ℓ_1-*APPROACHES TO OPTIMIZATION AND CONTROL*

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Statement of the problem

The control system:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

$x \in \mathbb{R}^n$ is the state vector

$y \in \mathbb{R}^l$ is the output

$u \in \mathbb{R}^p$ is the control

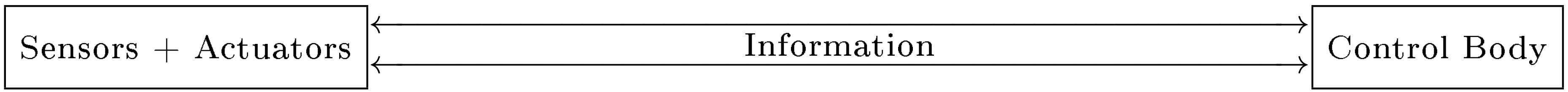
the pair (A, B) is controllable, the pair (A, C) is observable

The goal is to design a stabilizing controller K in the form

$$u = Kx \quad \text{or} \quad u = Ky$$

which minimizes one of the following criteria:

- the number of control inputs $\Longrightarrow$ synonymous with actuators
- the number of states used $\Longrightarrow$ synonymous with sensors/measurements
- the number of outputs used $\Longrightarrow$ synonymous with information transmitted



The Lyapunov approach

The quadratic form

$$V(x) = x^\top Q x, \quad Q \succ 0$$

is a Lyapunov function for the closed-loop system $\iff$

$$A_c^\top Q + Q A_c \prec 0$$

or equivalently

$$A_c P + P A_c^\top \prec 0, \quad P = Q^{-1}$$

For the state feedback $(u = Kx)$ we have $A_c = A + BK$ and arrive at

$$AP + PA^\top + BKP + PK^\top B^\top \prec 0, \quad P \succ 0$$

For the output feedback $(u = Ky)$ we have $A_c = A + BKC$ and obtain

$$AP + PA^\top + BKCP + PC^\top K^\top B^\top \prec 0, \quad P \succ 0$$

K and P are matrix variables

- The matrix inequalities are nonlinear and nonconvex in K and P
- Static output feedback design is generically hard

Sparse controller

- Zero rows in K:

$$u = \underbrace{\begin{pmatrix} \cdots\cdots\cdots\cdots\cdots \\ 0 & 0 & \cdots & 0 \\ \cdots\cdots\cdots\cdots\cdots \\ 0 & 0 & \cdots & 0 \\ \cdots\cdots\cdots\cdots\cdots \end{pmatrix}}_{K} x$$

$\Longrightarrow$ small number of control inputs used

- Zero columns in K:

$$u = \underbrace{\begin{pmatrix} \cdots & 0 & \cdots & 0 & \cdots \\ \cdots & 0 & \cdots & 0 & \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots & 0 & \cdots & 0 & \cdots \end{pmatrix}}_{K} x$$

$\Longrightarrow$ small number of states used

Recall the vector ℓ_0-norm

Let $x \in \mathbb{R}^n$

Define the ℓ_0-norm of the vector x as

$$\|x\|_0 \doteq \sum_{i=1}^{n} |\operatorname{sign} x_i|$$

where

$$\operatorname{sign} x = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ +1, & x > 0 \end{cases}$$

- $\|x\|_0$ is the number of nonzero components of the vector x
- $\|x\|_0$ is not a norm (no homogeneity!)
- The unit ball $\mathcal{S} = \left\{x \in \mathbb{R}^n : \ \|x\|_0 \leqslant 1\right\}$ is nonconvex (the coordinate axes)
- The nonconvex problem:

$$\|x\|_0 \longrightarrow \min \quad \text{s.t.} \quad g(x) \leqslant 0$$

ℓ_1-optimization

Let $x \in \mathbb{R}^n$

Recall the 1-norm of the vector:

$$\|x\|_1 \doteq \sum_{i=1}^{n} |x_i|$$

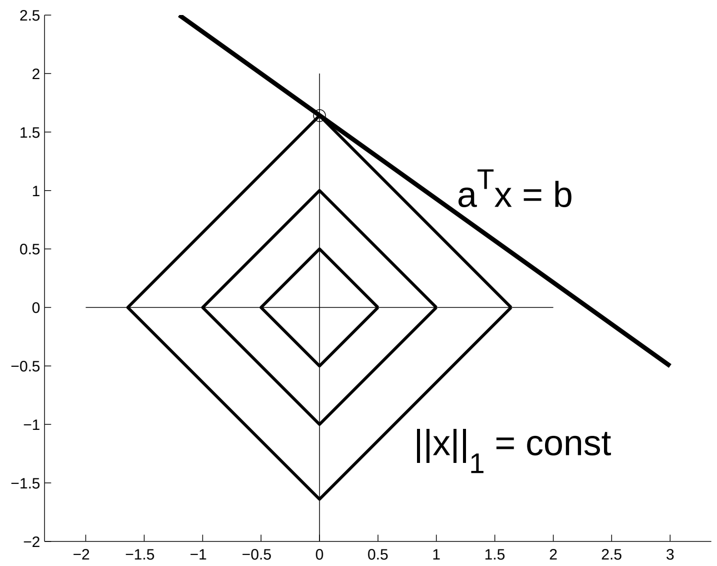
vector 1-norm minimization $\Longrightarrow$ zero components

Lemma 1. *If the problem*

$$\|x\|_1 \longrightarrow \min \quad s.t. \quad Ax = b,$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $m < n$, is feasible, then there exists a solution $\widehat{x}$ having no more than m nonzero components.

