Algorithms for constructing optimal strategies in nonlinear differential games with a goal set

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Joint work with Ivanov G.E.

Motivating example

Nonlinear pendulum with dynamics

$$\begin{cases} \dot{x}(t) = y(t) + 0.3v(t), \\ \dot{y}(t) = -\sin x(t) + u(t), \end{cases} \qquad t \in [0; 1].$$

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$$x_0 = (2.9, -2)$$

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Fixed time problem

$$\dot{x}(t) = a(t, x(t), u(t)) + b(t, x(t), v(t)), \quad t \in [0, \vartheta].$$

$$u : x(\vartheta) \in M, \qquad v : x(\vartheta) \notin M$$

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Time optimal problem

$$\dot{x}(t) = a(x(t), u(t)) + b(x(t), v(t)), \quad t \in [0, \vartheta].$$
$$u : \min \{ T \in [0, \vartheta] | \quad \exists t \in [0, T] : \quad x(t) \in M \}$$
$$v : \max \{ T \in [0, \vartheta] | \quad \forall t \in [0, T] : \quad x(t) \notin M \}$$

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- Our strategies will be piecewise-constant and will depend on position x

Optimal strategies

Fixed time problem

$$u_T^{\text{str}}: \quad \operatorname{dist}(x(\vartheta), M) \le \varepsilon \quad \forall v.$$
$$v_T^{\text{str}}: \quad x(\vartheta) \notin M \quad \forall u.$$

Optimal strategies

Fixed time problem

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Time optimal problem

$$u_T^{\mathrm{str}}: \quad \exists t \in [0, \vartheta]: \quad \mathrm{dist}\,(x(t), M) \leq \varepsilon \quad \forall v$$
$$v_T^{\mathrm{str}}: \quad \exists t_1 \in [0, \vartheta]: \quad x(t) \notin M \quad \forall t \in [0, t_1] \quad \forall u$$

Time discretization $x_{i+1} = x_i + \tau a(\tau_i, x_i, u) + \tau b(\tau_i, x_i, v)$ leads to error of $const \cdot \tau^2$

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For any $S \subset \mathbb{R}^n$, $i \in \overline{0, I-1}$ define A_i, B_i:

$$A_i S = \{x \in \mathbb{R}^n : \quad \exists u \in P : \quad x + \tau a(\tau_i, x, u) \in S\},$$
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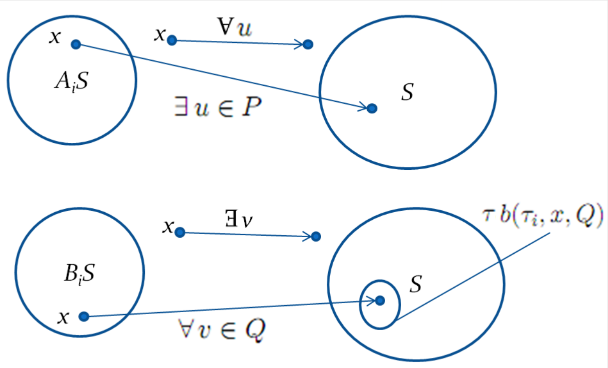

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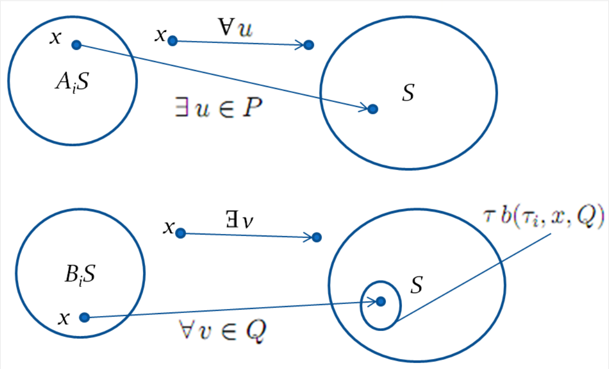

Note: No i for time-optimal problem.

Suppose existence of algorithms

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Sets $A_i S \in \Sigma$, $B_i S \in \Sigma$ for any $i \in \overline{0, I-1}$ s.t.

$$A_i(S \overset{*}{-} \mathfrak{B}_{\varepsilon_A}) \subset \tilde{A}_i S \subset A_i(S + \mathfrak{B}_{\varepsilon_A}),$$
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Vectors $\tilde{u}_i = \tilde{u}_i(x, S) \in P$ for any $i \in \overline{0, I-1}$, $x \in \tilde{A}_iS$ and $\tilde{v}_i = \tilde{v}_i(x, S) \in Q$ for any $i \in \overline{0, I-1}$ and $x \in \mathbb{R}^n \setminus (\tilde{B}_iS)$ s.t.

