Anisotropy-based control for descriptor systems via output feedback

A. A. Belov

Why Descriptor Systems?

H.S. Wang et al. H_∞ Control for Nonlinear Descriptor Systems, Springer-Verlag London Limited, 2006

SYSTEM MODELS

System dynamics described by equations

$$F(\dot{x}(t), x(t)) = 0 \tag{1}$$

ODE case

$$\dot{x}(t) = f(x(t)) \tag{2}$$

General (DAE) case

$$A(x(t))\dot{x}(t) + f(x(t)) = 0 \tag{3}$$

Each variable has a physical meaning - **description** variable.

Descriptor systems: definitions and applications.

Definition

Descriptor system (implicit system, differential-algebraic system) is a system where state variables describe physical processes.

Applications

- Electrical networks
- Chemistry and biology
- Economics
- Mechanics (multibody dynamical systems, constrained systems)
- Robotics
- Power systems
- Lagre scale systems

Descriptor systems: advantages and disadvantages.

Advantages

- Easy to find math model.
- State variables describe physical processes.
- Ease in operating with large scale systems.
- Number of descriptor systems is bigger than ordinary systems.
- Ordinary systems is a subset of descriptor systems.

Disadvantages

- Algebraic constraints in system equations.
- Singularity of math model.
- Impossible to use analysis and control technique developed for ordinary systems.

Descriptor systems and control problems

- Solvability, controllability, and stability (Boyarincev, Chistyakov, Shcheglova, Campbell, Cobb, Luenberger)
- Canonical and equivalent forms (Shcheglova, Campbell, Vafiadis, Lewis)
- Discretization and numerical solutions (Shcheglova, Gear, Varga, Mehrmann)
- Regularization and stabilization (Bunse-Gerstner, Mehrmann, Nichols, Kaczorek, L. Dai, Varga)
- Model reduction (Stykel, L. Zhang, J. Lam)
- Lyapunov theorems (Chistyakov, Shcheglova, Stykel)
- Observer construction (L. Dai, M. Hou, Paraskevopoulos)
- Robust control (L. Dai, S. Xu, J. Lam, Katayama, Yung, Wang)

Example: RC-chain

P. Kunkel, V. Mehrmann, Differential-Algebraic Equations Analysis and Numerical Solution, 2006

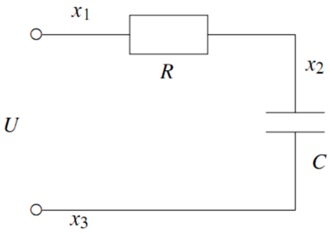

Рис.: RC-chain

Mathematical Model

Let x_i, $i = 1,2,3$ is potential. By Kirchhoff's 1rst law, the sum of the currents vanishes in each node. By choosing the zero potential as $x_3 = 0$ we obtain:

$$\begin{aligned} x_1 - x_3 - U &= 0 \\ C(\dot{x}_3 - \dot{x}_2) + (x_1 - x_2)/R &= 0 \\ x_3 &= 0 \end{aligned}$$

Discrete-time descriptor systems: features

State space equations

$$\begin{aligned} Ex(k+1) &= Ax(k)+Bu(k), \\ y(k) &= Cx(k)+Du(k), \end{aligned} \qquad (4)$$

E - singular matrix.

Differences from ordinary systems

- Regularity condition for existence and uniqueness of the solution – $\det(\lambda E - A) \neq 0$ for any λ.
- Noncausal behavior (i.e. the state vector $x(k)$ can depends on future $u(k+1)$, $u(k+2)$, ... , $u(k+i)$, i is called system index).
- Stability is a property only of dynamical subsystem ($\rho(E,A) < 1$, ρ is generalized spectral radius).
- Causal and stable system is called admissible system.

Causality

Definition

Descriptor system is called causal, if its solution $x(k)$ depends only on $u(k-1),...,u(0)$ и $x(k-1),...,x(0)$ with consistent initial condition $x(0)$.

Example

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(k+1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x(k) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(k),$$

The system has a solution

$$\begin{aligned} x_2(k) &= -u(k) \\ x_1(k) &= -u(k) - u(k+1) \end{aligned}$$

The system is causal if $\deg(\det(zE-A)) = rank(E)$

Descriptor systems affected by disturbances

LQG/H_2-optimization

Disturbance type - Gaussian white noise (output feedback approach - L. Dai 1989, state feedback approach - Nikoukhah et al., 1992, Katayama, 1993).