Example. $m = 1$, $n = 2$



Special matrix norms

Let $X \in \mathbb{R}^{n\times p}$

Introduce the following matrix norms:

$$\|X\|_{r_1} = \sum_{i=1}^{n} \max_{1\leqslant j\leqslant p} |x_{ij}|$$

$$\|X\|_{c_1} = \sum_{j=1}^{n} \max_{1\leqslant i\leqslant p} |x_{ij}|$$

r_1-norm minimization $\Longrightarrow$ zero rows in the matrix

c_1-norm minimization $\Longrightarrow$ zero columns in the matrix

Lemma 2. *If the problem*

$$\|X\|_{r_1} \longrightarrow \min \quad s.t. \quad AX = B,$$

where $A \in \mathbb{R}^{m\times n}$, $X \in \mathbb{R}^{n\times p}$, $B \in \mathbb{R}^{m\times p}$, $m < n$, is feasible, then there exists a solution $\widehat{X}$ having at most m nonzero rows.

- For $p = 1$ we arrive at Lemma 1
- A similar result is true for c_1-norm

Matrix multiplication: Retaining the structure

- Post-multiplication of a matrix with zero rows:

$$\begin{pmatrix} \cdots\cdots\cdots\cdots\cdots \\ \mathbf{0} \quad \mathbf{0} \quad \cdots \quad \mathbf{0} \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \end{pmatrix} \times \begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} = \begin{pmatrix} \cdots\cdots\cdots\cdots\cdots \\ \mathbf{0} \quad \mathbf{0} \quad \cdots \quad \mathbf{0} \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \end{pmatrix}$$

$\Longrightarrow$ keeps the zero rows at the same positions

- Pre-multiplication of a matrix with zero columns:

$$\begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} \times \begin{pmatrix} \cdots & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \mathbf{0} & \cdots \end{pmatrix} = \begin{pmatrix} \cdots & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \mathbf{0} & \cdots \end{pmatrix}$$

$\Longrightarrow$ keeps the zero columns at the same positions

r_1-optimization

Control system:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

The goal is to design a stabilizing controller $u = Kx$ with a reduced number of controls

The Lyapunov approach:

$$AP + PA^\top + BKP + PK^\top B^\top \prec 0, \quad P \succ 0$$

Introduce the matrix variable $Y = KP \implies$ LMI

$$AP + PA^\top + BY + Y^\top B^\top \prec 0, \quad P \succ 0$$

If the matrix Y has a reduced number of nonzero rows

$\Downarrow$

the controller matrix $K = YP^{-1}$ has a reduced number of nonzero rows

$\Downarrow$

a reduced number of the controls exploited

Linear matrix inequalities (LMIs)

The relation

$$AP + PA^\top + BY + Y^\top B^\top \prec 0$$

is referred to as a Linear Matrix Inequality (LMI) in the matrix variables P, Y

On the positive:

- convexity
- dimensions
- efficient solution tools
- minimization of a convex (linear) objective function subject to an LMI
- software
- wide range of applications

Sturm J.F. Using SeDuMi 1.02, a Matlab toolbox for optimization over symmetric cones (updated for version 1.05). URL `http://sedumi.ie.lehigh.edu/`

Löfberg J. YALMIP: Software for solving convex (and nonconvex) optimization problems. URL `http://control.ee.ethz.ch/~joloef/wiki/pmwiki.php`

Grant M., Boyd S. CVX: Matlab software for disciplined convex programming (web page and software). URL `http://stanford.edu/~boyd/cvx`.

r_1-optimization (cont.)

Proposition 1. *Let $\widehat{Y}$ and $\widehat{P}$ be the solution of the SDP*

$$\|Y\|_{r_1} \longrightarrow \min$$

subject to

$$AP + PA^\top + BY + Y^\top B^\top \prec 0, \qquad P \succ 0.$$

Then the matrix

$$\widehat{K} = \widehat{Y}\widehat{P}^{-1}$$

defines a row-sparse state feedback controller, so that some of the components of the control vector u are zeros.

- We detected the control inputs sufficient for stabilization
- The problem is reduced to an SDP
- Different type constraint

$$AP + PA^\top + BY + Y^\top B^\top \preccurlyeq -2\alpha P$$

guarantees the degree $\alpha > 0$ of stability of the closed-loop system

Example 1: Linearized dynamics of Bell201-A helicopter

$$A = \begin{pmatrix} -0.0046 & 0.038 & 0.3259 & -0.0045 & -0.402 & -0.073 & -9.81 & 0 \\ -0.1978 & -0.5667 & 0.357 & -0.0378 & -0.2149 & 0.5683 & 0 & 0 \\ 0.0039 & -0.0029 & -0.2947 & 0.007 & 0.2266 & 0.0148 & 0 & 0 \\ 0.0133 & -0.0014 & -0.4076 & -0.0654 & -0.4093 & 0.2674 & 0 & 9.81 \\ 0.0127 & -0.01 & -0.8152 & -0.0397 & -0.821 & 0.1442 & 0 & 0 \\ -0.0285 & -0.0232 & 0.1064 & 0.0709 & -0.2786 & -0.7396 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.0676 & 0.1221 & -0.0001 & -0.0016 \\ -1.1151 & 0.1055 & 0.0039 & 0.0035 \\ 0.0062 & -0.0682 & 0.001 & -0.0035 \\ -0.017 & 0.0049 & 0.1067 & 0.1692 \\ -0.0129 & 0.0106 & 0.2227 & 0.143 \\ 0.139 & 0.0059 & 0.0326 & -0.407 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Leibfritz F., Lipinski W. Description of the benchmark examples in COMPleib 1.0. Technical report. University of Trier, 2003. URL: `www.complib.de`

