On a Numerical Method of Solving Conflict Control Problems

Dmitri Kornev

Ural Federal University

Четвертая традиционная молодежная школа
«Управление, информация и оптимизация»
Звенигород,
17–24 июня 2012 года

Problem Statement

Consider a zero-sum differential game

Dynamic Equation

$$\dot{x} = A(t)x + f(t, u, v),\ t_0 \leqslant t < \vartheta,$$
$$x \in \mathbb{R}^n,\ u \in P \subset \mathbb{R}^r,\ v \in Q \subset \mathbb{R}^s, \quad (1)$$

Initial Condition

$$x(t_0) = x_0 \in \mathbb{R}^n, \quad (2)$$

Quality Index

$$\gamma = \mu_1\Big(D_1\big(x(t_1) - c_1\big), \ldots, D_N\big(x(t_N) - c_N\big)\Big). \quad (3)$$

$D_i - d_i \times n$-matrixes, $c_i \in \mathbb{R}^n$,
$t_i \in [t_0, \vartheta]$ $(t_{i+1} > t_i,\ i = \overline{1, N-1},\ t_N = \vartheta)$.

Problem Statement

Quality Index

$$\gamma = \mu_1\Big(D_1\big(x(t_1) - c_1\big), \ldots, D_N\big(x(t_N) - c_N\big)\Big).$$

Quality Index is Positional

It is assumed that there exist such norms $\mu_i\big(y_i, \ldots, y_N\big)$ and $\sigma_i(y_i, \beta)$, that

$$\mu_i(y_i, \ldots, y_N) = \sigma_i(y_i, \beta),$$
$$\beta = \mu_{i+1}(y_{i+1}, \ldots, y_N),$$
$$i = 1, \ldots, N - 1.$$

Differential Game Formalizations

Saddle Point Condition in a Small Game

$$\min_{u \in P} \max_{v \in Q} \langle m, f(t,u,v) \rangle = \max_{v \in Q} \min_{u \in P} \langle m, f(t,u,v) \rangle, m \in \mathbb{R}^n, t \in [t_0, \vartheta]$$

Condition is Valid

(a) Pure positional strategies $u(t,x,\varepsilon)$, $v(t,x,\varepsilon)$;

Condition is Not Valid

(b) Strategy $u(t,x,\varepsilon)$ of the first player — counter strategy $v(t,x,u,\varepsilon)$ of the second player;

(c) Counter strategy $u(t,x,v,\varepsilon)$ of the first player — strategy $v(t,x,\varepsilon)$ of the second player;

(d) Mixed strategies S^u and S^v.

Mixed Strategies

Initial System

$$\dot{x} = A(t)x + f(t, u, v),\ t_0 \leqslant t < \vartheta,$$

$$x \in \mathbb{R}^n,\ u \in P \subset \mathbb{R}^r,\ v \in Q \subset \mathbb{R}^s,$$

Auxilary System

$$\dot{y} = A(t)y + \sum_{l=1}^{N_\varepsilon} \sum_{j=1}^{M_\varepsilon} f(t, u^{[l]}, v^{[j]}) p_l^* q_j^*,\ t_0 \leqslant t < \vartheta,$$

$$x \in \mathbb{R}^n,\ u \in U^*(\cdot) \subset P \subset \mathbb{R}^r,\ v \in V^*(\cdot) \subset Q \subset \mathbb{R}^s,$$

Aims of the Work

- Extension of the numeric method and its software implementation of computing the value of the game (a) to the cases (b)–(d).
- Construction of optimal control laws of the first and second players.

Auxilary Constructions

Consider a partition of the control interval $[t_0, \vartheta]$, that contains the instants t_i:

$$\Delta_k = \{\tau_j : \tau_1 = t_0,\ \tau_j < \tau_{j+1},\ j = 1,\ \ldots, k,\ \tau_{k+1} = \vartheta\}.$$

For $m \in \mathbb{R}^n$, $j = 1, \ldots, k$, set

$$\Delta\psi_j(m) =$$

$$\begin{cases} \int\limits_{\tau_j}^{\tau_{j+1}} \max\limits_{v \in Q} \min\limits_{u \in P} \langle m, X[\vartheta, \tau] f(\tau, u, v) \rangle d\tau, \ \text{(a), (c)} \\ \int\limits_{\tau_j}^{\tau_{j+1}} \min\limits_{u \in P} \max\limits_{v \in Q} \langle m, X[\vartheta, \tau] f(\tau, u, v) \rangle d\tau, \ \text{(b)} \\ \int\limits_{\tau_j}^{\tau_{j+1}} \min\limits_{p^*} \max\limits_{q^*} \langle m, X[\vartheta, \tau] \sum\limits_{l=1}^{N_\varepsilon} \sum\limits_{j=1}^{M_\varepsilon} f(\tau, u^{[l]}, v^{[j]}) p_l^* q_j^* \rangle d\tau, \ \text{(d)} \end{cases}$$

$$m \in \mathbb{R}^n, \quad j = \overline{1, k}.$$

Sets G_j and functions $\varphi_j(m)$

Step by step, in the reverse order, starting from the last point of the partition Δ, define sets G_j of vectors $m \in \mathbb{R}^n$ and scalar functions $\varphi_j(m)$, $m \in G_j$. $\varphi_j(m)$, $m \in G_j$.

