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Алгоритмы зеркального спуска в задачах выпуклой оптимизации и управления

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ПЛАН

1. Введение: Метод зеркального спуска (МЗС).
2. Общее рассмотрение и задача выпуклой стохастической оптимизации.
3. Оценивание главного вектора стохастической матрицы.
4. Многорукий бандит.
5. Заключение.
6. Некоторые ссылки.

Введение

МЗС — метод градиентного типа для решения задач выпуклой оптимизации, использующий основное движение в сопряженном (двойственном) пространстве и отображающий получаемую траекторию в исходное пространство (множество):

1. Nemirovski and Yudin (1979/1983): [1]
2. Ben-Tal, Margalit, and Nemirovski (2001): [2]
3. Beck and Teboulle (2003): [3]
4. Nesterov (2005, 2007): [4], [5]
5. Юдицкий, Назин, Цыбаков, и Ваятис (2005): [6]
6. Juditsky, Lan, Nemirovski, and Shapiro (2007): [7]

1 Идея МЗС (непрерывное время) [1]

Рассмотрим исходно-двойственный метод — МЗС:

$$\dot{\xi}(t) = -\nabla_x f(x(t)), \quad \xi(0) = \xi_0, \tag{1}$$

$$x(t) = \nabla_\xi W(\xi(t)), \quad t \geq 0. \tag{2}$$

Здесь:

f — минимизируемая выпуклая функция на банаховом пространстве E,

W — равномерно дифференцируемая выпуклая функция на сопряженном пространстве E^* .

Так, в “евклидовом” случае

$$W(\xi) = \frac{1}{2}\|\xi\|_2^2$$

получаем стандартный градиентный метод:

$$\dot{x}(t) = -\nabla_x f(x(t)) \,.$$

Проведем небольшой анализ. Пусть

$$x^* = \arg\min f(x).$$

Имеем кандидата на функцию Ляпунова

$$W_*(\xi) \triangleq W(\xi) - < \xi, x^* >,$$

поскольку

$$
\begin{aligned}
\frac{dW_*(\xi(t))}{dt} &= \; < \dot{\xi}(t), \nabla_\xi W(\xi(t)) - x^* > && (3) \\
&= \; - < \nabla_x f(x(t)), x(t) - x^* > && (4) \\
&\leq \; f(x^*) - f(x(t)) && (5) \\
&\leq \; 0\,, && (6)
\end{aligned}
$$

то есть функция $W_*(\xi)$ убывает вдоль траектории $\{\xi(t)\}$. Далее, из (3)–(5) имеем

$$
f(x(t)) - f(x^*) \;\leq\; < \dot{\xi}(t), x^* > - \frac{dW(\xi(t))}{dt}\,, \qquad (7)
$$

и, предполагая, что

$$\xi(0) = 0, \quad W(0) = 0,$$

и интегрируя по $t \in [0, T]$, получаем

$$\begin{aligned} \int_0^T f(x(t))dt - Tf(x^*) &\leq \; < \xi(T), x^* > -W(\xi(T)) && (8) \\ &\leq \; V(x^*), && (9) \end{aligned}$$

где учитывается преобразование Лежандра:

$$V(x) \triangleq \sup_{\xi}\{< \xi, x > -W(\xi)\}.$$

Введем оценку $\widehat{x}(T)$, усредненную по траектории:

$$\widehat{x}(T) \triangleq \frac{1}{T}\int_0^T x(t)dt\,.$$

Тогда в силу выпуклости функции $f(x)$ и леммы Иенсена из (8)–(9) получаем

$$f(\widehat{x}(T)) - f(x^*) \quad \leq \quad \frac{1}{T}\,V(x^*). \tag{10}$$

Некоторые выводы:

- МЗС содержит параметр — функцию $W : E^* \to \mathbb{R}$, обеспечивающую функцию Ляпунова $W_* : E^* \to \mathbb{R}$; в частности, МЗС реализует стандартный градиентный метод; следовательно, эта дополнительная степень свободы потенциально может улучшить точность алгоритма.

- МЗС приводит к усредненной оценке $\widehat{x}(t)$ — среднему текущих оценок на данном интервале времени.