Example 1 (cont.)

$$\widehat{Y} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -0.0368 & -0.0368 & 0.0370 & 0.0351 & -0.0333 & 0.0369 & 0.0370 & -0.0370 \\ -0.0067 & 0.0075 & -0.0076 & 0.0074 & -0.0076 & -0.0076 & -0.0076 & -0.0076 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$$\Downarrow$$

$$\widehat{K} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -0.0383 & 0.0046 & 4.7623 & -0.0178 & -2.3057 & -0.2035 & 2.2435 & -3.5635 \\ 0.0091 & 0.0094 & -0.7096 & -0.0057 & -0.7954 & -0.0939 & -0.8332 & -0.7581 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

Only two (u_2 and u_3) out of the four control inputs are in play

c_1-optimization: State feedback

Control system:

$$\dot{x} = Ax + u$$

(the dimensions of the state vector and the control vector coincide)

The goal is to design a static state feedback $u = Kx$ that uses a reduced number of states

The Lyapunov approach:

$$A^\top Q + QA + QK + K^\top Q \prec 0, \quad Q \succ 0$$

Introduce the matrix variable $Y = QK \implies$ LMI

$$A^\top Q + QA + Y + Y^\top \prec 0, \quad Q \succ 0$$

Proposition 2. *Let $\widehat{Y}$ and $\widehat{Q}$ be the solution of the SDP*

$$\|Y\|_{c_1} \longrightarrow \min \quad \text{s.t.} \quad A^\top Q + QA + Y + Y^\top \prec 0, \quad Q \succ 0.$$

Then the matrix

$$\widehat{K} = \widehat{Q}^{-1}\widehat{Y}$$

defines a column-sparse state feedback controller, so that some of the components of the state vector x are not used.

- We detected the states sufficient for stabilization

Example 2

$\dot{x} = Ax + u$

$$A = \begin{pmatrix} 0 & 13 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 13 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$\widehat{Y} = \begin{pmatrix} -0.1930 & \mathbf{0} & \mathbf{0} & -0.0144 \\ -0.1930 & \mathbf{0} & \mathbf{0} & -0.0719 \\ -0.1930 & \mathbf{0} & \mathbf{0} & 0.0721 \\ -0.0878 & \mathbf{0} & \mathbf{0} & -0.0721 \end{pmatrix} \implies \widehat{K} = \begin{pmatrix} -2.5325 & \mathbf{0} & \mathbf{0} & 0.0319 \\ -0.4642 & \mathbf{0} & \mathbf{0} & -0.0143 \\ -1.3742 & \mathbf{0} & \mathbf{0} & 0.2368 \\ -0.7718 & \mathbf{0} & \mathbf{0} & -0.8593 \end{pmatrix}$$

Only two (x_1 and x_4) out of the four states are in play

$$\widehat{Y} = \begin{pmatrix} -0.5661 & 0 & 0 & \mathbf{0} \\ -0.5654 & 0 & 0 & \mathbf{0} \\ -0.5660 & 0 & 0 & \mathbf{0} \\ 0.0482 & 0 & 0 & \mathbf{0} \end{pmatrix} \implies \widehat{K} = \begin{pmatrix} -3.4039 & 0 & 0 & \mathbf{0} \\ -0.6387 & 0 & 0 & \mathbf{0} \\ -1.6738 & 0 & 0 & \mathbf{0} \\ -1.7367 & 0 & 0 & \mathbf{0} \end{pmatrix}$$

Only one (x_1) out of the four states is in play!

c_1-optimization: Output feedback

Control system:

$$\dot{x} = Ax + u$$
$$y = Cx$$

The goal is to design a static output feedback $u = Ky$ with a reduced numbers of outputs

The Lyapunov approach:

$$A^\top Q + QA + QKC + C^\top K^\top Q \prec 0, \quad Q \succ 0$$

Introduce the matrix variable $Y = QK \implies$ LMI

$$A^\top Q + QA + YC + C^\top Y^\top \prec 0, \quad Q \succ 0$$

Proposition 3. *Let $\widehat{Y}$ and $\widehat{Q}$ be the solution of the SDP*

$$\|Y\|_{c_1} \longrightarrow \min \quad \text{s.t.} \quad A^\top Q + QA + YC + C^\top Y^\top \prec 0, \quad Q \succ 0.$$

Then the matrix

$$\widehat{K} = \widehat{Q}^{-1}\widehat{Y}$$

defines a column-sparse output feedback controller, so that some of the components of the output vector y are not used.

- With $B = I$, the SOF design is doable
- We detected the outputs sufficient for stabilization

Example 3

$$A = \begin{pmatrix} 0 & 13 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 13 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\widehat{Y} = \begin{pmatrix} -0.1167 & \mathbf{0} & -0.0729 \\ -0.1167 & \mathbf{0} & -0.0729 \\ -0.1167 & \mathbf{0} & 0.0729 \\ -0.0397 & \mathbf{0} & -0.0729 \end{pmatrix} \Longrightarrow \widehat{K} = \begin{pmatrix} -1.5180 & \mathbf{0} & -0.3871 \\ -0.2754 & \mathbf{0} & -0.0898 \\ -0.8551 & \mathbf{0} & -0.0050 \\ -0.3385 & \mathbf{0} & -0.9296 \end{pmatrix}$$

Only two (y_1 and y_3) out of the three outputs are in play

$$\widehat{Y} = \begin{pmatrix} -0.3013 & 0 & \mathbf{0} \\ -0.3009 & 0 & \mathbf{0} \\ -0.3012 & 0 & \mathbf{0} \\ 0.0881 & 0 & \mathbf{0} \end{pmatrix} \Longrightarrow \widehat{K} = \begin{pmatrix} -2.2317 & 0 & \mathbf{0} \\ -0.4130 & 0 & \mathbf{0} \\ -1.1678 & 0 & \mathbf{0} \\ -0.8547 & 0 & \mathbf{0} \end{pmatrix}$$

Only one (y_1) out of the three outputs is in play!

