Two-stage Bilevel Quantile Formulation of Logistic Problem

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1. The problem formulation and the terminology

We suggest a mathematical model for planning of a railway transport hub activity. For modeling we use the two-stage bilevel stochastic programming problem with quantile objective function. In the first stage, we choice a number of locomotives for the transport hub and transport prices. In the second stage, when a random transport demand realization turns out to be known, locomotives of the transport hub and leased locomotives distribute for all railway directions. We use a bilevel problem for modeling of a competition between the railway transport hub and a road carrier. The criterion of optimality is losses which arise during the transport hub activity.

Leader is a railway transport hub which takes into account the follower's (competitor's) optimal solution when it choice own solution.
Follower is a competitor (road carrier) which takes into account the leader's (transport hub's) solution when it choice own solution.

2. The first stage problem (symbols)

c is maintenance cost of one locomotive for planning period;

$X \triangleq (\overrightarrow{X}_1, \overrightarrow{X}_2, \dots, \overrightarrow{X}_n, \overleftarrow{X}_1, \overleftarrow{X}_2, \dots, \overleftarrow{X}_n)^{\mathrm{T}}$ is a random demand with realizations $x \triangleq (\overrightarrow{x}_1, \overrightarrow{x}_2, \dots, \overrightarrow{x}_n, \overleftarrow{x}_1, \overleftarrow{x}_2, \dots, \overleftarrow{x}_n)^{\mathrm{T}} \in \mathbb{R}^{2n}$ where
$\overrightarrow{X}_i$ is a demand (in tons) for transportation from the considered transport hub to the i-th transport hub,
$\overleftarrow{X}_i$ is a demand (in tons) for transportation from the i-th transport hub to the considered transport hub,
n is a number of directions from the considered transport hub to other transport hubs;

$u_0 \in \mathbb{Z}$ is a number of locomotives for the transport hub (the first stage solution);
$u \in \mathbb{R}^n$ is a vector, where u_i is a price for transportation of one ton in the i-th direction (the first stage solution);

$\bar{u}_0$ is a maximum available number of locomotives for the maintenance at the transport hub;
$\bar{u}_i$ is a maximum possible price for transportation of one ton in the i-th direction, $i = \overline{1, n}$.

3. The first stage problem

$\Phi(u_0, u, x)$ is losses (an income with inverse sign) of the transport hub (an optimal value of the second stage objective function);

$\Phi_\alpha(u_0, u) \triangleq \min\{\varphi \mid \mathbf{P}\{\Phi(u_0, u, X) \leqslant \varphi\} \geqslant \alpha\}$ is losses of the transport hub which cannot be exceeded with probability α.

The first stage problem:

$$cu_0 + \Phi_\alpha(u_0, u) \to \min_{u, u_0} \tag{1}$$

subject to

$$0 \leqslant u_i \leqslant \bar{u}_i, \quad i = \overline{0, n}.$$

Let $(u_0^*,\ u^*)$ be an optimal solution of problem (1).

4. The second stage problem (symbols)

$y \triangleq (\overrightarrow{y}_1^{\mathrm{T}}, \overleftarrow{y}_1^{\mathrm{T}}, \overrightarrow{y}_2^{\mathrm{T}}, \overleftarrow{y}_2^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{Z}^{4n}$ is the second stage solution (a leader's solution),

$\overrightarrow{y}_{1i}$ is a number of own locomotives which transport cargo from the considered transport hub to the i-th transport hub, $i = \overline{1, n}$;

$\overleftarrow{y}_{1i}$ is a number of own locomotives which transport cargo from the i-th transport hub to the considered transport hub, $i = \overline{1, n}$;

$\overrightarrow{y}_{2i}$ is a number of leased locomotives which transport cargo from the considered transport hub to the i-th transport hub, $i = \overline{1, n}$;

$\overleftarrow{y}_{2i}$ is a number of leased locomotives which transport cargo from the i-th transport hub to the considered transport hub, $i = \overline{1, n}$;

β is a coefficient of customer preferences;

m is a cargo weight (in tons) which one locomotive transports;

d_i is a rent price of one leased locomotive in the i-th direction; $i = \overline{1, n}$;

s_i^t is a cost value of transportation of one train in the i-th direction; $i = \overline{1, n}$;

