Modified barycentric method

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Problem statement

Given n points

$$a \leq x_1 < x_2 < \cdots < x_n \leq b$$

and corresponding function values

$$f(x_1), \ldots, f(x_n),$$

it is necessary to construct the interpolation of function in the interval $[a, b]$.

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Rational interpolation

Let $\mathcal{D}_m$ be a set of rational polynomials

$$D_m(x) = \frac{a_0 + a_1 x + \cdots + a_m x^m}{1 + b_1 x + \cdots + b_m x^m}$$

where m is a polynomial order.
Then

$$\inf_{D \in \mathcal{D}_m} \sup_{x \in [-1,1]} \big| |x| - D(x) \big| \leq \exp(-\sqrt{m}).$$

In practice a_i, $i = 0, \ldots, m$, b_j, $j = 1, \ldots, m$ are defined by linear equations:

$$\sum_{j=0}^{m} a_j x_i^j - f(x_i) \sum_{k=1}^{m} b_k x_i^k = f(x_i).$$

The control of denominator zeros existence is necessary.

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Barycentric interpolation

The barycentric interpolation formula [Floater 2007] is

$$r(x) = \sum_{i=1}^{n-d-1} \pi_i(x) p_i(x) \Big/ \sum_{i=1}^{n-d-1} \pi_i(x),$$

where

$$\pi_i(x) = \frac{(-1)^i}{(x - x_i)(x - x_{i+1}) \dots (x - x_{i+d})}.$$

It is easy to show that

$$r(x_i) = f(x_i), \quad i = 1, \dots, n.$$

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Theorems

Theorem (Floater 2007)

The function $r(x)$ doesn't have poles in $\mathbb{R}$.

Theorem (Floater 2007)

Let $h = \max_i |x_{i-1} - x_i|$ and the norm of $u(x),\ x \in [a, b]$ is defined as $\|u\|_C = \sup_{x\in[a,b]} |u(x)|$. Let $d \geq 1$, $f \in C^{d+2}[a, b]$. If $n - d$ is odd, then

$$\|r - f\|_C \leq h^{d+1} \frac{(b - a)\|f^{(d+2)}\|_C}{d + 2}.$$

If $n - d$ is even, then

$$\|r - f\|_C \leq h^{d+1} \left(\frac{(b - a)\|f^{(d+2)}\|_C}{d + 2} + \frac{\|f^{(d+1)}\|_C}{d + 1} \right).$$

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The drawback of barycentric interpolation

The accuracy of approximation is low if $f(x)$ is discontinuous. The reason is the "long memory" of the method: $|\pi_k(x) p_k(x)|$ decreases slowly as x move away from the interval $[x_k, x_{k+d}]$.

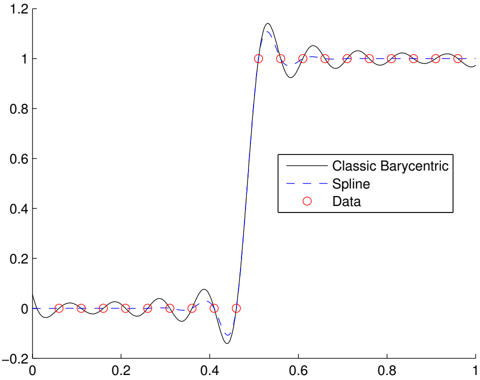

Figure: Gibbs effect for splines and barycentric method

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Modification.

The modified barycentric formula is

$$r_{\text{mod}}(x) = \sum_{i=1}^{n-d-1} \frac{\pi_i(x) p_i(x)}{W_i(x)} \Big/ \sum_{i=1}^{n-d-1} \frac{\pi_i(x)}{W_i(x)}.$$

It is easy to show that $r_{\text{mod}}(x_k) = f(x_k)$.

Theorem

Suppose that for each $i = 1, \ldots, n$ and $x \in [x_i, x_{i+1}]$ the following conditions are valid

1. $W_{k+1}(x) \geq W_k(x)$ *when* $k \geq i+1$,
2. $W_{k-1}(x) \geq W_k(x)$ *when* $k \leq i-d$.

Then $\sup_x |r_{mod}(x)| < \infty$.

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Weights computation

Let $l(x)$ be a linear interpolation on the interval $[x_1, x_n]$ given the data $\{x_k, f(x_k)\}$, $k = 1, \dots n$.

- Mean deviation of $p_i(x)$ from $l(x)$ in $[x_k, x_{k+1}]$

$$d_i(x) = \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} [p_i(u) - l(u)]^2 \, \mathrm{d}u,$$

$$x \in (x_k, x_{k+1}].$$

So $d_i(x)$ is constant in $(x_k, x_{k+1}]$.