Design of low-dimensional output

Control system:

$$\dot{x} = Ax + Bu$$

The goal is to design a linear low-dimensional output

$$y = Cx$$

(with few rows in the matrix C) and the associated linear static output feedback

$$u = Ky$$

Motivation: low-dimensional output $\implies$ low capacity of the control channel

Recall the Lyapunov approach:

$$AP + PA^\top + BKP + PK^\top B^\top \prec 0$$

Introduce $\quad Y = KP \quad \implies$ LMI

$$AP + PA^\top + BY + Y^\top B^\top \prec 0$$

Design of low-dimensional output (cont.)

Assume the matrix Y is column-sparse:

$\Downarrow$

$$u = Kx = \underbrace{\begin{pmatrix} \times & 0 & \times & 0 & \times \\ \times & 0 & \times & 0 & \times \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \times & 0 & \times & 0 & \times \end{pmatrix}}_{Y} \underbrace{\begin{pmatrix} \times & \times & \cdots & \times \\ \cdots & \cdots & \cdots & \cdots \\ \times & \times & \cdots & \times \\ \cdots & \cdots & \cdots & \cdots \\ \times & \times & \cdots & \times \end{pmatrix}}_{P^{-1}} x = \widetilde{K} \underbrace{\widetilde{C} x}_{y}$$

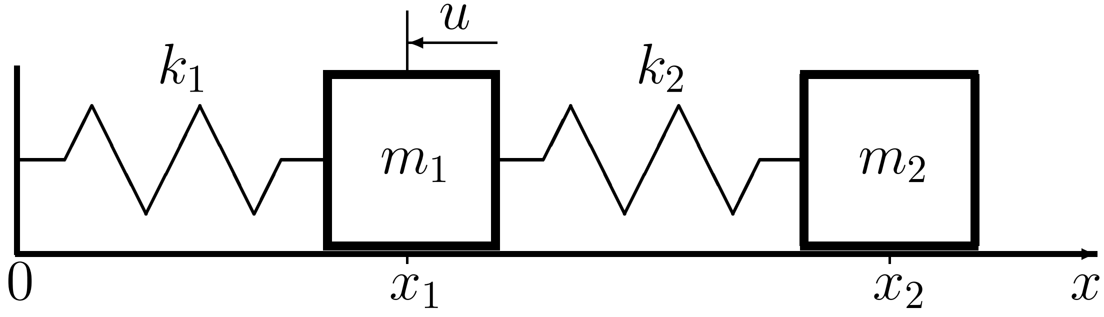
Proposition 4. *Let $\widehat{Y}$ and $\widehat{P}$ be the solution of the SDP*

$$\|Y\|_{c_1} \longrightarrow \min \quad \text{s.t.} \quad AP + PA^\top + BY + Y^\top B^\top \prec 0, \quad P \succ 0.$$

Let the matrix $\widetilde{K}$ be composed of the nonzero columns of $\widehat{Y}$, and the matrix $\widetilde{C}$ be composed of the rows of $\widehat{P}^{-1}$ having the same indices. Then the quantity $y = \widetilde{C}x$ represents a low-dimensional output of the system and the output feedback is given by $u = \widetilde{K}y$.

- We constructed a specific low-dimensional output
- The requirement $B = I$ is removed

Example 4: Double pendulum



- A benchmark for various feedback design methods
- The left body is governed by the control input
- The continuous-time model:

$$\dot{x}_1 = v_1$$

$$\dot{x}_2 = v_2$$

$$\dot{v}_1 = -\frac{k_1 + k_2}{m_1} x_1 + \frac{k_2}{m_1} x_2 + \frac{1}{m_1} u$$

$$\dot{v}_2 = \frac{k_2}{m_2} x_1 - \frac{k_2}{m_2} x_2$$

(below, the parameters of the system are set to unity)

Example 4 (cont.)

Here Y is the row-vector $\Longrightarrow$ we minimize not $\|Y\|_{C_1}$ but simply $\|Y\|_1$

$$\widehat{Y} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & -0.4231 & \mathbf{0} \end{pmatrix}$$

$$\widehat{P}^{-1} = \begin{pmatrix} 0.7255 & -0.2422 & 0.0709 & 0.1077 \\ -0.2422 & 0.4809 & -0.0353 & -0.0118 \\ \mathbf{0.0709} & \mathbf{-0.0353} & \mathbf{0.4727} & \mathbf{0.2411} \\ 0.1077 & -0.0118 & 0.2411 & 0.7280 \end{pmatrix}$$

Stabilizing controller for 1-dimensional output:

$$\widetilde{K} = -0.4231$$

Output matrix:

$$\widetilde{C} = \begin{pmatrix} 0.0709 & -0.0353 & 0.4727 & 0.2411 \end{pmatrix}$$

Example 5: Triple pendulum

$$\widehat{Y} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -0.5008 & 0.0409 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -0.2892 & -0.1616 \end{pmatrix}$$

