix} Z & Y \\ Y^{\mathrm{T}} & P \end{pmatrix}.$$

By the Schur formula, for $P > 0$ the inequality $H \geqslant 0$ is equivalent to $Z \geqslant YP^{-1}Y^{\mathrm{T}}$. Then, minimization of the function $f(P, Y)$ is equivalent to the minimization of $\mathrm{tr}\,[CPC^{\mathrm{T}} + B_2ZB_2^{\mathrm{T}}]$ under the constraints

$$\begin{pmatrix} Z & Y \\ Y^{\mathrm{T}} & P \end{pmatrix} \geqslant 0,$$

whence (18) follows with regard for (5), which proves the theorem. □

Proof of Lemma 1. Since $u = Kx$, the constraint on the control $\|u\| \leqslant \mu$ is representable as

$$x^{\mathrm{T}}K^{\mathrm{T}}Kx \leqslant \mu^2.$$

We consider an ellipsoid with the matrix $P = Q^{-1} > 0$ which is state invariant for the closed-loop system (15). To satisfy the control constraints, we require that

$$x^{\mathrm{T}}K^{\mathrm{T}}Kx \leqslant \mu^2 \qquad \forall x: \quad x^{\mathrm{T}}Qx \leqslant 1 \tag{A.12}$$

be satisfied. It is the classical S-procedure for two quadratic forms. According to Assertion A.1, for satisfaction of (A.12) in case (a) it is necessary and sufficient that there exist a number $\tau \geqslant 0$ such that

$$K^{\mathrm{T}}K \leqslant \tau Q, \quad \tau \leqslant \mu^2.$$

Since it is the minimal invariant ellipsoids that are of interest to us,

$$\tau = \tau_{\max} = \mu^2.$$

Now, let $Y = KP$. Then, $K = YQ$ and

$$QY^{\mathrm{T}}YQ \leqslant \mu^2 Q.$$

By pre-multiplying and post-multiplying the resulting inequality by P, we get

$$Y^{\mathrm{T}}Y \leqslant \mu^2 P,$$

which by means of the Schur formula is representable as (20), which proves the lemma. □

Proof of Lemma 2. For the exogenous disturbances to "push" the system trajectories to the boundaries of the invariant ellipsoid, we require that

$$\dot{V}(x) \longrightarrow \max,$$

where

$$V(x) = x^{\mathrm{T}}\widehat{P}^{-1}x$$

be the Lyapunov function constructed on the solutions of system (13). Since

$$\dot{V}(x) = x^{\mathrm{T}}\left(A_c^{\mathrm{T}}\widehat{P}^{-1} + \widehat{P}^{-1}A_c\right)x + 2w^{\mathrm{T}}D^{\mathrm{T}}\widehat{P}^{-1}x, \quad A_c = A + B_1\widehat{K},$$

we get the problem

$$\max_{(w,w)=1} \left(w, D^{\mathrm{T}} \widehat{P}^{-1} x\right)$$

with the obvious solution

$$w^*(t) = \frac{D^{\mathrm{T}} \widehat{P}^{-1} x(t)}{\|D^{\mathrm{T}} \widehat{P}^{-1} x(t)\|}.$$

In particular, for $m = 1$ we get

$$w^*(t) = \mathrm{sgn}\left(D^{\mathrm{T}} \widehat{P}^{-1} x(t)\right),$$

which proves the lemma. □

Proof of Theorem 4. We use Theorem 2 to determine the minimal output-invariant ellipsoid for the closed-loop system (23) with the bounded exogenous disturbances $\|w_k\| \leqslant 1$. Then, the problem is rearranged in the minimization

$$\mathrm{tr}\left[(C + B_2 K) P (C + B_2 K)^{\mathrm{T}}\right] \longrightarrow \min \tag{A.13}$$

under the constraints (5) and

$$\frac{1}{\alpha}(A + B_1 K) P (A + B_1 K)^{\mathrm{T}} - P + \frac{1}{1-\alpha} D D^{\mathrm{T}} \leqslant 0, \quad \alpha > 0. \tag{A.14}$$