- Обеспечивается неасимптотическая верхняя граница на превышение уровня минимизируемой функции $f(\widehat{x}(t))$ над ее минимумом $f(x^*)$; эта верхняя граница типа $O(T^{-1})$ прямо зависит от $V(x^*)$; значит, рассматриваемый класс функций f должен обеспечить конечную ее верхнюю грань. (В связи с этим в дальнейшем рассматривается минимизация на заданном выпуклом компакте.)

- Из предыдущего видна роль преобразования Лежандра.

2 A Generalized View-Point

Proxy functions. Denote by $E = \ell_1^M$ the space $\mathbb{R}^M$ with the 1-norm

$$\|z\|_1 = \sum_{j=1}^{M} |z^{(j)}|$$

and by $E^* = \ell_\infty^M$ the dual space which is $\mathbb{R}^M$ equipped with the sup-norm

$$\|z\|_\infty = \max_{\|\theta\|_1=1} z^T\theta = \max_{1\le j\le M} \left|z^{(j)}\right|, \quad \forall\, z \in E^*.$$

Let Θ be a convex, closed set in E. For a given parameter $\beta > 0$ and a convex function $V : \Theta \to \mathbb{R}$, we call β-*conjugate* function of V the Legendre–Fenchel type transform of βV:

$$\forall\, z \in E^*, \quad W_\beta(z) = \sup_{\theta \in \Theta} \left\{ -z^T\theta - \beta V(\theta) \right\} . \qquad (11)$$

Assumption (L). *A convex function* $V : \Theta \to \mathbb{R}$ *is such that its* β*-conjugate* W_β *is continuously differentiable on* E^* *and its gradient* ∇W_β *satisfies*

$$\|\nabla W_\beta(z) - \nabla W_\beta(\tilde{z})\|_1 \leq \frac{1}{\alpha\beta}\|z - \tilde{z}\|_\infty, \quad \forall\, z, \tilde{z} \in E^*,\ \beta > 0,$$

where $\alpha > 0$ *is a constant independent of* β.

Assumption (L) relates to the strong convexity w.r.t. 1-norm:

$$V(sx+(1-s)y) \leq sV(x)+(1-s)V(y)-\frac{\alpha}{2}s(1-s)\|x-y\|_1^2 \tag{12}$$

for all $x, y \in \Theta$ and any $s \in [0,1]$.

The following proposition sums up some properties of β-conjugates and, in particular, yields a sufficient condition for Assumption (L).

Proposition 1. *Let function* $V : \Theta \to \mathbb{R}$ *be convex and* $\beta > 0$. *Then, the* β*-conjugate* W_β *of* V *has the following properties.*

1. *The function* $W_\beta : E^* \to \mathbb{R}$ *is convex and has a conjugate* βV, *i.e.,*

$$\forall\, \theta \in \Theta, \quad \beta V(\theta) = \sup_{z \in E^*} \left\{ -z^T\theta - W_\beta(z) \right\} .$$

2. *If function* V *is* α*-strongly convex with respect to the* 1*-norm then*

   (i) *Assumption (L) holds true,*

   (ii) $\operatorname*{argmax}_{\theta \in \Theta} \left\{ -z^T\theta - \beta V(\theta) \right\} = -\nabla W_\beta(z) \in \Theta$.

Definition 1. *We call* $V : \Theta \to \mathbb{R}_+$ proxy function *if it is convex, and*

(i) there exists a point $\theta_* \in \Theta$ *such that* $\min_{\theta \in \Theta} V(\theta) = V(\theta_*)$,

(ii) Assumption (L) holds true.

Example: Consider a standard simplex $\Theta = \Theta_M$ and an entropy-type proxy function

$$V(\theta) = \ln(M) + \sum_{j=1}^{M} \theta^{(j)} \ln \theta^{(j)} \qquad (13)$$

(where $0 \ln 0 \triangleq 0$) which has a single minimizer $\theta_* = (1/M, \ldots, 1/M)^T$ with $V(\theta_*) = 0$. It is directly checked that this function is α-strongly convex w.r.t. the 1-norm, with the parameter $\alpha = 1$. This leads to a β-conjugate function $W_\beta(z)$ of (14) and to a Mirror Descent Algorithm:

Indeed, for $\beta > 0$, one get function

$$W_\beta(z) = \beta \, \ln \left(\frac{1}{M} \sum_{k=1}^{M} e^{-z^{(k)}/\beta} \right), \quad z \in \mathbb{R}^M, \tag{14}$$

with partial derivatives relating to a Gibbs distribution on the coordinates of vector $z = (z^{(1)}, \ldots, z^{(M)})^T$, with β being a "temperature" parameter:

$$-\frac{\partial W_\beta(z)}{\partial z^{(j)}} = e^{-z^{(j)}/\beta} \left(\sum_{k=1}^{M} e^{-z^{(k)}/\beta} \right)^{-1}, \; j = 1, \ldots, M. \tag{15}$$

□

Convex Stochastic Optimization Problem

$$A(\theta) \triangleq \mathbb{E}\, Q(\theta, Z) \;\to \min_{\theta \in \Theta}$$

with loss function $Q : \Theta \times \mathcal{Z} \to \mathbb{R}_+$ being such that the random function $Q(\cdot\,, Z) : \Theta \to \mathbb{R}_+$ is convex a.s., on a convex closed set $\Theta \subset \mathbb{R}^M$.

Let a learning sample be given in the form of an i.i.d. sequence $(Z_1, \ldots, Z_{t-1})$, where each Z_i has the same distribution as Z.

Denote stochastic subgradients

$$u_i(\theta) = \nabla_\theta Q(\theta, Z_i)\,, \quad i = 1, 2, \ldots, \tag{16}$$

which are measurable functions on $\Theta \times \mathcal{Z}$ such that, for any $\theta \in \Theta$, the expectation $\mathbb{E}\, u_i(\theta)$ belongs to the subdifferential of the function $A(\theta)$.

Mirror Descent Algorithm (MDA)

The algorithm is defined as follows:

- Fix the initial value $\zeta_0 = 0 \in \mathbb{R}^M$.
- For $i = 1, \ldots, t-1$, do the recursive update

$$\begin{aligned} \zeta_i &= \zeta_{i-1} + \gamma_i u_i(\theta_{i-1}), \\ \theta_i &= -\nabla W_{\beta_i}(\zeta_i) . \end{aligned} \tag{17}$$

- Output at iteration t the following convex combination:

$$\widehat{\theta}_t = \sum_{i=1}^{t} \gamma_i \theta_{i-1} \left(\sum_{i=1}^{t} \gamma_i \right)^{-1} . \tag{18}$$

A Particular Case of the Algorithm

Let

$$\gamma_i \equiv 1\,, \quad \beta_i = \beta_0 \sqrt{i+1} \quad (i \geq 1), \quad \beta_0 > 0\,. \tag{19}$$

Then the algorithm becomes simpler and can be implemented in the following recursive form:

$$\begin{aligned}
\zeta_i &= \zeta_{i-1} + u_i(\theta_{i-1})\ , & (20)\\
\theta_i &= -\nabla W_{\beta_i}(\zeta_i)\ , & (21)\\
\widehat{\theta}_i &= \widehat{\theta}_{i-1} - \frac{1}{i}\left(\widehat{\theta}_{i-1} - \theta_{i-1}\right), \quad i = 1, 2, \ldots, & (22)
\end{aligned}$$

with initial value $\zeta_0 = 0$.

Theorem 0. *Assume that*

$$\sup_{\theta\in\Theta} \mathbb{E}\,\|\nabla_\theta\, Q(\theta,Z)\|_\infty^2 \le L_{\Theta,Q}^2 \ , \tag{23}$$

where $L_{\Theta,Q} \in (0,+\infty)$. *Let* V *be a proxy function on* Θ *satisfying Assumption (L) with a parameter* $\alpha > 0$, *and assume that there exists* $\theta_A^* \in \operatorname*{Argmin}_{\theta\in\Theta} A(\theta)$. *Furthermore, let* $V(\theta_A^*) \le \overline{V} < +\infty$, *and we set* $\beta_0 = L_{\Theta,Q}\,(\alpha\,\overline{V}\,)^{-1/2}$.