$\Downarrow$

$$\widetilde{K} = \begin{pmatrix} -0.5008 & 0.0409 \\ 0.2892 & -0.1616 \end{pmatrix}$$

$$\widetilde{C} = \begin{pmatrix} -0.0568 & 0.0752 & -0.0076 & 0.1876 & 0.5015 & 0.1907 \\ 0.0700 & 0.0004 & 0.0174 & 0.0018 & 0.1907 & 0.6919 \end{pmatrix}$$

Let

$$\widehat{Y} = \begin{pmatrix} 0 & 0 & 0 & 0 & -0.5008 & \mathbf{0} \\ 0 & 0 & 0 & 0 & -0.2892 & \mathbf{0} \end{pmatrix}$$

$\Downarrow$

$$\widetilde{K} = \begin{pmatrix} -0.5008 \\ 0.2892 \end{pmatrix}$$

$$\widetilde{C} = \begin{pmatrix} -0.0568 & 0.0752 & -0.0076 & 0.1876 & 0.5015 & 0.1907 \end{pmatrix}$$

Example 6

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 13 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We obtain:

$$\widehat{Y} = \begin{pmatrix} 1.7400 & -15.6830 & \mathbf{0} \end{pmatrix}$$

$$\Downarrow$$

$$\widetilde{K} = \begin{pmatrix} 1.7400 & -15.6830 \end{pmatrix}, \qquad \widetilde{C} = \begin{pmatrix} 0.4000 & 0.1527 & 0.0127 \\ 0.1527 & 0.8994 & 0.2368 \end{pmatrix}$$

Let

$$\widehat{Y} = \begin{pmatrix} \mathbf{0} & -15.6830 & 0 \end{pmatrix}$$

$$\Downarrow$$

$$\widetilde{K} = -15.6830, \qquad \widetilde{C} = \begin{pmatrix} 0.1527 & 0.8994 & 0.2368 \end{pmatrix}$$

Syrmos V.L., Abdallah C.T., Dorato P., Grigoriadis K. Static output feedback: a survey // Automatica. 1997. Vol. 33. P. 125–137.

Extensions and further research

- discrete time systems
- robust formulations: $A = A_0 + F\Delta H, \quad \|\Delta\| \leqslant 1$
- optimal control
  - linear quadratic optimization
  - H_∞-optimization
  - rejection of exogenous disturbances

General scheme for sparse solutions of optimal control problems

- Initialization: solve the original problem to obtain the optimal value
- Zero row/column detection: solve the r_1/c_1-optimization problem with a relaxed bound on the performance to detect candidate zero rows/columns
- Optimization: solve the original problem over the set of row/column-structured controllers

A reduced number of controls/outputs is obtained at the expense of the (usually very small) loss of optimality

Filtering problem

Dynamical system:

$$\dot{x} = Ax + D_1 w$$
$$y = Cx + D_2 w$$

$x \in \mathbb{R}^n$ is the state vector (not available!)

$y \in \mathbb{R}^l$ is the output

$w \in \mathbb{R}^m$ is the L_∞-bounded exogenous disturbance (noise):

$$\|w(t)\| \leqslant 1 \qquad \forall t \geqslant 0$$

Goal: design a linear stationary filter with the Luenberger observer structure:

$$\dot{\widehat{x}} = A\widehat{x} + L(y - C\widehat{x}), \qquad \widehat{x} \text{ is the state estimate}$$

Can be formulated as finding the minimal invariant ellipsoid which comprises the error

$$e = x - \widehat{x}$$

Can be solved using LMIs!

Invariant ellipsiods

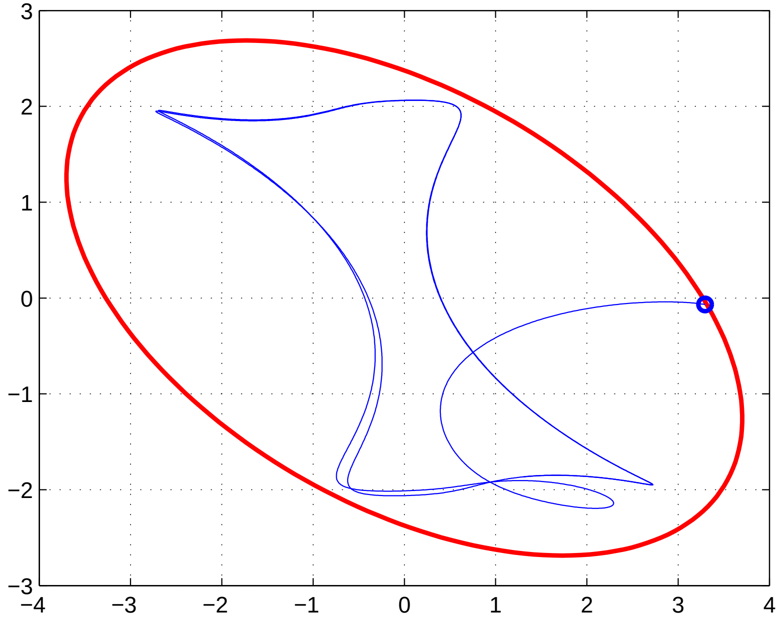
- The ellipsoid

$$\mathcal{E}_x = \{x \in \mathbb{R}^n : \quad x^\top P^{-1} x \leqslant 1\}, \quad P \succ 0$$

is said to be invariant for the system above if

$$x(0) \in \mathcal{E}_x \implies x(t) \in \mathcal{E}_x \quad \forall t \geqslant 0$$

i. e. starting at any point in $\mathcal{E}_x$, the trajectory of the system is guaranteed to remain inside $\mathcal{E}_x$ for all admissible disturbances.