The variables P and K occur nonlinearly in the matrix inequality (A.14), but after the change $Y = KP$ it takes the linear form (25). Now, with regard for the introduced variable Y the objective function in (A.13) is rearranged in

$$f(P, Y) = \mathrm{tr}\left[C P C^{\mathrm{T}} + B_2 Y P^{-1} Y^{\mathrm{T}} B_2^{\mathrm{T}}\right].$$

To reduce problem (A.13) to the minimization of the linear function, we consider the matrix

$$H = \begin{pmatrix} Z & Y \\ Y^{\mathrm{T}} & P \end{pmatrix}.$$

By the Schur formula, for $P > 0$ the inequality $H \geqslant 0$ is equivalent to $Z \geqslant Y P^{-1} Y^{\mathrm{T}}$. Then, minimization of the function $f(P, Y)$ is equivalent to the minimization of $\mathrm{tr}\left[C P C^{\mathrm{T}} + B_2 Z B_2^{\mathrm{T}}\right]$ under the constraint

$$\begin{pmatrix} Z & Y \\ Y^{\mathrm{T}} & P \end{pmatrix} \geqslant 0,$$

whence (26) follows with regard for (5), which proves the theorem. □

Proof of Lemma 3. For the exogenous disturbances to "push" the system trajectories to the boundary of the invariant ellipsoid, we require that

$$V(x_{k+1}) \longrightarrow \max,$$

where

$$V(x_k) = x_k^{\mathrm{T}} \widehat{P}^{-1} x_k$$

is the Lyapunov function constructed on the solutions of system (21). Since

$$V(x_{k+1}) = x_k^{\mathrm{T}} A_c^{\mathrm{T}} \widehat{P}^{-1} A_c x_k + 2w_k^{\mathrm{T}} D^{\mathrm{T}} \widehat{P}^{-1} A_c x_k + w_k^{\mathrm{T}} D^{\mathrm{T}} \widehat{P}^{-1} D w_k,$$
$$A_c = A + B_1 \widehat{K},$$

we get the problem

$$\max_{(w_k, w_k)=1} \Big[(w_k, D^{\mathrm{T}} \widehat{P}^{-1} A_c x_k) + (w_k, D^{\mathrm{T}} \widehat{P}^{-1} D w_k) \Big],$$

which has the obvious solution

$$w_k^* = \operatorname{sgn} \left(D^{\mathrm{T}} \widehat{P}^{-1} (A + B_1 \widehat{K}) x_k \right),$$

provided that the vector of exogenous disturbances is one-dimensional, which proves the lemma. □