Then, with sequences $(\gamma_i)_{i\geq 1}$ and $(\beta_i)_{i\geq 1}$ from (19), for any integer $t \geq 1$, the estimate $\widehat{\theta}_t$ being defined in (17)–(18) with stochastic subgradients (16) satisfies inequality

$$\mathbb{E}\, A(\widehat{\theta}_t) - \min_{\theta\in\Theta} A(\theta) \leq 2\, L_{\Theta,Q} \left(\alpha^{-1}\overline{V}\right)^{1/2} \frac{\sqrt{t+1}}{t}\,. \qquad (24)$$

In particular, if Θ is a convex compact set, we can take $\overline{V} = \max_{\theta\in\Theta} V(\theta)$. □

Proof: see [6].

3 Main Eigenvalue Estimation to a Stochastic Matrix.

Let $A = \|a_{ij}\|_{N\times N}$ be a given left stochastic matrix, N be *a large number*. Denote $\Theta_N \subset \mathbb{R}^N$, the standard simplex.

Our goal: We are to approximate a positive solution to a linear system equations

$$Ax = x\,, \quad x \in \Theta_N\,. \tag{25}$$

Motivation: Calculations for PageRank problem.

Notations

- Let $\mathcal{A}_N$ be a set of all left stochastic $N \times N$-matrices A.
- $A^{(j)}$ and $A_{(i)}$ mean j-column and i-row in the matrix A.
- Given a matrix $A \in \mathcal{A}_N$, define set

$$X_* \triangleq \operatorname*{Arg\,min}_{x \in \Theta_N} \|Ax - x\|_2 = \{x \in \Theta_N \ : \ Ax = x\} \tag{26}$$

being convex compact of all solutions $x_* \in X_*$.

- Define risk functions

$$\mathcal{R}_A(x) \triangleq \frac{1}{2}\|Ax - x\|_2^2 , \quad x \in \mathbb{R}^N , \tag{27}$$

$$\mathcal{Q}_A(x) \triangleq \|Ax - x\|_2 , \quad x \in \mathbb{R}^N . \tag{28}$$

Related Optimization Problem

Minimize risk function $\mathcal{R}_A(x)$ on simplex Θ_N . This gives

$$\nabla_x \mathcal{R}_A(x) = A^T A x - A^T x - A x + x \,. \tag{29}$$

Stochastic Gradients by Randomization

On iteration $k \geq 1$, one can prove

$$\mathbb{E}\left(\zeta_k \mid x_1, \ldots, x_k\right) = \left. \nabla_x \mathcal{R}_A(x) \right|_{x=x_k} \,, \tag{30}$$

where x_t means the result of t^{th} iteration, $t = 1, \ldots, k$,

$$\zeta_k \triangleq \left(A_{(\xi_k)}\right)^T - \left(A_{(\eta_k)}\right)^T - A^{(\eta_k)} + x_k \,; \tag{31}$$

having the two random indexes $\xi_k\,, \eta_k \in \{1, \ldots, N\}$ with

$$\mathbb{P}\left(\eta_k = j \mid x_1, \ldots, x_k\right) = x_k^{(j)}\,, \quad j = 1, \ldots, N\,, \tag{32}$$

and

$$\mathbb{P}\left(\xi_k = i \mid x_1, \eta_1, \ldots, x_k, \eta_k\right) = a_{i\eta_k}\,, \quad i = 1, \ldots, N\,. \tag{33}$$

Important bounds hold

$$\begin{aligned} \|\zeta_k\|_\infty &\leq \|\left(A_{(\xi_k)}\right)^T - \left(A_{(\eta_k)}\right)^T\|_\infty + \|x_k - A^{(\eta_k)}\|_\infty & (34) \\ &\leq 2\,. & (35) \end{aligned}$$

Optimization MD Algorithms

- Fix $x_0 \in \Theta_N$ and $\psi_0 = 0 \in \mathbb{R}^N$. Fix positive $(\gamma_k)_{k \geq 1}$, $(\beta_k)_{k \geq 1}$, and horizon $n > 1$.
- For $k = 0, \ldots, n-1$ generate η_k and ξ_k by (32) and (33); then calculate stochastic gradient ζ_k (31), and iterate

$$\begin{aligned} \psi_k &= \psi_{k-1} + \gamma_k \zeta_k \, , \\ x_k &= -\nabla W_{\beta_k}(\psi_k) \, . \end{aligned} \tag{36}$$

