Ellipsoid estimation of reachability sets

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Motivation

Approximation convex set by one or several simple geometrical figures

Recent techniques:

- *Rectangular polytopes;*
- ***Ellipsoid*** *(!)*
- *Zonotope*
- …

Publications

- A. B. Kurzhanski and I. Valyi. “Ellipsoidal Calculus for Estimation and Control”, Birkhauser, Boston, 1997.
- A. B. Kurzhanski, P. Varaiya. “Reachability Analysis for Uncertain Systems —the Ellipsoidal Technique”, Journal of Dynamics of Continuous, Discrete andImpulsive Systems, Ser. B., v. 9, № 3, 2002, pp. 347–367.
- A. A. Kurzhanskiy and P. Varaiya. “Ellipsoidal Toolbox manual”, 2006-2008.

Basic notions

Ellipsoid $\mathcal{E}(q, Q)$ *in* $\mathbf{R}^n$ *with center* q *and shape matrix* Q *is the set*

$$\mathcal{E}(q, Q) = \{x \in \mathbf{R}^n \mid \langle (x - q), Q^{-1}(x - q)\rangle \leq 1\},$$

wherein Q *is positive definite (*$Q = Q^T$ *and* $\langle x, Qx\rangle > 0$ *for all nonzero* $x \in \mathbf{R}^n$*).*

The support function of a set $\mathcal{X} \subseteq \mathbf{R}^n$ *is*

$$\rho(l \mid \mathcal{X}) = \sup_{x \in \mathcal{X}} \langle l, x\rangle.$$

In particular, the support function of the ellipsoid is

$$\rho(l \mid \mathcal{E}(q, Q)) = \langle l, q\rangle + \langle l, Ql\rangle^{1/2}.$$

Operations with Ellipsoids

- Affine transformation of ellipsoid;
- Geometric sum of finite number of ellipsoids;
- Geometric difference of two ellipsoids;
- Intersection of finite number of ellipsoids.

Affine Transformation

The simplest operation with ellipsoids is an affine transformation. Let ellipsoid $\mathcal{E}(q, Q) \subseteq \mathbf{R}^n$, matrix $A \in \mathbf{R}^{m \times n}$ and vector $b \in \mathbf{R}^m$. Then

$$A\mathcal{E}(q, Q) + b = \mathcal{E}(Aq + b, AQA^T).$$

Thus, ellipsoids are preserved under affine transformation.

Geometric Sum

Consider the geometric sum in which $\mathcal{X}_1, \cdots, \mathcal{X}_k$ are nondegenerate ellipsoids $\mathcal{E}(q_1, Q_1), \cdots,$ $\mathcal{E}(q_k, Q_k) \subseteq \mathbf{R}^n$. The resulting set is not generally an ellipsoid. However, it can be tightly approximated by the parametrized families of external and internal ellipsoids.

Let parameter l be some nonzero vector in $\mathbf{R}^n$. Then the external approximation $\mathcal{E}(q, Q_l^+)$ and the internal approximation $\mathcal{E}(q, Q_l^-)$ of the sum $\mathcal{E}(q_1, Q_1) \oplus \cdots \oplus \mathcal{E}(q_k, Q_k)$ are *tight* along direction l, i.e.,

$$\mathcal{E}(q, Q_l^-) \subseteq \mathcal{E}(q_1, Q_1) \oplus \cdots \oplus \mathcal{E}(q_k, Q_k) \subseteq \mathcal{E}(q, Q_l^+)$$

and

$$\rho(\pm l \mid \mathcal{E}(q, Q_l^-)) = \rho(\pm l \mid \mathcal{E}(q_1, Q_1) \oplus \cdots \oplus \mathcal{E}(q_k, Q_k)) = \rho(\pm l \mid \mathcal{E}(q, Q_l^+)).$$

Geometric Sum

Here the center q is

$$q = q_1 + \cdots + q_k,$$

the shape matrix of the external ellipsoid Q_l^+ is

$$Q_l^+ = \left(\langle l, Q_1 l\rangle^{1/2} + \cdots + \langle l, Q_k l\rangle^{1/2}\right)\left(\frac{1}{\langle l, Q_1 l\rangle^{1/2}} Q_1 + \cdots + \frac{1}{\langle l, Q_k l\rangle^{1/2}} Q_k\right),$$

and the shape matrix of the internal ellipsoid Q_l^- is

$$Q_l^- = \left(Q_1^{1/2} + S_2 Q_2^{1/2} + \cdots + S_k Q_k^{1/2}\right)^T \left(Q_1^{1/2} + S_2 Q_2^{1/2} + \cdots + S_k Q_k^{1/2}\right),$$

with matrices S_i, $i = 2, \cdots, k$, being orthogonal ($S_i S_i^T = I$) and such that vectors $Q_1^{1/2} l, S_2 Q_2^{1/2} l, \cdots, S_k Q_k^{1/2} l$ are parallel.