$\tilde{s}_i^t$ is a cost value of transportation of one locomotive without cargo in the i-th direction; $i = \overline{1, n}$;

$z_1^* \triangleq (\overrightarrow{z}_{11}^*, \overrightarrow{z}_{12}^*, \ldots, \overrightarrow{z}_{1n}^*, \overleftarrow{z}_{11}^*, \overleftarrow{z}_{12}^*, \ldots, \overleftarrow{z}_{1n}^*)^{\mathrm{T}}$ where $\overrightarrow{z}_{1i}^*$, $\overleftarrow{z}_{1i}^*$ are optimal follower's traffic volumes in the corresponding directions, $i = \overline{1, n}$;

$z_2^* \triangleq (z_{21}^*, z_{22}^*, \ldots, z_{2n}^*)^{\mathrm{T}}$ where z_{2i}^* is a follower's optimal transport price (for one ton) in the i-th direction, $i = \overline{1, n}$.

5. The second stage problem (the leader's problem in the optimistic position)

Leader's loss function:

$$\Phi(u_0, u, x) \triangleq \min_{y} \min_{(z_1^*, z_2^*) \in Z(u,y,x)} \sum_{i=1}^{n} ((s_i^t - mu_i)(\overrightarrow{y}_{1i} + \overleftarrow{y}_{1i} + \\ + \overrightarrow{y}_{2i} + \overleftarrow{y}_{2i}) + d_i(\overrightarrow{y}_{2i} + \overleftarrow{y}_{2i}) + \tilde{s}_i^t |\overrightarrow{y}_{1i} - \overleftarrow{y}_{1i}|) \quad (2)$$

subject to

$$\sum_{i=1}^{n} \max\{\overrightarrow{y}_{1i}, \overleftarrow{y}_{1i}\} \leqslant u_0; \quad (3)$$

$$m(\overrightarrow{y}_{1i} + \overrightarrow{y}_{2i}) \leqslant \overrightarrow{x}_i, \quad i = \overline{1, n}; \quad (4)$$

$$m(\overleftarrow{y}_{1i} + \overleftarrow{y}_{2i}) \leqslant \overleftarrow{x}_i, \quad i = \overline{1, n}; \quad (5)$$

$$m(\overrightarrow{y}_{1i} + \overrightarrow{y}_{2i}) \leqslant \overrightarrow{x}_i - \overrightarrow{z}^*_{1i} \text{ if } u_i \geqslant \beta z^*_{2i},\ i = \overline{1, n}; \quad (6)$$

$$m(\overleftarrow{y}_{1i} + \overleftarrow{y}_{2i}) \leqslant \overleftarrow{x}_i - \overleftarrow{z}^*_{1i} \text{ if } u_i \geqslant \beta z^*_{2i},\ i = \overline{1, n}; \quad (7)$$

$$\overrightarrow{y}_{li} \geqslant 0, \overleftarrow{y}_{li} \geqslant 0, \quad l = 1, 2;\ i = \overline{1, n}, \quad (8)$$

where $Z(u, y, x)$ is the set of follower's optimal solutions.

6. The follower's problem (symbols)

(z_1, z_2) is a follower's solution where

$z_1 \triangleq (\overrightarrow{z}_{11}, \overrightarrow{z}_{12}, \ldots, \overrightarrow{z}_{1n}, \overleftarrow{z}_{11}, \overleftarrow{z}_{12}, \ldots, \overleftarrow{z}_{1n})^{\mathrm{T}}$, $\overrightarrow{z}_{1i}$, $\overleftarrow{z}_{1i}$ are follower's traffic volumes (in tons) in the corresponding directions, $i = \overline{1, n}$;
z_{2i} is a follower's transport price (for one ton) in the i-th direction, $i = \overline{1, n}$;

$\bar{z}_1$ is a maximum cargo weight which the follower can transport;

s_i^a is a cost value of transportation of one ton for the follower in the i-th direction, $i = \overline{1, n}$;

$\tilde{s}_i^a$ is a cost value of transportation of an empty transport, which can carry one ton of cargo, in the i-th direction, $i = \overline{1, n}$.