$$x + \tau a(\tau_i, x, \tilde{u}_i) \in S + \mathfrak{B}_{\varepsilon_u}$$
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Sets $D_u S \in \Sigma$, $D_v S \in \Sigma$ s.t.

$$S \overset{*}{-} \mathfrak{B}_{\Delta_u + \varepsilon_D} \subset D_u S \subset S \overset{*}{-} \mathfrak{B}_{\Delta_u}, \; S + \mathfrak{B}_{\Delta_v} \subset D_v S \subset S + \mathfrak{B}_{\Delta_v + \varepsilon_D},$$

where $\Delta_u = c_1\tau^2 + \varepsilon_B + c_2\varepsilon_u, \Delta_v = c_3\tau^2 + \varepsilon_A + c_4\varepsilon_v$

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$$S \stackrel{*}{-} \mathfrak{B}_{\Delta_u + \varepsilon_D} \subset D_u S \subset S \stackrel{*}{-} \mathfrak{B}_{\Delta_u},\ S + \mathfrak{B}_{\Delta_v} \subset D_v S \subset S + \mathfrak{B}_{\Delta_v + \varepsilon_D},$$

where $\Delta_u = c_1 \tau^2 + \varepsilon_B + c_2 \varepsilon_u, \Delta_v = c_3 \tau^2 + \varepsilon_A + c_4 \varepsilon_v$
Note: No i for time-optimal problem.

Fixed time problem

Define $M_l^u \in \Sigma$, $M_l^v \in \Sigma$ s.t.

$$M + \mathfrak{B}_{\varepsilon-\varepsilon_M} \subset M_l^u \subset M + \mathfrak{B}_{\varepsilon}$$
$$M \subset M_l^v \subset M + \mathfrak{B}_{\varepsilon_M}.$$

Fixed time problem

Define $M_I^u \in \Sigma$, $M_I^v \in \Sigma$ s.t.

$$M + \mathfrak{B}_{\varepsilon - \varepsilon_M} \subset M_I^u \subset M + \mathfrak{B}_{\varepsilon}$$
$$M \subset M_I^v \subset M + \mathfrak{B}_{\varepsilon_M}.$$

Moving from $i = I$ to $i = 0$ calculate sets $\{M_i^u\}_{i=0}^{I-1}$, $\{M_i^v\}_{i=0}^{I-1}$ s.t.

$$M_i^u = \tilde{A}_i \tilde{B}_i D_u M_{i+1}^u$$

$$M_i^v = \tilde{B}_i \tilde{A}_i D_v M_{i+1}^v$$

Fixed time problem

Define $M_l^u \in \Sigma$, $M_l^v \in \Sigma$ s.t.

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$$M \subset M_l^v \subset M + \mathfrak{B}_{\varepsilon_M}.$$

Moving from $i = l$ to $i = 0$ calculate sets $\{M_i^u\}_{i=0}^{l-1}$, $\{M_i^v\}_{i=0}^{l-1}$ s.t.

$$M_i^u = \tilde{A}_i \tilde{B}_i D_u M_{i+1}^u$$

$$M_i^v = \tilde{B}_i \tilde{A}_i D_v M_{i+1}^v$$

Calculate strategies for any $i \in \overline{0, l-1}$ and $x \in \mathbb{R}^n$ s.t.

$$u_i(x) = \tilde{u}_i(x, \tilde{B}_i D_u M_{i+1}^u), (*)$$
$$v_i(x) = \tilde{v}_i(x, \tilde{A}_i D_v M_{i+1}^v). (**)$$

Time-optimal problem

Define $M_0^u \in \Sigma$, $M_0^v \in \Sigma$ s.t.

$$M + \mathfrak{B}_{\varepsilon - \varepsilon_M} \subset M_0^u \subset M + \mathfrak{B}_{\varepsilon},$$
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Moving from $i = 0$ to $i = I - 1$ calculate sets $\{M_i^u\}_{i=0}^{I-1}$, $\{M_i^v\}_{i=0}^{I-1}$ s.t.

$$M_{i+1}^u = \tilde{A}\tilde{B}D_u M_i^u \cup M_i^u,$$
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Calculate strategies for any $x \in \mathbb{R}^n$ s.t.

$$u^{\mathrm{str}}(x) = \tilde{u}(x, \tilde{B}D_u M_i^u),$$
$$v^{\mathrm{str}}(x) = \tilde{v}(x, \tilde{A}D_v M_i^v).$$

Fixed time problem

Theorem

Let $x_0 \in M_0^u$, $u_T^{\mathrm{str}} = \{u_i\}_{i=0}^{l-1}$ *defined by* $(*)$. *Then* u_T^{str} *guarantees* ε*-capture on* $[0, \vartheta]$ *for* $x(0) = x_0$.