- The attractivity property:

$$x(0) \notin \mathcal{E}_x \implies x(t) \to \mathcal{E}_x \text{ as } t \to \infty$$

- The ellipsoid

$$\mathcal{E}_z = \{z \in \mathbb{R}^l : \quad z^\top (CPC^\top)^{-1} z \leq 1\}$$

is said to be bounding for the output variable $z = Cx$, associated with $\mathcal{E}_x$.

- Key point: the invariance condition can be formulated as an LMI.

Theorem 1

Let $\widehat{Q}$ and $\widehat{Y}$ be the solution of the minimization problem

$$\operatorname{tr} H \longrightarrow \min$$

subject to the constraints

$$\begin{pmatrix} A^\top Q + QA + \alpha Q - YC - C^\top Y^\top & QD_1 - YD_2 \\ D_1^\top Q - D_2^\top Y^\top & -\alpha I \end{pmatrix} \preccurlyeq 0,$$

$$\begin{pmatrix} H & I \\ I & Q \end{pmatrix} \succcurlyeq 0,$$

with respect to the matrix variables $Q = Q^\top$, Y, $H = H^\top$, and the scalar parameter α.

Then

$$\widehat{P} = \widehat{Q}^{-1}$$

is the matrix of the minimal invariant ellipsoid for the error, and

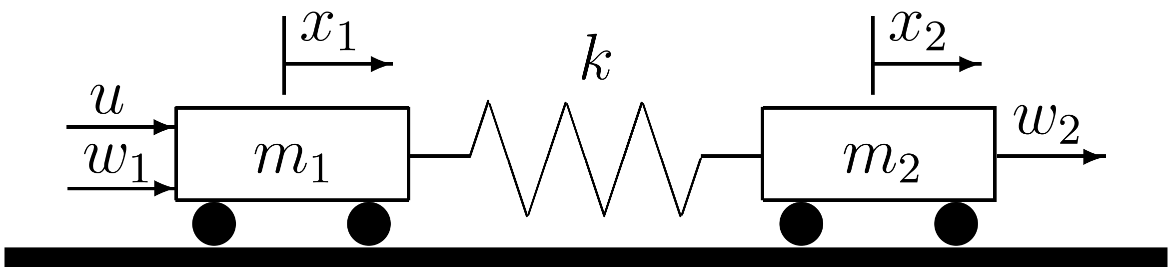
$$\widehat{L} = \widehat{Q}^{-1}\widehat{Y}$$

is the matrix of the optimal filter.

Polyak B.T., Topunov M.V. Filtering under nonrandom disturbances: the method of invariant ellipsoids // Doklady Mathematics. 2008. Vol. 77. No. 1. P. 158–162.

Example 7: Two-mass-spring system

- Two rigid bodies having masses m_1 and m_2 are linked together by a spring with elasticity coefficient k. They are allowed to slide without friction along a fixed horizontal rod.



- The bodies are subject to exogenous disturbances $w = \begin{pmatrix} w_1 & w_2 \end{pmatrix}^\top$ for which the only information available is boundedness at any time instant. The left body is governed by the control input aimed at compensating the effect of exogenous disturbances.

- The continuous-time model of the disturbed oscillations of the system:

$$\dot{x}_1 = v_1$$

$$\dot{x}_2 = v_2$$

$$\dot{v}_1 = -\frac{k}{m_1}x_1 + \frac{k}{m_2}x_2 + \frac{1}{m_1}u + \frac{1}{m_1}w_1$$

$$\dot{v}_2 = \frac{k}{m_2}x_1 - \frac{k}{m_2}x_2 + \frac{1}{m_2}w_2$$

(below, the parameters of the system are set to unity)

Example 7 (cont.)

- observable output: $y = \begin{pmatrix} x_1 + w_3 & x_2 \end{pmatrix}^\top$

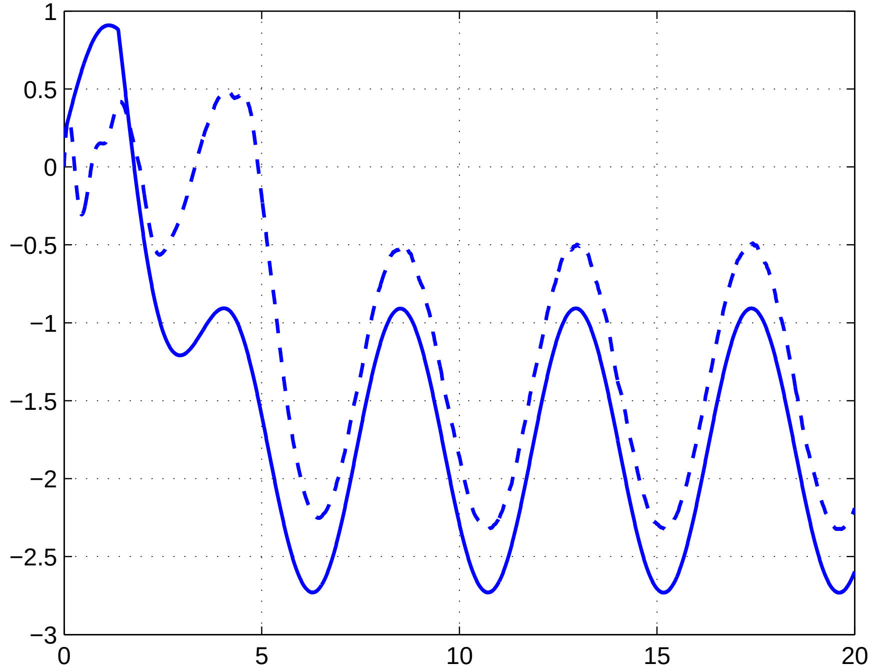
- controllable output: $y_1 = \begin{pmatrix} v_1 & v_2 \end{pmatrix}^\top$