H_∞-optimization

Disturbance type - disturbance is square-integrable ("finite energy") (output feedback approach - unknown, state feedback approach - Xu and Yang, 2000)

Anisotropy-based optimization

Disturbance type - colored noise (state feedback approach - Belov, 2011)

Mean anisotropy of gaussian sequence and anisotropic norm

Real systems are affected by the noisy signals. Assumption on a noise kind - noise W is a stationary gaussian sequence, generated by a linear admissible filter G. Mean anisotropy of a signal characterizes its spectral color.

Definition

Mean anisotropy of a sequence $W = G \otimes V$ is defined as

$$\overline{A}(G) \equiv -\frac{1}{4\pi} \int_{\Omega} \ln\det\left(\frac{m}{\|G\|_2^2}\widehat{G}(\omega)\left(\widehat{G}(\omega)\right)^*\right) d\omega. \qquad (5)$$

a-anisotropic norm

Let $P(z) \in H_\infty^{p\times m}$,i.e. $P(z)$ has a finite H_∞ norm a-anisotropic norm of P is defined as

$$|||P|||_a = \sup_G \{\|PG\|_2 / \|G\|_2 : \ G \in \mathbf{G}_a\}, \qquad (6)$$

$$\mathbf{G}_a = \left\{G \in H_2^{m\times m} : \ \overline{A}(G) \le a\right\} \qquad (7)$$

Problem statement

$$\begin{aligned} Ex(k+1) &= Ax(k) + B_1 w(k) + B_2 u(k), \\ z(k) &= C_1 x(k) + D_{11} w(k) + D_{12} u(k), \quad (8) \\ y(k) &= C_2 x(k) + D_{21} w(k), \quad (9) \end{aligned}$$

$w(k) \in \mathbb{R}^{m_1}$ - disturbance with known $\overline{A}(W) = a$, $u(k) \in \mathbb{R}^{m_2}$ - control input. Let the system be causal controllable, causal observable, and stabilizable.

Problem is to find a dynamical controller $K \in \mathscr{K}$ which admissibilize the system and minimize a-anisotropic norm of closed-loop system

$$|||\mathscr{F}(F,K)|||_a = \sup \left\{ \frac{\|\mathscr{F}(F,K)G\|_2}{\|G\|_2} : G \in \mathbb{G}_a \right\} \underset{K \in \mathscr{K}}{\to} \inf, \quad (10)$$

Controller structure

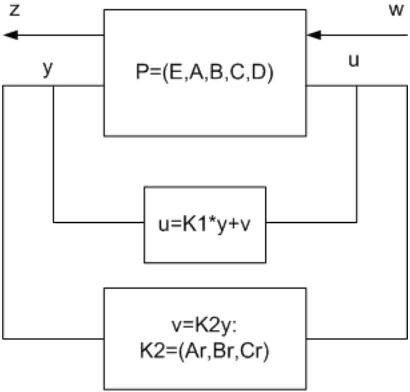

Рис.: Controller structure

Problem solution step 1: causalization procedure

The system is causal observable and causal controllable, i.e. there exists a control law $u(k) = K_1 y(k)$, such that the pair is $(E, A + B_2 K_1 C_2)$ causal.

Causalization algorithm

Equivalent transformation:

$$\begin{aligned} x_1(k+1) &= A_{11}x_1(k) + A_{12}x_2(k) + B_1 u(k), \\ 0 &= A_{21}x_1(k) + A_{22}x_2(k) + B_2 u(k), \\ y(k) &= C_1 x_1(k) + C_2 x_2(k) + Du(k), \end{aligned} \tag{11}$$

where $WEV = \mathrm{diag}(I_r, 0)$,

$$WAV = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad WB = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad CV = \begin{pmatrix} C_1 & C_2 \end{pmatrix}. \tag{12}$$

Problem solution step 1: causalization procedure

The system is noncausal, therefore matrix A_{22} is singular.

Using svd decomposition we have $SA_{22}U = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$. With left and right multiplication by S and U equation $(A_{22} + B_{22}K_1C_{22})$, turns to

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} + \tilde{B}_{22}K_1\tilde{C}_{22}, \tag{13}$$

where $\tilde{B}_{22} = SB_{22}$ и $\tilde{C}_{22} = C_{22}U$.

$$\tilde{B}_{22}K_1\tilde{C}_{22} = I_{n-nf}. \tag{14}$$

Finally

$$K_1 = \tilde{B}_{22}^{+}\tilde{C}_{22}^{+}, \tag{15}$$

where M^{+} is pseudo inverse matrix.