REFERENCES

1. Bulgakov, B.V., On Accumulation of Disturbances in the Linear Systems with Constant Parameters, *Dokl. Akad. Nauk SSSR*, 1946, vol. 5, no. 5, pp. 339–342.
2. Ulanov, G.M., *Dinamicheskaya tochnost' i kompensatsiya vozmushchenii v sistemakh avtomaticheskogo upravleniya* (Dynamic Precision and Compensation of Disturbances in the Automatic Control Systems), Moscow: Mashinostroenie, 1971.
3. Yakubovich, E.D., Solution of the Optimal Control Problem for the Linear Discrete Systems, *Avtom. Telemekh.*, 1975, no. 9, pp. 73–79.
4. Barabanov, A.E., Optimal Control of the Nonminimum-phase Discrete Plant with Arbitrary Bounded Noise, *Vestn. Leningr. Gos. Univ., Ser. Mat.*, 1980, vol. 13, pp. 119–120.
5. Vidyasagar, M., Optimal Rejection of Persistent Bounded Disturbances, *IEEE Trans. Automat. Control*, 1986, vol. 31, pp. 527–535.
6. Barabanov, A.E. and Granichin, O.N., Optimal Controller for Linear Plants with Bounded Noise, *Avtom. Telemekh.*, 1984, no. 5, pp. 39–46.
7. Dahleh, M.A. and Pearson, J.B., l_1-Optimal Feedback Controllers for MIMO Discrete-time Systems, *IEEE Trans. Automat. Control*, 1987, vol. 32, pp. 314–322.
8. Glover, D. and Schweppe, F., Control of Linear Dynamic Systems with Set Constrained Disturbances, *IEEE Trans. Automat. Control*, 1971, vol. 16, pp. 411–423.
9. Bertsekas, D.P. and Rhodes, I.B., On the Minimax Reachability of Target Sets and Target Tubes, *Automatica*, 1971, vol. 7, pp. 233–247.
10. Elia, N. and Dahleh, M.A., Minimization of the Worst Case Peak-to-Peak Gain via Dynamic Programming: State Feedback Case, *IEEE Trans. Automat. Control*, 2000, vol. 45, pp. 687–701.
11. Polyak, B.T. and Shcherbakov, P.S., *Robastnaya ustoichivost' i upravlenie* (Robust Stability and Control), Moscow: Nauka, 2002.
12. Polyak, B.T. and Shcherbakov, P.S., Difficult Problems of the Linear Control Theory. Some Approaches, *Avtom. Telemekh.*, 2005, no. 5, pp. 7–46.
13. Schweppe, F.C., *Uncertain Dynamic Systems*, New Jersey: Prentice Hall, 1973.
14. Bertsekas, D.P. and Rhodes, I.B., Recursive State Estimation for a Set-membership Description of Uncertainty, *IEEE Trans. Automat. Control*, 1971, vol. 16, pp. 117–128.
15. Kurzhanskii, A.B., *Upravlenie i nablyudenie v usloviyakh neopredelennosti* (Control and Observation under Uncertainty), Moscow: Nauka, 1977.
16. Chernous'ko, F.L., *Otsenivanie fazovogo sostoyaniya dinamicheskikh sistem* (Estimation of the Phase State of the Dynamic Systems), Moscow: Nauka, 1988.
17. Blanchini, F., Set Invariance in Control—A Survey, *Automatica*, 1999, vol. 35, pp. 1747–1767.

18. Nazin, A.V., Nazin, S.A., and Polyak, B.T., On Convergence of External Ellipsoidal Approximations of the Reachability Domains of Discrete Dynamic Linear Systems, *Avtom. Telemekh.*, 2004, no. 8, pp. 39–61.

19. Boyd, S., El Ghaoui, L., Ferron, E., and Balakrishnan, V., *Linear Matrix Inequalities in System and Control Theory*, Philadelphia: SIAM, 1994.

20. Abedor, J., Nagpal, K., and Poola, K., A Linear Matrix Inequality Approach to Peak-to-Peak Gain Minimization, *Int. J. Robust Nonlinear Control*, 1996, vol. 6, pp. 899–927.

21. Blanchini, F. and Sznaier, M., Persistent Disturbance Rejection via Static State Feedback, *IEEE Trans. Automat. Control*, 1995, vol. 40, pp. 1127–1131.

22. Polyak, B.T. and Topunov, M.V., Rejection of the Bounded Exogenous Disturbances by the Example of the Double-Pendulum Problem, *Abstracts of Papers at the IX E.S. Pyatnitskii Int. Workshop "Stability and Oscillations of the Nonlinear Control Systems, May 31–June 2, 2006*, Moscow: Inst. Probl. Upravlen., 2006, pp. 213–214.

23. Polyak, B.T., Nazin, A.V., Topunov, M.V., and Nazin, S.A., Rejection of Bounded Disturbances via Invariant Ellipsoids Technique, in *Proc. 45th IEEE Conf. Decision Control*, San Diego, 2006.

24. Venkatesh, S. and Dahleh, M., Does Star Norm Capture l_1 Norm? in *Proc. Am. Control Conf.*, 1995, pp. 944–945.

25. Polyak, B.T., Convexity of Quadratic Transformations and Its Use in Control and Optimization, *J. Optim. Theory Appl.*, 1998, vol. 99, pp. 553–583.

26. Gusev, S.V. and Likhtarnikov, A.L., Kalman–Popov–Yakubovich Lemma and the S-procedure: A Historical Essay, *Avtom. Telemekh.*, 2006, no. 11, pp. 77–121.

27. Horn, R. and Johnson, C., *Matrix Analysis*, New York: Cambridge Univ. Press, 1985. Translated under the title *Matrichnyi analiz*, Moscow: Mir, 1989.

This paper was recommended for publication by A.P. Kurdyukov, a member of the Editorial Board