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Check of Existence of Input-Output Map Realization

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Traditional Youth School “Control, Information and Optimization”
2012, Zvenigorod

Realization Problem

Input-Output Map

$$y_i^{k_i} = \varphi_i\left(t, y, \dot{y}, \ldots, y^{(k-1)}, u, \dot{u}, \ldots, u^{(s)}\right), \quad i = \overline{1, p}, \quad y \in R^p, u \in R^m \qquad (1)$$

Realization problem is to transform system (1) into state-space form

$$\dot{x} = f(t, x, u, \dot{u}, \ldots, u^{(r_0)}), \quad x \in R^n, \qquad (2)$$

$$y = h(t, x, u, \dot{u}, \ldots u^{(r)}) \qquad (3)$$

To do this we need the following functions

$$x = X(t, y, \dot{y}, \ldots, y^{(k-1)}, u, \dot{u}, \ldots, u^{(s_1)}), \qquad (4)$$

The required view of (2)

$$\dot{x} = f(t, x, u), \quad x \in R^n, \quad n = k_1 + \ldots + k_p \qquad (5)$$

Realization Example

$$\ddot{y} = y + 2y\dot{y} + \dot{u}$$

$$\downarrow$$

$$x_1 = y,$$
$$x_2 = \dot{y} - u$$

$$\downarrow$$

$$\dot{x}_1 = x_2 + u,$$
$$\dot{x}_2 = x_1 + 2x_1x_2 + 2x_1u$$

Main Definitions

F is the ring of smooth functions depending on t, y, u and an arbitrary quantity of their derivatives.

$$H_0 = span_F\{dt, dy, d\dot{y}, \ldots, dy^{(k-1)}, du, d\dot{u}, \ldots du^{(s-1)}\} \tag{6}$$

$$H_{k+1} = \{\omega \in H_k : \dot{\omega} \in H_k\} \tag{7}$$

$$D = \frac{\partial}{\partial t} + \sum_{\alpha=1}^{p} \sum_{l_\alpha=0}^{k_\alpha-2} y_\alpha^{(l_\alpha+1)} \frac{\partial}{\partial y_\alpha^{(l_\alpha)}} + \sum_{\alpha=1}^{p} \varphi_\alpha \frac{\partial}{\partial y_\alpha^{(k_\alpha-1)}} + \sum_{\beta=1}^{m} \sum_{k=0}^{2s-2} u_\beta^{(k+1)} \frac{\partial}{\partial u_\beta^{(k)}} \tag{8}$$

$$D_i = span\{ad_D B_1, \ldots, ad_D B_m, \ldots, ad_D^i B_1, \ldots, ad_D^i B_m\}, \quad i = \overline{1, s}, \tag{9}$$

where $B_\beta = \dfrac{\partial}{\partial u_\beta^{(s)}}, \quad \beta = \overline{1, m}$

Conditions of Realization Existence

Theorem 1 (Chetverikov, Krischenko)
State-space realization (5), (3) for the input–output map (1) locally exists if and only if module H_s has a basis consisting of exact 1-forms (i.e. H_s is **integrable**). As X in the change of variables (4) we can choose functions whose differentials are these exact 1-forms.

If such realization exists then $n = k_1 + \ldots + k_p$.

Theorem 2 (Chetverikov, Krischenko)
State-space realization (2), (3) for the input–output map (1) exists if and only if distribution D_s is **integrable**. As the set of X components (change of variables) we can choose the set of first integrals for D_s.

Ways of Checking Integrability

- Symbolic integration of distribution with systems of computer algebra
- Frobenus conditions for distributions

Regular $\widetilde{D} = span_F\{X_0, \ldots, X_k\}$ is integrable if and only if

$$[X_i, X_j] \in \widetilde{D}, \quad 0 \le i < j \le k \qquad (10)$$

- Frobenius conditions for codistributions

If $\{\omega_0, \omega_1, \ldots \omega_n\}$ is a basis of regular codistribution H_s then H_s is integrable if and only if

$$d\omega_i \wedge \omega_0 \wedge \ldots \wedge \omega_n = 0, \quad i = \overline{0, n} \qquad (11)$$

Basis Construction: H_0 Decomposition

Theorem 3. For an arbitrary 1-form $\omega \in H_0$

$$\omega = a_0 dt + \sum_{i=1}^{p} \sum_{l_i=0}^{k_i-1} a_i^{l_i} dy_i^{(l_i)} + \sum_{q=1}^{m} \sum_{k=0}^{s-1} b_q^k du_q^{(k)} \qquad (12)$$

exists unique decomposition

$$\omega = \omega_{s+1} + \sum_{q=1}^{m} \sum_{k=0}^{s-1} c_q^k du_q^{(k)} \, , \qquad (13)$$

with c_q^k can be found from recurrent relations.

Remark. Coefficients c_q^k do not depend on derivatives of $a_i^{l_i}$ and b_q^k

Basis construction

Theorem 3. As a basis we can choose the following system

$$dt, \omega_1^0, \dots, \omega_1^{k_1-1}, \omega_2^0, \dots, \omega_p^{k_p-1},$$

$$\omega_i^{l_i} = dy_i^{l_i} - \sum_{q=1}^{m} \sum_{k=0}^{s-1} c_{i,q}^{l_i,k} du_q^{(k)}, \quad i = \overline{1,p}, l_i = \overline{0, k_i - 1},$$

where $c_{i,q}^{l_i,k} = c_q^k(dy_i^{l_i})$ – coefficients in decomposition of $dy_i^{l_i}$ according to (8).

$$\Downarrow$$

We can check Frobenius conditions

$$d\omega_i \wedge \omega_0 \wedge \dots \wedge \omega_n = 0, \quad i = \overline{0,n}$$

Simplicity, Reliability

How can we find the rank of Functional matrix?

We cannot rely on math programs

$$A := \begin{bmatrix} 100000 & 0 \\ 0 & 0.00001 \end{bmatrix}$$

$$LinearAlgebra:\text{-}Rank(A) = 1$$

Simplified Model of Crane

$$\frac{d^2 y}{dt^2} = -\frac{g \sin y}{R} - \frac{2\dot{y}}{R}\dot{R} - \frac{\cos y}{R}\ddot{D}$$

$$\downarrow$$

Basis + Frobenius theorem

$$\frac{2\cos y}{R^2} - \frac{\cos y}{R^2} \equiv 0$$

$$\downarrow$$

We cannot reduce the order of control derivatives

Fragile Realization

$$\frac{d^2 y}{dt^2} = -\frac{g \sin y}{R} - \frac{\dot{y}}{R}\dot{R} - \frac{\cos y}{R}\ddot{D}$$

$$x_1 = y, \quad x_2 = \dot{D} + \frac{R\dot{y}}{\cos y}$$

$$\downarrow$$

$$\dot{x}_1 = \cos x_1 \frac{x_2 - \dot{D}}{R},$$

$$\dot{x}_2 = \operatorname{tg} x_1 (x_2 - \dot{D} - g)$$

Conclusions

- Module H_0 decomposition is made
- Analytical expressions for H_s basis are found
- Basis of H_s is used to check integrability of the codistribution and existence of a realization