7. The follower's problem

Follower's problem:

$$Z(u,y,x) \triangleq \operatorname{Arg}\min_{z_1,z_2} \sum_{i=1}^{n} \Big((s_i^a - z_{2i})(\overrightarrow{z}_{1i} + \overleftarrow{z}_{1i}) + \\ + \tilde{s}_i^a |\overrightarrow{z}_{1i} - \overleftarrow{z}_{1i}| \Big) \quad (9)$$

subject to

$$\sum_{i=1}^{n} \max\{\overrightarrow{z}_{1i}, \overleftarrow{z}_{1i}\} \leqslant \bar{z}_1; \quad (10)$$

$$0 \leqslant z_{2i} \leqslant \bar{u}_i, \quad i = \overline{1,n}; \quad (11)$$

$$\overrightarrow{z}_{1i} \leqslant \overrightarrow{x}_i, \quad i = \overline{1,n}; \quad (12)$$

$$\overleftarrow{z}_{1i} \leqslant \overleftarrow{x}_i, \quad i = \overline{1,n}; \quad (13)$$

$$\overrightarrow{z}_{1i} \leqslant \overrightarrow{x}_i - m(\overrightarrow{y}_{1i} + \overrightarrow{y}_{2i}) \text{ if } \beta z_{2i} > u_i, \quad i = \overline{1,n}; \quad (14)$$

$$\overleftarrow{z}_{1i} \leqslant \overleftarrow{x}_i - m(\overleftarrow{y}_{1i} + \overleftarrow{y}_{2i}) \text{ if } \beta z_{2i} > u_i, \quad i = \overline{1,n}; \quad (15)$$

$$\overrightarrow{z}_{1i} \geqslant 0, \ \overleftarrow{z}_{1i} \geqslant 0, \quad i = \overline{1,n}. \quad (16)$$

8. Investigation of the problem in the case of one direction

Let $n = 1$, $\overleftarrow{X} \equiv 0$, $\tilde{s}^t = 0$, $\tilde{s}^a = 0$ (there is only one direction and there are only transportations from the considered transport hub) then

$$(\overrightarrow{z}^*_{11}, \overrightarrow{z}^*_2) = \begin{cases} (\min\{f(u_1,x),u_0\}, f(u_1,x) - \min\{f(u_1,x),u_0\}) \\ \text{ if } s^t - mu_1 + d \leqslant 0; \\ (\min\{f(u_1,x),u_0\}, 0) \text{ if } 0 \leqslant s^t - mu_1 + d \leqslant d; \\ (0,0) \text{ if } s^t - mu_1 \geqslant 0, \end{cases}$$

$$\Phi(u_0,u_1,x) = \min\left\{(s^t - mu_1)\min\{f(u_1,x),u_0\}, 0\right\} + \min\left\{(s^t - mu_1 + d)(f(u_1,x) - \min\{u_0, f(u_1,x)\}), 0\right\}. \quad (17)$$

$$f(u_1,x) \triangleq \left[\frac{\overrightarrow{x} - g(u_1,x)}{m}\right],$$

$$g(u_1,x) \triangleq \begin{cases} \max\left\{\frac{\min\left\{\frac{u_1}{\beta}, \bar{u}_1\right\} - s^a}{\bar{u}_1 - s^a}\min\{\bar{z}_1, \overrightarrow{x}\}, 0\right\} \text{ if } \bar{u}_1 > s^a; \\ 0, \text{otherwise.} \end{cases}$$

$[\cdot]$ is an integer part.

Properties of $\Phi(u_0, u_1, x)$:
1) it does not increase on $\overrightarrow{x} \in [0, +\infty)$;
2) it is right-continuous on $\overrightarrow{x} \in [0, +\infty)$.

From Properties 1) and 2) it follows that

$$\Phi_\alpha(u_0, u_1) = \Phi(u_0, u_1, \bar{x}_\alpha), \tag{18}$$

where $\bar{x}_\alpha \triangleq (\overrightarrow{x}_\alpha, 0)^{\mathrm{T}}$, $\overrightarrow{x}_\alpha \triangleq -\min\{\varphi\colon \mathbf{P}\{-\overrightarrow{X} \leqslant \varphi\} \geqslant \alpha\}$.