- Output n^{th} iteration of convex combination

$$\widehat{x}_n = \frac{\sum_{k=1}^n \gamma_k x_{k-1}}{\sum_{k=1}^n \gamma_k} \, . \tag{37}$$

▲

Here function $W_\beta(z)$ and its gradient $\nabla W_\beta(\cdot)$ being Gibbs potential are as follows: $\forall\, z \in \mathbb{R}^N$,

$$W_\beta(z) \;=\; \beta \ln \left(\frac{1}{N} \sum_{k=1}^{N} e^{-z^{(k)}/\beta} \right), \tag{38}$$

$$\frac{\partial W_\beta(z)}{\partial z^{(j)}} \;=\; -e^{-z^{(j)}/\beta} \left(\sum_{k=1}^{N} e^{-z^{(k)}/\beta} \right)^{-1}, \; j = 1, \ldots, N \tag{39}$$

Remark: Another conjugate function $W_\beta(z) = \frac{\beta}{2}\|z\|_2^2$ would give Euclidean potential $\nabla W_\beta(z) = \beta z$ which leads to an ordinary stochastic gradient algorithm (projected SA) with time averaging. Cf Polyak–Juditsky SA with averaging.

Main Results I: Uniform Upper Bounds

Theorem 1. *Let $N \geq 2$, and let estimation $\widehat{x}_n$ be defined by randomized algorithm (36)–(39) with stochastic gradient ζ_k (31) and the parameters*

$$\gamma_k \equiv 1\,, \quad \beta_k = \beta_0\sqrt{k+1}\,, \quad \beta_0 = \ 2(\ln N)^{-1/2}. \tag{40}$$

Then, under arbitrary iteration number $n \geq 1$, one holds

$$\mathbb{E}\,\|A\widehat{x}_n - \widehat{x}_n\|_2^2 \leq 8\,(\ln N)^{1/2}\,\frac{\sqrt{n+1}}{n}\,. \tag{41}$$

Remark: The projected SA (i.e., the MDA with Euclidean potential) would give Upper Bound like $O(N/n)$, instead of $O(\sqrt{\ln N/n})$ (41). For instance, condition $\sqrt{(\ln N)/n} \geq N/n$ implies $n \geq N^2/(\ln N)$.

4 Задача о многоруком бандите.

Представлено на 17th IFAC World Congress:

1. Juditsky, A., A.V. Nazin, A.B. Tsybakov, N. Vayatis. Gap-free Bounds for Stochastic Multi-Armed Bandit. *Proc. 17th IFAC World Congress, Seoul, Korea, 6–11 July 2008, pp.11560–11563*.

Let $X = \{x(1), \ldots, x(N)\}$ be a set of N available actions. At each time $t = 1, 2, \ldots,$ we have to choose sequentially an action $x_t \in X$. We denote by η_t the observable (instantaneous) loss for the choice of x_t, and introduce the average loss up to horizon T which is to be minimized:

$$\Phi_T = \frac{1}{T} \sum_{t=1}^{T} \eta_t \,. \tag{42}$$

A strategy $\mathcal{U}$ is a sequence of rules for the choice x_t at times $t = 1, \dots, T$. In the stochastic setup that we consider here, the sequence of losses $(\eta_t)_{t \geq 1}$ is a stochastic process and x_t is a measurable function (random, in general) depending only on the vector of past decisions and losses $(x_1, \dots, x_{t-1}; \eta_1, \dots, \eta_{t-1})$.

Any strategy $\mathcal{U}$ generates a flow of σ-algebras $\mathcal{F}_t = \sigma\{x_1, \dots, x_t; \eta_1, \dots, \eta_t\}$, $t \geq 1$ (for brevity we do not indicate the dependence of $\mathcal{F}_t$ on $\mathcal{U}$). Throughout the paper we denote by $z^{(j)}$ the jth component of vector $z \in \mathbb{R}^N$.

Two basic assumptions:

A1. With probability 1, the conditional expectations satisfy

$$\mathbb{E}\{\eta_t \,|\, \mathcal{F}_{t-1}\,,\, x_t = x(k)\} = a_k,\;\; k = 1, \ldots, N, \qquad (43)$$

where $a_k \in \mathbb{R}$ are unknown deterministic values.