- For $j = k+1$, set

$$G_{k+1} = \{m = 0\}, \quad \varphi_{k+1}(m) = 0, \ m \in G_{k+1}.$$

- For current j
  - if $\tau_{j+1} \neq t_i$:

$$G_j = G_{j+1}, \quad \varphi^*_{j+1}(m) = \varphi(m), \ m \in G_j,$$

  - if $\tau_{j+1} = t_i$:

$$\begin{aligned} G_j &= \big\{m = \nu m_* + X^T[t_i, \vartheta] D_i^T l \mid l \in \mathbb{R}^{p_i}, \\ &\qquad \sigma_i^*(l, \nu) \leqslant 1, \ \nu \in [0,1], \ m_* \in G_{j+1}\big\}, \\ \varphi^*_{j+1}(m) &= \max_{m_*, \nu, l} \big[\nu \varphi_{j+1}(m_*) - \langle l, D_i c_i \rangle\big], \ m \in G_j, \end{aligned}$$

Определение множеств G_j и функций $\varphi_j(m)$

Finally, in any case, set

$$\psi_j(m) = \Delta\psi_j(m) + \varphi^*_{j+1}(m),\ m \in G_j,$$

$$\varphi_j(m) = \{\psi_j\}^*_{G_j}(m).$$

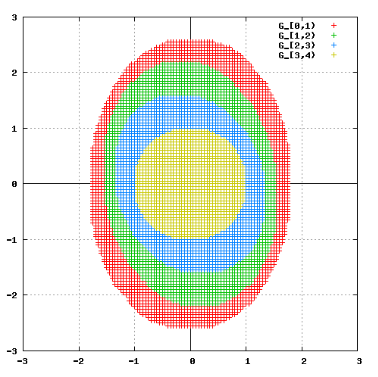
Game Value

$$e_j(x) = \max_{m \in G_j} \big[\langle m, X[\vartheta, \tau_j] x \rangle + \varphi_j(m) \big], \quad j = \overline{1, k+1},\ x \in \mathbb{R}^n.$$

$$\rho(t_0, x_0) \approx e_1(x_0).$$

Data Organization

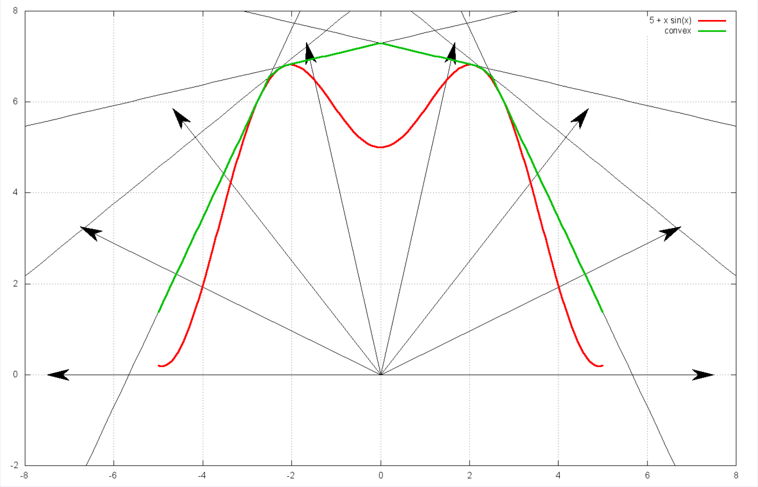
Pixel Method



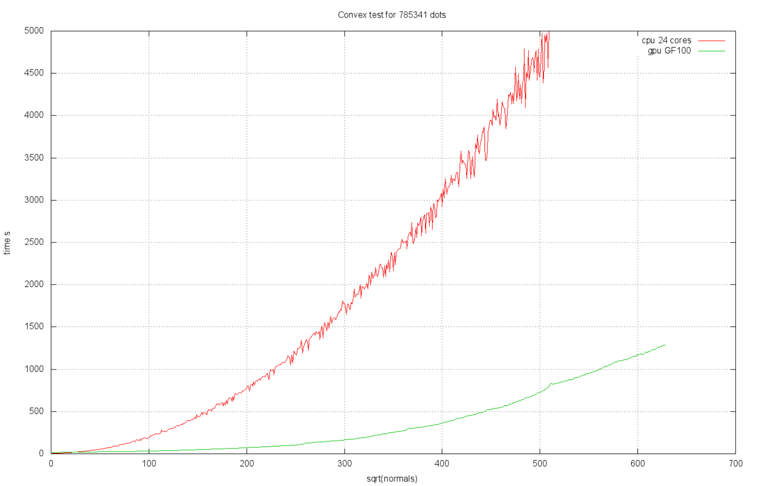
Data Structures

- Functions of time are stored in arrays.
- Functions of vectors m are stored in hash tables.