Problem solution step 2: transformation to normal system

Applying control law $u(k) = K_1 y(k) + v(k)$ we get

$$\begin{aligned} Ex(k+1) &= (A + B_2 K_1 C_2)x(k) + (B_1 + B_2 K_1 D_{21})w(k) + B_2 v(k), \\ z(k) &= (C_1 + D_{12} K_1 C_2)x(k) + (D_{11} + D_{12} K_1 D_{21})w(k) + D_{12} v(k), \\ y(k) &= C_2 x(k) + D_{21} w(k), \end{aligned}$$

Using svd decomposition the system is r.s.e to

$$\begin{aligned} \tilde{x}(k+1) &= \widetilde{A}\tilde{x}(k) + \widetilde{B}_1 w(k) + \widetilde{B}_2 v(k), \\ z(k) &= \widetilde{C_1}\tilde{x}(k) + \widetilde{D}_{11} w(k) + D_{12} v(k), \\ y(k) &= \widetilde{C}_2 \tilde{x}(k) + D_{21} w(k), \end{aligned}$$

$$\widetilde{A} = \hat{A}_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21}, \widetilde{B}_i = \hat{B}_{i1} - \hat{A}_{22}^{-1}\hat{B}_{i2},$$
$$\widetilde{C}_i = \hat{C}_{i1} - \hat{C}_{i2}\hat{A}_{22}^{-1}\hat{A}_{21}, \widetilde{D}_{ij} = \hat{D}_{ij} - \hat{C}_{i2}\hat{A}_{22}^{-1}B_2;$$

where

$$\hat{A} = (A + B_2 K_1 C_2), \ \ \hat{B}_1 = (B_1 + B_2 K_1 D_{21}), \ \ \hat{B}_2 = B_2,$$
$$\hat{C}_1 = (C_1 + D_{12} K_1 C_2), \ \ \hat{D}_{11} = (D_{11} + D_{12} K_1 D_{21}), \ \ \hat{D}_{12} = D_{12}, \ \ \hat{C}_2 = C_2, \ \ \hat{D}_{21} = D_{21}.$$

Problem solution step 3: dynamic controller synthesis

After transformation to normal system it is possible to apply standard anisotropy-based optimization procedure.
K_2 has a state-space realization:

$$K_2 \sim \begin{pmatrix} \widetilde{A} + \widetilde{B}_2 N_1 - \Lambda \widetilde{C}_2 & \widetilde{B}_1 M + \widetilde{B}_2 N_2 & \Lambda \\ 0 & \bar{A} & \bar{B} \\ I & 0 & 0 \end{pmatrix}$$

where $\bar{A} = \widetilde{A} + \widetilde{B}_1 M + \widetilde{B}_2 \bar{C} - \Lambda(\widetilde{C}_2 + \widetilde{D}_{21} M)$, $\bar{B} = \Lambda$, $M = L_1 + L_2$, $\bar{C} = N_1 + N_2$.
Matrices M, Λ, N can be found from the solution of 3 Riccati equations, equation of a special case and Lyapunov equation from Vladimirov et al. (1995).

Problem solution step 3: dynamic controller synthesis

- Synthesis of worst case generating filter (WCGF) which provides an input signal with mean anisotropy a. 1 Riccati equation for filter (A, B, C, D) realization, 1 equation of special case to connect with level of mean anisotropy, 1 Lyapunov equation to compute H_2-norm for WCGF.
- Synthesis of dynamical controller based on 1-step prediction. 1 Riccati equation to find $\hat{(}A)$ and $\hat{(}B)$ realizations for controller K_2.
- Synthesis of state feedback controller for "weighted"H_2-problem. 1 Riccati equation to find $\hat{C}$ realization for K_2.

Conclusions

- The anisotropy-based control synthesis is generalized at class of descriptor systems.
- The solution of problem consists of three steps: causalization, equivalent transformation and normal controller synthesis.
- The controller includes two parts: the first one is static feedback control to causalize the system, the second one is dynamical controller to stabilize the dynamical part of the descriptor system with given criterion.

Thank you!