- matrix of the optimal filter:

$$\widehat{L} \approx \begin{pmatrix} 1.4091 & 90.7141 \\ 0.0033 & 6.5556 \\ 1.0413 & 52.7907 \\ 0.1475 & 90.5180 \end{pmatrix}$$

- solid line: the trajectory of $v_1(t)$

- dotted line: the trajectory of $\widehat{v}_1(t)$



Filtering: Sparse approach

The goal is to minimize the number of outputs exploited in the filter design

If the matrix Y has a reduced number of nonzero columns

$\Downarrow$

matrix of the filter $L = Q^{-1}Y$ has a reduced number of nonzero columns

$\Downarrow$

filtering via reduced number of outputs

Example 8: Bell201-A helicopter

$$\dot{x} = Ax + D_1 w$$

$$y = Cx + D_2 w$$

$$A = \begin{pmatrix} -0.0046 & 0.038 & 0.3259 & -0.0045 & -0.402 & -0.073 & -9.81 & 0 \\ -0.1978 & -0.5667 & 0.357 & -0.0378 & -0.2149 & 0.5683 & 0 & 0 \\ 0.0039 & -0.0029 & -0.2947 & 0.007 & 0.2266 & 0.0148 & 0 & 0 \\ 0.0133 & -0.0014 & -0.4076 & -0.0654 & -0.4093 & 0.2674 & 0 & 9.81 \\ 0.0127 & -0.01 & -0.8152 & -0.0397 & -0.821 & 0.1442 & 0 & 0 \\ -0.0285 & -0.0232 & 0.1064 & 0.0709 & -0.2786 & -0.7396 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$D_1 = \begin{pmatrix} 0.0676 \\ -1.1151 \\ 0.0062 \\ -0.017 \\ -0.0129 \\ 0.139 \\ 0 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 \\ 0.1 \\ 0 \\ 0 \\ 0.05 \\ 0 \end{pmatrix}$$

Example 8 (cont.)

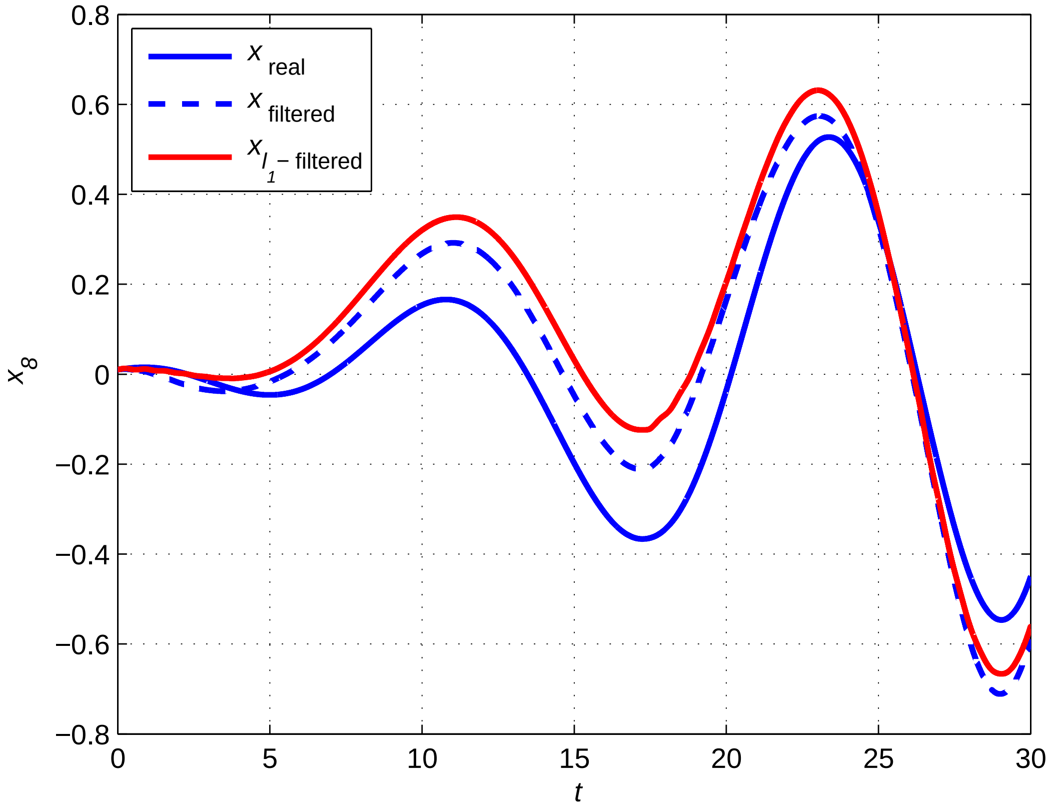
$$\widehat{Y} = \begin{pmatrix} -3.1276 & -28.7212 & 3.7161 & 1.4481 & \mathbf{0} & \mathbf{0} \\ -1.8945 & -94.7964 & 0 & 0 & \mathbf{0} & \mathbf{0} \\ 3.5134 & 2.0043 & 3.5650 & 1.1829 & \mathbf{0} & \mathbf{0} \\ -0.4208 & -3.9019 & 90.7763 & 3.2087 & \mathbf{0} & \mathbf{0} \\ -3.0936 & -0.9977 & 5.0174 & -1.4784 & \mathbf{0} & \mathbf{0} \\ 4.3478 & 11.7556 & 0 & -4.3076 & \mathbf{0} & \mathbf{0} \\ 68.4681 & 5.9099 & 5.1968 & -8.1004 & \mathbf{0} & \mathbf{0} \\ 2.3962 & 0.8696 & 1.3265 & -0.1294 & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$$\Downarrow$$