Theorem

Let $x_0 \in \mathbb{R}^n \setminus M_0^v$, $v_T^{\mathrm{str}} = \{v_i\}_{i=0}^{l-1}$ *defined by* $(**)$. *Then* v_T^{str} *guarantees evasion on* $[0, \vartheta]$ *for* $x(0) = x_0$.

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$$\varepsilon_0 = \left(C_1\tau + 2\varepsilon_M + (3\varepsilon_A + 3\varepsilon_B + 2\varepsilon_D + 2\varepsilon_u + 2\varepsilon_v)I\right).$$

Theorem

Let $\varepsilon \geq \varepsilon_0$. Then $(u_T^{\mathrm{str}}, v_T^{\mathrm{str}})$ is ε-optimal.

Idea of algorithm

Minkowski sum and difference of two sets $X \subset \mathbb{R}^n$ и $Y \subset \mathbb{R}^n$:

$$X + Y = \{x + y : x \in X,\ y \in Y\},$$
$$X \overset{*}{-} Y = \{x \in \mathbb{R}^n : x + Y \subset X\}.$$

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Note that if $a(t, x, u) = a(t, u)$, $b(t, x, v) = b(t, v)$ then

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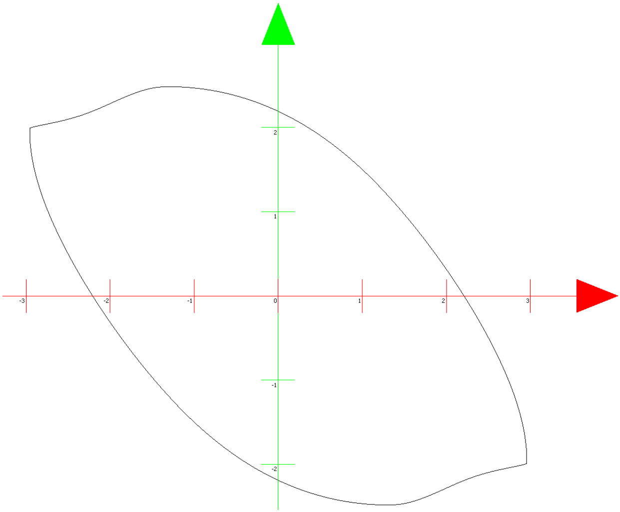
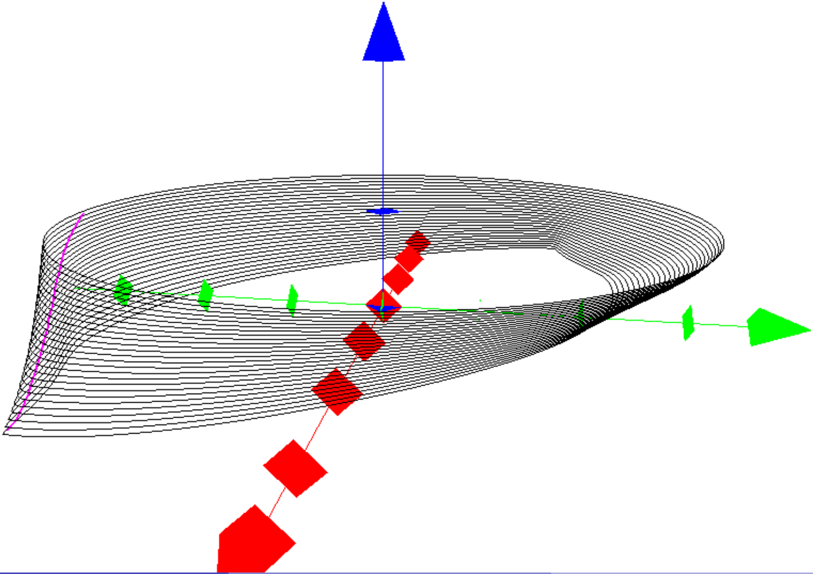
From now on $n = 2$ and Σ contains all polygons with length of all edges not greater than h.

Numerical results 1

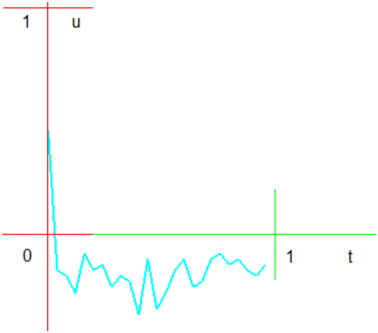


Visualisation thanks to Kirill Chuvilin.

Numerical results 2



Numerical results 3



Thank you!