Varying vector l we get exact external and internal approximations,

$$\bigcup_{\langle l, l\rangle = 1} \mathcal{E}(q, Q_l^-) = \mathcal{E}(q_1, Q_1) \oplus \cdots \oplus \mathcal{E}(q_k, Q_k) = \bigcap_{\langle l, l\rangle = 1} \mathcal{E}(q, Q_l^+).$$

Geometric Difference

Consider the geometric difference in which the sets $\mathcal{X}_1$ and $\mathcal{X}_2$ are nondegenerate ellipsoids $\mathcal{E}(q_1, Q_1)$ and $\mathcal{E}(q_2, Q_2)$. We say that ellipsoid $\mathcal{E}(q_1, Q_1)$ is *bigger* than ellipsoid $\mathcal{E}(q_2, Q_2)$ if

$$\mathcal{E}(0, Q_2) \subseteq \mathcal{E}(0, Q_1).$$

Given two ellipsoids $\mathcal{E}(q_1, Q_1)$ and $\mathcal{E}(q_2, Q_2)$ with $\mathcal{E}(0, Q_2) \subseteq \mathcal{E}(0, Q_1)$, l is a bad direction if

$$\frac{\langle l, Q_1 l\rangle^{1/2}}{\langle l, Q_2 l\rangle^{1/2}} > r,$$

in which r is a minimal root of the equation

$$\mathbf{det}(Q_1 - rQ_2) = 0.$$

To find r, compute

$$r = \frac{1}{\max(\mathbf{diag}(TQ_2T^T))}.$$

If l is *not* a bad direction, we can find tight external and internal ellipsoidal approximations $\mathcal{E}(q, Q_l^+)$ and $\mathcal{E}(q, Q_l^-)$ such that

$$\mathcal{E}(q, Q_l^-) \subseteq \mathcal{E}(q_1, Q_1) \dot{-} \mathcal{E}(q_2, Q_2) \subseteq \mathcal{E}(q, Q_l^+)$$

Geometric Difference

The center q is

$$q = q_1 - q_2;$$

the shape matrix of the internal ellipsoid Q_l^- is

$$Q_l^- = \left(1 - \frac{\langle l, Q_1 l\rangle^{1/2}}{\langle l, Q_2 l\rangle^{1/2}}\right) Q_1 + \left(1 - \frac{\langle l, Q_2 l\rangle^{1/2}}{\langle l, Q_1 l\rangle^{1/2}}\right) Q_2;$$

and the shape matrix of the external ellipsoid Q_l^+ is

$$Q_l^+ = \left(Q_1^{1/2} + SQ_2^{1/2}\right)^T \left(Q_1^{1/2} + SQ_2^{1/2}\right).$$

Here S is an orthogonal matrix such that vectors $Q_1^{1/2} l$ and $SQ_2^{1/2} l$ are parallel.

Running l over all unit directions that are not bad, we get

$$\bigcup_{\langle l, l\rangle = 1} \mathcal{E}(q, Q_l^-) = \mathcal{E}(q_1, Q_1) \dot{-} \mathcal{E}(q_2, Q_2) = \bigcap_{\langle l, l\rangle = 1} \mathcal{E}(q, Q_l^+).$$

Geometric Diff-Sum, Sum-Diff

Given ellipsoids $\mathcal{E}(q_1, Q_1)$, $\mathcal{E}(q_2, Q_2)$ and $\mathcal{E}(q_3, Q_3)$, it is possible to compute families of external and internal approximating ellipsoids for

$$\mathcal{E}(q_1, Q_1) \dot{-} \mathcal{E}(q_2, Q_2) \oplus \mathcal{E}(q_3, Q_3)$$

parametrized by direction l, if this set is nonempty ($\mathcal{E}(0, Q_2) \subseteq \mathcal{E}(0, Q_1)$).

As a result, we get

$$\mathcal{E}(q_1 - q_2 + q_3, Q_l^-) \subseteq \mathcal{E}(q_1, Q_1) \dot{-} \mathcal{E}(q_2, Q_2) \oplus \mathcal{E}(q_3, Q_3) \subseteq \mathcal{E}(q_1 - q_2 + q_3, Q_l^+)$$

and

$$\rho(\pm l \mid \mathcal{E}(q_1 - q_2 + q_3, Q_l^-)) = \rho(\pm l \mid \mathcal{E}(q_1, Q_1) \dot{-} \mathcal{E}(q_2, Q_2) \oplus \mathcal{E}(q_3, Q_3)) = \rho(\pm l \mid \mathcal{E}(q_1 - q_2 + q_3, Q_l^+)).$$