Deterministic equivalent:

$$cu_0 + \Phi(u_0, u_1, \bar{x}_\alpha) \to \min_{u_0, u_1} \tag{19}$$

subject to

$$0 \leqslant u_i \leqslant \bar{u}_i, \quad i = 0, 1.$$

10. Investigation of the problem in the case of competition absence

In the case of competition absence (that is $\bar{z}_1 = 0$), the second stage problem can be reduce to an integer linear programming problem

$$c_2^{\mathrm{T}} \tilde{y} \to \min_{\tilde{y} \in \mathbb{R}^{5n}} \qquad (20)$$

subject to $B_2 \tilde{y} \geqslant \bar{x}, \quad y \geqslant 0$,
where $\tilde{y} \triangleq (y^{\mathrm{T}}, \psi^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{5n}$, $\bar{x} \triangleq (0, \ldots, 0, x^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{2^n + 4n}$, c_2, B_2, A_2 are matrixes and vectors of corresponding dimensions which depend on parameters of the problem, $\psi \in \mathbb{R}^n$ is a vector of complementary variables.
Let us consider single-stage stochastic linear programming problem with quantile objective function $\hat{\Phi}_\alpha(u_0)$ for loss function

$$\hat{\Phi}(u_0, x) \triangleq \max_{j = \overline{1, J}} \{c u_0 - (v^j)^{\mathrm{T}} A_2 - (v^j)^{\mathrm{T}} \bar{x}\}, \qquad (21)$$

where v^j are vertexes of the set $V \triangleq \{B_2^{\mathrm{T}} v \leqslant c_2,\ v \geqslant 0\}$. Let $\hat{u}^*$ be an optimal solution of the problem of minimizing the function $\hat{\Phi}_\alpha(u_0)$.
The following ratio is true.

$$0 \leqslant c^{\mathrm{T}} u_0^* + \Phi_\alpha(u_0^*, u^*) - \hat{\Phi}_\alpha(\hat{u}_0^*) \leqslant \sum_{i=1}^{n} (4(m\bar{u}_i - s_i^t) - 2d_i + \tilde{s}_i^t). \qquad (22)$$

In this case, $u^* = \bar{u}$.

Case of one direction

Source data: $m = 5\,000$ [t], $\bar{u}_1 = 80$ [\$], $c = 2\,000$ [\$], $d = 5\,000$ [\$], $s^t = 300\,000$ [\$], $\tilde{s}^t = 0$, $\beta = 1$, $s^a = 50$ [\$], $\tilde{s}^a = 0$, $\bar{z}_1 = 100\,000$ [t], $\bar{u}_0 = 400$ [psc.], $\overrightarrow{X} \sim \mathcal{N}(500\,000, 24\,000^2)$, $\overleftarrow{X} \equiv 0$.

Simulation results:

α	u_0^*, psc.	u_1^*, \$	φ^*, \$
0,8	76	80,00	$-7{,}452 \cdot 10^6$
0,9	74	79,77	$-7{,}170 \cdot 10^6$
0,95	72	74,94	$-7{,}056 \cdot 10^6$

Case of two directions without competition

Source data:

$m = 5000$ [t], $\bar{u}_1 = 80$ [\$], $\bar{u}_2 = 100$ [\$], $c = 2\,000$ [\$], $d_1 = 5\,000$ [\$], $d_2 = 8\,000$ [\$], $s_1^t = 300\,000$ [\$], $s_2^t = 350\,000$ [\$], $\tilde{s}_1^t = 30\,000$ [\$], $\tilde{s}_2^t = 35\,000$ [\$], $\bar{z}_1 = 0$, $\bar{u}_0 = 400$ [pcs.], $\overrightarrow{X}_1 \sim \mathcal{N}(500\,000, 24\,000^2)$, $\overrightarrow{X}_2 \sim \mathcal{N}(400\,000, 21\,000^2)$, $\overleftarrow{X}_1 \sim \mathcal{N}(600\,000, 27\,000^2)$, $\overleftarrow{X}_2 \sim \mathcal{N}(350\,000, 26\,000^2)$. Random variables $\overrightarrow{X}_1$, $\overrightarrow{X}_2$, $\overleftarrow{X}_1$, $\overleftarrow{X}_2$ are independent.

Simulation results:

α	u_0^*, pcs.	(u_1^*, u_2^*), \$	φ^*, \$
0,8	166	(80, 100)	$-41{,}891 \cdot 10^6$
0,9	163	(80, 100)	$-41{,}336 \cdot 10^6$
0,95	162	(80, 100)	$-40{,}904 \cdot 10^6$