$$\widehat{L} = \begin{pmatrix} -1.4942 & 0.6760 & 0.0516 & 0.5915 & \mathbf{0} & \mathbf{0} \\ 0.7861 & -11.1510 & -0.3269 & -0.2488 & \mathbf{0} & \mathbf{0} \\ 0.7362 & 0.0620 & -0.0171 & 0.0698 & \mathbf{0} & \mathbf{0} \\ -0.8909 & -0.1700 & 9.9416 & 0.4196 & \mathbf{0} & \mathbf{0} \\ -0.8065 & -0.1290 & 1.0820 & -0.1051 & \mathbf{0} & \mathbf{0} \\ 0.3321 & 1.3900 & 0.0163 & -0.4009 & \mathbf{0} & \mathbf{0} \\ 8.0532 & 0 & -0.5776 & -0.9769 & \mathbf{0} & \mathbf{0} \\ 0.1134 & 0 & 0.3173 & -0.0097 & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

Two (y_5 and y_6) out of the six outputs are not in play!

Example 8 (cont.)



The loss in performance is just 6.5%!

Optimal control: Linear quadratic regulator (LQR)

Problem: For the system

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0,$$

find $u = Kx$ such that

$$J = \int_0^\infty (x^\top R x + u^\top S u)\, dt \longrightarrow \min, \qquad R, S \succ 0$$

Theorem 2. *Let $\widehat{P}$, $\widehat{Y}$, $\widehat{\gamma}$ be the solution of the SDP*

$$\gamma \longrightarrow \min$$

subject to

$$\begin{pmatrix} AP + PA^\top + BY + Y^\top B^\top & P & Y^\top \\ * & -R^{-1} & 0 \\ * & * & -S^{-1} \end{pmatrix} \preccurlyeq 0, \qquad \begin{pmatrix} \gamma & x_0^\top \\ x_0 & P \end{pmatrix} \succcurlyeq 0.$$

Then the optimal gain matrix is given by

$$\widehat{K} = \widehat{Y}\widehat{P}^{-1}$$

and the associated optimal value of the functional is

$$J^* = \widehat{\gamma} = x_0^\top \widehat{P}^{-1} x_0.$$

LQR: Sparse formulation

How much worse a sparse controller is?

Introduce a scalar relaxation coefficient $\alpha > 1$ and solve the r_1-optimization problem

$$\|Y\|_{r_1} \longrightarrow \min$$

subject to

$$\begin{pmatrix} AP + PA^\top + BY + Y^\top B^\top & P & Y^\top \\ * & -R^{-1} & 0 \\ * & * & -S^{-1} \end{pmatrix} \preccurlyeq 0, \qquad \begin{pmatrix} \gamma & x_0^\top \\ x_0 & P \end{pmatrix} \succcurlyeq 0,$$

and

$$\gamma \preccurlyeq \alpha J^*$$

in the variables P, Y, γ.

Due to Proposition 1, expect appearance of zero rows in the solution $\widetilde{Y}$

Now:

i) fix this zero-row structure, and

ii) solve the SDP as in Theorem 2, w.r.t. P, Y, γ,

but with the variable Y being sparse ($\widetilde{Y}$-structured)

Example 9: Bell201-A helicopter

For $R = S = I$ and initial $x_0 = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}^\top$

$$K_{\mathrm{opt}} = \begin{pmatrix} -0.0832 & 0.6154 & -1.0050 & 0.0259 & -0.1685 & -0.0213 & -0.9011 & 0.2389 \\ -0.9284 & -0.0447 & 18.9601 & -0.1157 & 0.1597 & 0.1222 & 20.6687 & -3.7814 \\ 0.1896 & 0.0225 & -1.3921 & -0.6600 & -6.2891 & -0.7592 & -4.7282 & -11.2563 \\ 0.0176 & 0.0652 & 1.2860 & -0.4619 & -2.9303 & 0.0900 & -0.1442 & -6.3896 \end{pmatrix}$$

Admit $\alpha = 1.25$, detection step gives zero rows 1 and 4

Optimization step:

$$K_{\mathrm{sparse}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -0.9634 & -0.0317 & 19.5523 & -0.1405 & -0.1253 & 0.1533 & 21.2335 & -4.3546 \\ 0.1979 & 0.0401 & -0.6174 & -0.7753 & -7.5970 & -0.7700 & -4.8240 & -13.6827 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

Performance:

$$J_{\mathrm{sparse}} \approx 1.0529 J^*,$$

i.e., the degradation is about 5%

Magnitude of the controllers: $\|K_{\mathrm{opt}}\| \approx 28.5695$, $\|K_{\mathrm{sparse}}\| \approx 29.2932$

Degrees of stability of the closed loop systems: $\sigma_{\mathrm{opt}} \approx 0.4002$, $\sigma_{\mathrm{sparse}} \approx 0.3976$

Final remarks

We proposed:

- new statements of the problems
- new approach
- low computational complexity
- high efficacy

Thank you for your attention!

Grazie dell'attenzione!

Спасибо за внимание!