The value a_k characterizes the expected loss for deciding to take the action $x_t = x(k)$ at time t. Assumption A1 says that this loss should not depend on t.

A2. The second conditional moment of the loss η_t is a.s. bounded by a constant:

$$\mathbb{E}\{\eta_t^2 \,|\, \mathcal{F}_{t-1}\,,\, x_t\} \leq \sigma^2 < \infty\,. \qquad (44)$$

It is easy to prove (see, e.g., [12]) that under these assumptions all the limiting points of the average loss sequence $(\Phi_t)_{t\geq 1}$ cannot be almost surely (a.s.) less than

$$a_{\min} \triangleq \min_{k=1,\ldots,N} a_k\,.$$

Thus, the problem is to design a strategy $\mathcal{U}^*$ which has the asymptotically minimal average loss:

$$\Phi_T \to a_{\min} \quad \text{as} \quad T \to \infty\,, \tag{45}$$

in an appropriate probability sense.

We study here *convergence in mean*, trying to get the rate of convergence

$$\mathbb{E}(\Phi_T) \to a_{\min}$$

as fast as possible.
In particular, we provide *non-asymptotic* upper bounds for the expected excess risk $\mathbb{E}(\Phi_T) - a_{\min}$ that are close, up to logarithmic factors, to the lower bound of the order $\sqrt{N/T}$ proved for arbitrary N by (see Theorem 6.11 in [14]).

We will suppose that the following assumption on the loss sequence $(\eta_t)_{t\geq 1}$ holds:

A3. The losses are nonnegative: $\eta_t \geq 0$ a.s.

Below we propose a randomized decision strategy in which, at each step $t+1$, the action x_{t+1} is drawn according to a distribution $p_t \triangleq \left(p_t^{(1)}, \dots, p_t^{(N)}\right)^\top$ over X where:

$$p_t^{(k)} \triangleq \mathbb{P}(x_{t+1} = x(k) \,|\, \mathcal{F}_t)\,, \quad k = 1, \dots, N\,. \qquad (46)$$

The update of the distribution p_t over time is given by the MDA.

Denote by Θ the simplex of all probability vectors over X:

$$\Theta \triangleq \left\{ p \in \mathbb{R}_+^N \,\middle|\, \sum\nolimits_{k=1}^{N} p^{(k)} = 1 \right\} . \tag{47}$$

We then define the mean (over the set of actions) loss function A on Θ:

$$A(p) = \sum_{k=1}^{N} a_k p^{(k)} = a^\top p \, , \quad p \in \Theta \, , \tag{48}$$

where $a = (a_1, \ldots, a_N)^\top$. Since p_t is a random vector, the quantity $A(p_t)$ is a random variable. The update rule for the probability distribution p_t uses a stochastic gradient of A.

The expected average loss equals to the average over time of the expectations $\mathbb{E}A(p_t)$, that is

$$\mathbb{E}(\Phi_T) = \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}(\mathbb{E}(\eta_t \mid x_t\,, \mathcal{F}_{t-1})) = \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}(A(p_{t-1}))\ . \tag{49}$$

Theorem. *Let assumptions A1-A2-A3 be satisfied and let the conditional distributions $(p_t)_{t\geq 0}$ be defined by the MDA. Then, for any horizon $T \geq 1$,*

$$\mathbb{E}\left(\Phi_T\right) - a_{\min} \leq 2\,\sigma\,\frac{\sqrt{(T+1)N\ln N}}{T}\ . \tag{50}$$

Remark: The known information lower bound (see [14], Theorem 6.11) differs from the upper bound (50) by logarithmic term $\sqrt{\ln N}$.

5 Заключение.

И другие задачи, например:

- Робастный PageRank [Поляк, Юдицкий (2011)]:

$$f(x) \triangleq \|Ax\|_2 + \varepsilon\|x\|_2 \to \min_{x\in\Theta_N},$$

  где $\varepsilon > 0$ задано, Θ_N — стандартный симплекс в $\mathbb{R}^N$, $A = P - I$, P — известная стохастическая матрица $N \times N$, I — единичная.

- Распознавание образов (классификация с учителем) [6]

- Управление конечными марковскими цепями [16], [17]

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