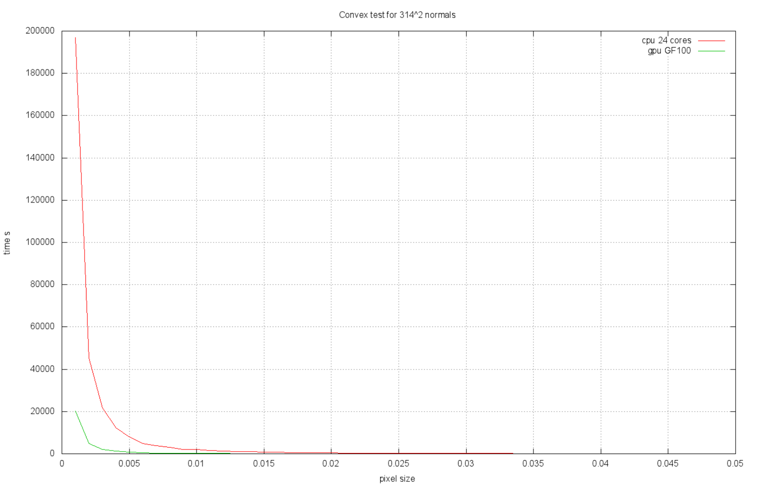
Speed-up of the Upper Convex Hulls Construction of Functions $\psi_j(m)$



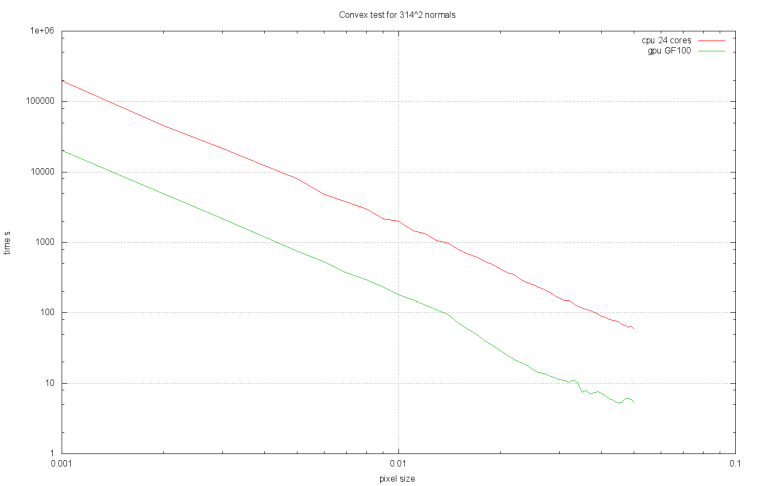
Speed-up of the Upper Convex Hulls Construction of Functions $\psi_j(m)$



Speed-up of the Upper Convex Hulls Construction of Functions $\psi_j(m)$



Speed-up of the Upper Convex Hulls Construction of Functions $\psi_j(m)$



Example

Dynamic Equation

$$\ddot{x} = \frac{4\,u}{1 + e^{\,8(t-2)}} + \frac{(u+v)^2}{2} + \frac{2\,v}{1 + e^{\,8(3-t)}}, \quad t_0 = 0 \leqslant t < \vartheta = 4,$$

$$u \in P = \{-1, 1\},\ v \in Q = \{-1, 1\},$$

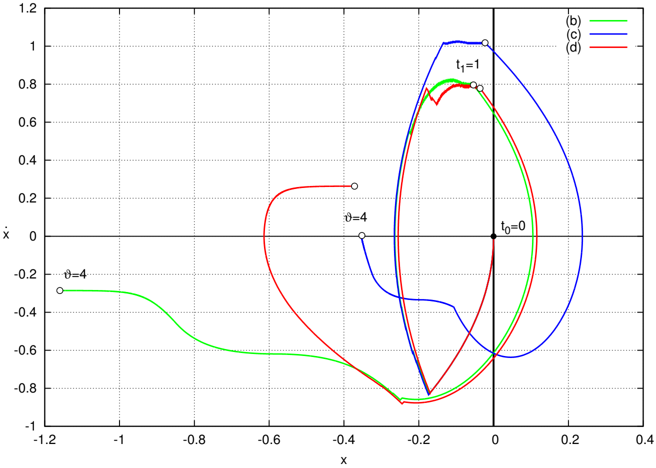
Initial Condition

$$x(0) = 0,\ \dot{x}(0) = 0,$$

Quality Index

$$\gamma = \sqrt{x^2(1) + x^2(4)}.$$

Example



$$\delta = 0.001$$
$$\Delta_m = 0.01$$
$$\Delta_l = 0.01$$
$$\Delta_\nu = 0.01$$
$$\Delta_\phi = \pi/314$$