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Weights computation

- To fulfil the conditions of poles absence we construct the envelope $D_i(x)$ of $d_i(x)$.

$$D_i(x) = \left\{ \begin{array}{ll} \max\limits_{i \le s < k-d} d_s(x), & i < k-d, \\ d_i(x), & i = k-d, \ldots, k+1, \\ \max\limits_{k+1 < s < i} d_s(x), & i > k+1. \end{array} \right\}.$$

- Finally

$$W_i(x) = D_i(x) + D,$$

where

$$D = \frac{1}{x_n - x_1} \int_{x_1}^{x_n} [f_0(u) - l(u)]^2 du;$$

and

$$f_0(x) = \sum_{i=1}^{n} f(x_i)(x - x_i)^{-4} \Big/ \sum_{i=1}^{n} (x - x_i)^{-4}$$

– the approximation of piecewise-constant interpolation.

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The damping

Such weights definition allows to damp the summands $\pi_i(x)p_i(x)$ if $p_i(x)$ deviates strongly from the linear interpolation for $x \in [x_k, x_{k+1}]$. The constant D is a large value. And the sufficient damping occurs when $D_i(x) \gg D$.

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Modified versus classic barycentric method

Smooth function

$$f(x) = \tanh[6(x-0.5)]/\cosh[8(x-0.5)].$$

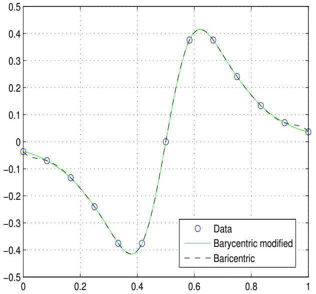

Figure: Interpolation

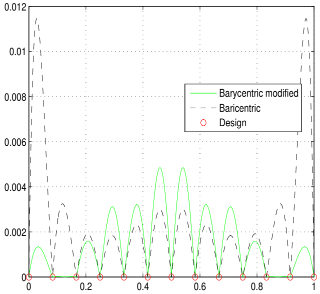

Figure: Errors

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Modified versus standard barycentric method

Discontinuous function

$$f(x) = [-16(x-0.2)^2 + 4] \times \mathbf{1}\{x < 0.4\} + \sin(2\pi X) \times \mathbf{1}\{x >= 0.4\} \times \mathbf{1}\{x < 0.8\} + 0.8 \times \mathbf{1}\{x >= 0.8\}.$$

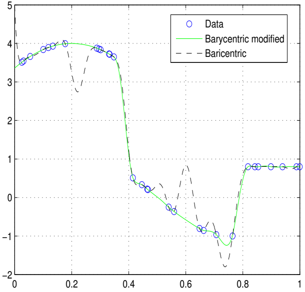

Figure: Interpolation.

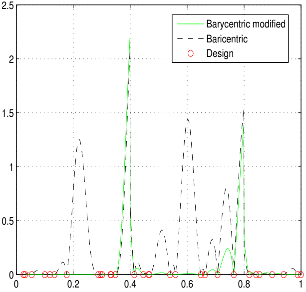

Figure: Errors.

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Comparison of methods

Function are defined in $x \in [0, 1]$. Smooth functions

$$
\begin{aligned}
f_1(x) =& \sin(2\pi x), \\
f_2(x) =& (1 + 25x^2)^{-1}, \\
f_3(x) =& \sinh[2\exp(-60(x - 0.8)^2] - 3\exp[-70(x - 0.5)^2] \\
& + 4\exp[-80(x - 0.3)^2)].
\end{aligned}
$$

Functions $f_4(\cdot)$, $f_5(\cdot)$, $f_6(\cdot)$ have singularities:

$$
\begin{aligned}
f_4(x) =& \mathbf{1}\{x \geq 0.5\}, \\
f_5(x) =& -4\sin(3x) \times \mathbf{1}\{x < 0.36\} \\
& - (1.5 - 4x)^2 \times \mathbf{1}\{X > 0.36\}, \\
f_6(x) =& 1 - |2x - 0.5|.
\end{aligned}
$$

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Comparison of methods. MSE errors.

methods	f_1	f_2	f_3	f_4	f_5	f_6
n=20						
Bar. mod.	1e-3	6e-3	0.68	0.04	0.12	2e-3
Bar. stand.	8e-4	1e-4	1.01	0.09	0.45	6e-3
spline	1e-3	5e-3	0.67	0.05	0.19	3e-3
Spl. ten.	7e-3	5e-3	0.74	0.04	0.11	6e-4
Krig. exp.	4e-2	3e-2	0.83	0.04	0.13	3e-3
Krig. gaus.	4e-7	2e-4	0.52	0.08	0.32	1e-2
n=80						
Bar. mod.	6e-6	9e-6	2e-3	8e-3	0.03	7e-5
Bar. stand.	1e-5	3e-7	2e-4	8e-2	0.21	9e-4
Spline	1e-5	4e-6	8e-3	1e-2	0.04	1e-4
Spl. tens.	3e-5	5e-6	1e-3	7e-3	0.03	3e-5
Krig. exp.	4e-3	1e-3	8e-2	9e-3	0.03	1e-4
Krig. gaus.	7e-9	1e-8	1e-4	6e-2	0.14	4e-3

Conclusion

The modification of barycentric interpolating method is proposed. It allows to increase the interpolation accuracy of discontinuous underlying function without sufficient decrease of interpolation quality in case of smooth underlying function.