Running l over all unit vectors that are not bad, this translates to

$$\bigcup_{\langle l, l \rangle = 1} \mathcal{E}(q_1 - q_2 + q_3, Q_l^-) = \mathcal{E}(q_1, Q_1) \dot{-} \mathcal{E}(q_2, Q_2) \oplus \mathcal{E}(q_3, Q_3) = \bigcap_{\langle l, l \rangle = 1} \mathcal{E}(q_1 - q_2 + q_3, Q_l^+).$$

Intersection of Ellipsoid and Ellipsoid

Given two nondegenerate ellipsoids $\mathcal{E}(q_1, Q_1)$ and $\mathcal{E}(q_2, Q_2)$:

$$\mathcal{E}(q_1, Q_1) \cap \mathcal{E}(q_2, Q_2) \neq \emptyset.$$

This intersection can be approximated by ellipsoids from the outside and from the inside. Trivially, both $\mathcal{E}(q_1, Q_1)$ and $\mathcal{E}(q_2, Q_2)$ are external approximations of this intersection.

Define matrices

$$W_1 = Q_1^{-1}, \qquad W_2 = Q_2^{-1}.$$

Minimal volume external ellipsoidal approximation $\mathcal{E}(q+, Q^+)$ of the intersection $\mathcal{E}(q_1, Q_1) \cap \mathcal{E}(q_2, Q_2)$ is determined from the set of equations:

$$\begin{aligned}
Q^+ &= \alpha X^{-1} \\
X &= \pi W_1 + (1-\pi) W_2 \\
\alpha &= 1 - \pi(1-\pi)\langle (q_2 - q_1), W_2 X^{-1} W_1 (q_2 - q_1) \rangle
\end{aligned}$$

$$\begin{aligned}
q^+ &= X^{-1}(\pi W_1 q_1 + (1-\pi) W_2 q_2) \\
0 &= \alpha(\mathbf{det}(X))^2 \mathbf{trace}(X^{-1}(W_1 - W_2)) \\
&- n(\mathbf{det}(X))^2 \big(2\langle q^+, W_1 q_1 - W_2 q_2 \rangle + \langle q^+, (W_2 - W_1) q^+ \rangle \\
&\quad - \langle q_1, W_1 q_1 \rangle + \langle q_2, W_2 q_2 \rangle \big),
\end{aligned}$$

Reachability

Consider the system:

$$\dot{x} = Ax(t) + Bu(t) + Cv(t)$$

in which $x \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}^m$ is the control and $v \in \mathbf{R}^d$ is the disturbance.
A, B, C take their values in $\mathbf{R}^{n\times n}$, $\mathbf{R}^{n\times m}$ and $\mathbf{R}^{n\times d}$ respectively.
$u(t, x(t)) \in \mathcal{E}(p, P)$ and $v(t) \in \mathcal{E}(q, Q)$.
The set of initial conditions is assumed to be the ellipsoid $\mathcal{E}(x_0, X_0)$.

$$\mathcal{X}(t, t_0, \mathcal{E}(x_0, X_0)) = \mathcal{X}(t, \tau, \mathcal{X}(\tau, t_0, \mathcal{E}(x_0, X_0))), \quad t_0 \leq \tau \leq t.$$

External estimation

The reach set can be approximated by the parametrized families of external and internal ellipsoids, $\mathcal{E}(x_c(t), X_l^+(t))$ and $\mathcal{E}(x_c(t), X_l^-(t))$ respectively:

$$\mathcal{E}(x_c(t), X_l^-(t)) \subseteq \mathcal{X}(t, t_0, \mathcal{E}(x_0, X_0)) \subseteq \mathcal{E}(x_c(t), X_l^+(t)).$$

The trajectory of the center is governed by the equation

$$\dot{x}_c(t) = Ax_c(t) + Bp(t) + Gq(t), x_c(t_0) = x_0.$$

The equation for the shape matrix of the external ellipsoid is

$$\begin{aligned}\dot{X}_l^+(t) &= AX_l^+(t) + X_l^+(t)A^T + \pi_l(t)X_l^+(t) + \frac{1}{\pi_l(t)}BPB^T \\ &- X_l^{+1/2}(t)S_l(GQG^T)^{1/2} - (GQG^T)^{1/2}S_l^T X_l^{+1/2}(t), \\ &X_l^+(t_0) = X_0,\end{aligned}$$

in which

$$\pi_l(t) = \frac{\langle l, \Phi(t_0, t)BPB^T \Phi^T(t_0, t)l\rangle^{1/2}}{\langle l, \Phi(t_0, t)X_l^+(t)\Phi^T(t_0, t)l\rangle^{1/2}},$$

and matrix S_l is orthogonal ($S_l S_l^T = I$), determined from the equation

$$S_l(CQG^T)^{1/2}\Phi^T(t_0, t)l = \frac{\langle l, \Phi(t_0, t)GQG^T\Phi^T(t_0, t)l\rangle^{1/2}}{\langle l, \Phi(t_0, t)X_l^+(t)\Phi^T(t_0, t)l\rangle^{1/2}} X_l^{+1/2}(t)\Phi^T(t_0, t)l.$$