$$\rho_b \approx 1.13, \ \gamma_b = \sqrt{(-0.053)^2 + (-1.159)^2} \approx 1.16$$

$$\rho_c \approx 0.34, \ \gamma_c = \sqrt{(-0.022)^2 + (-0.351)^2} \approx 0.35$$

$$\rho_d \approx 0.36, \ \gamma_d = \sqrt{(-0.036)^2 + (-0.371)^2} \approx 0.37$$

References

- Krasovskii, N.N. Control of dynamic system. Problem about the minimum guaranteed result. Moscow, Nauka, 1985.
- Krasovskii, A.N. About positional minimax control. In PMM. 1980, T. 1980. №4.
- Krasovskii A.N., Krasovskii N.N. Control under Lack of Information. Berlin etc.: Birkhäuser, 1995. 322 p.
- Lukoyanov, N.Yu. To the question of calculating the value of a differential game with positional functional. In PMM. 1988. T. 62. №2. P. 188–198.

- Krasovskii, A.N. About positional minimax control. In PMM. 1980, T. 1980. №4.

THANK YOU !

Mixed Strategies

Mixed Strategy S^u

$$S^u(\cdot) = \{U(\cdot), p(\cdot); U^*(\cdot), p^*(\cdot); V^*(\cdot), q^*(\cdot)\},$$

Mixed Strategies

Mixed Strategy S^u

$$S^u(\cdot) = \{U(\cdot), p(\cdot); U^*(\cdot), p^*(\cdot); V^*(\cdot), q^*(\cdot)\},$$

$$\begin{aligned} U(\cdot) &= \{U(\varepsilon), \varepsilon > 0\} = \{u^{[l]} \in P, l = 1, \ldots, N_\varepsilon\}, \\ U^*(\cdot) &= U(\cdot), \end{aligned}$$

Mixed Strategies

Mixed Strategy S^u

$$S^u(\cdot) = \{U(\cdot), p(\cdot); U^*(\cdot), p^*(\cdot); V^*(\cdot), q^*(\cdot)\},$$

$$\begin{aligned}
U(\cdot) &= \{U(\varepsilon), \varepsilon > 0\} = \{u^{[l]} \in P, l = 1, \ldots, N_\varepsilon\}, \\
U^*(\cdot) &= U(\cdot), \\
\\
p(\cdot) &= \{p(t, x, y, \varepsilon), t \in [t_0, \vartheta], x \in \mathbb{R}^n, y \in \mathbb{R}^n, \varepsilon > 0\}, \\
p(t, x, y, \varepsilon) &= \{p_l(t, x, y, \varepsilon) \geqslant 0, l = 1, \ldots, N_\varepsilon; \\
&\qquad \sum_{l=1}^{N_\varepsilon} p_l(t, x, y, \varepsilon) = 1\}, \\
p^*(\cdot) &= \{p^*(t, x, y, \varepsilon)\},
\end{aligned}$$

Mixed Strategies

Mixed Strategy S^u

$$S^u(\cdot) = \{U(\cdot), p(\cdot); U^*(\cdot), p^*(\cdot); V^*(\cdot), q^*(\cdot)\},$$

$$\begin{aligned}
U(\cdot) &= \{U(\varepsilon), \varepsilon > 0\} = \{u^{[l]} \in P, l = 1, \ldots, N_\varepsilon\}, \\
U^*(\cdot) &= U(\cdot), \\
\\
p(\cdot) &= \{p(t, x, y, \varepsilon), t \in [t_0, \vartheta], x \in \mathbb{R}^n, y \in \mathbb{R}^n, \varepsilon > 0\}, \\
p(t, x, y, \varepsilon) &= \{p_l(t, x, y, \varepsilon) \geqslant 0, l = 1, \ldots, N_\varepsilon; \\
&\qquad \sum_{l=1}^{N_\varepsilon} p_l(t, x, y, \varepsilon) = 1\}, \\
p^*(\cdot) &= \{p^*(t, x, y, \varepsilon)\}, \\
\\
V^*(\cdot) &= \{V^*(\varepsilon), \varepsilon > 0\} = \{v^{[j]} \in Q, j = 1, \ldots, M_\varepsilon\}, \\
q^*(\cdot) &= \{q^*(t, x, y, \varepsilon)\}.
\end{aligned}$$