Internal estimation

The equation for the shape matrix of the internal ellipsoid is

$$\begin{aligned}\dot{X}_l^-(t) &= AX_l^-(t) + X_l^-(t)A^T + X_l^{-1/2}(t)T_l(BPB^T)^{1/2} + \\ &\quad (BPB^T)^{1/2}T_l^T(t)X_l^{-1/2}(t) - \eta_l(t)X_l^-(t) - \frac{1}{\eta_l(t)}GQG^T,\\ X_l^-(t_0) &= X_0,\end{aligned}$$

in which

$$\eta_l(t) = \frac{\langle l, \Phi(t_0,t)CQG^T\Phi^T(t_0,t)l\rangle^{1/2}}{\langle l, \Phi(t_0,t)X_l^+(t)\Phi^T(t_0,t)l\rangle^{1/2}},$$

and matrix $T_l(t)$ is orthogonal, determined from the equation

$$T_l(t)(BPB^T)^{1/2}\Phi^T(t_0,t)l = \frac{\langle l, \Phi(t_0,t)BPB^T\Phi^T(t_0,t)l\rangle^{1/2}}{\langle l, \Phi(t_0,t)X_l^-(t)\Phi^T(t_0,t)l\rangle^{1/2}} X_l^{-1/2}(t)\Phi^T(t_0,t)l.$$

$$\bigcup_{\langle l,l\rangle=1} \mathcal{E}(x_c(t), X_l^-(t)) = \mathcal{X}(t,t_0,\mathcal{E}(x_0,X_0)) = \bigcap_{\langle l,l\rangle=1} \mathcal{E}(x_c(t), X_l^+(t)).$$

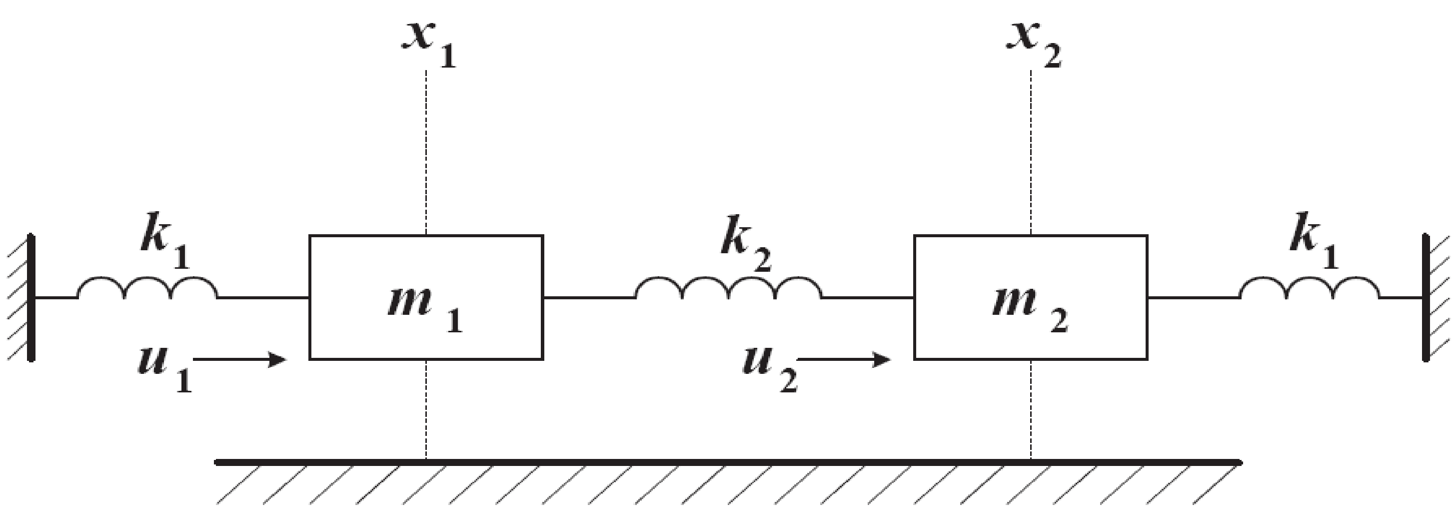
System without Disturbance

$$\begin{aligned} m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 &= u_1, \\ m_2\ddot{x}_2 - k_2 x_1 + (k_1 + k_2)x_2 &= u_2. \end{aligned}$$

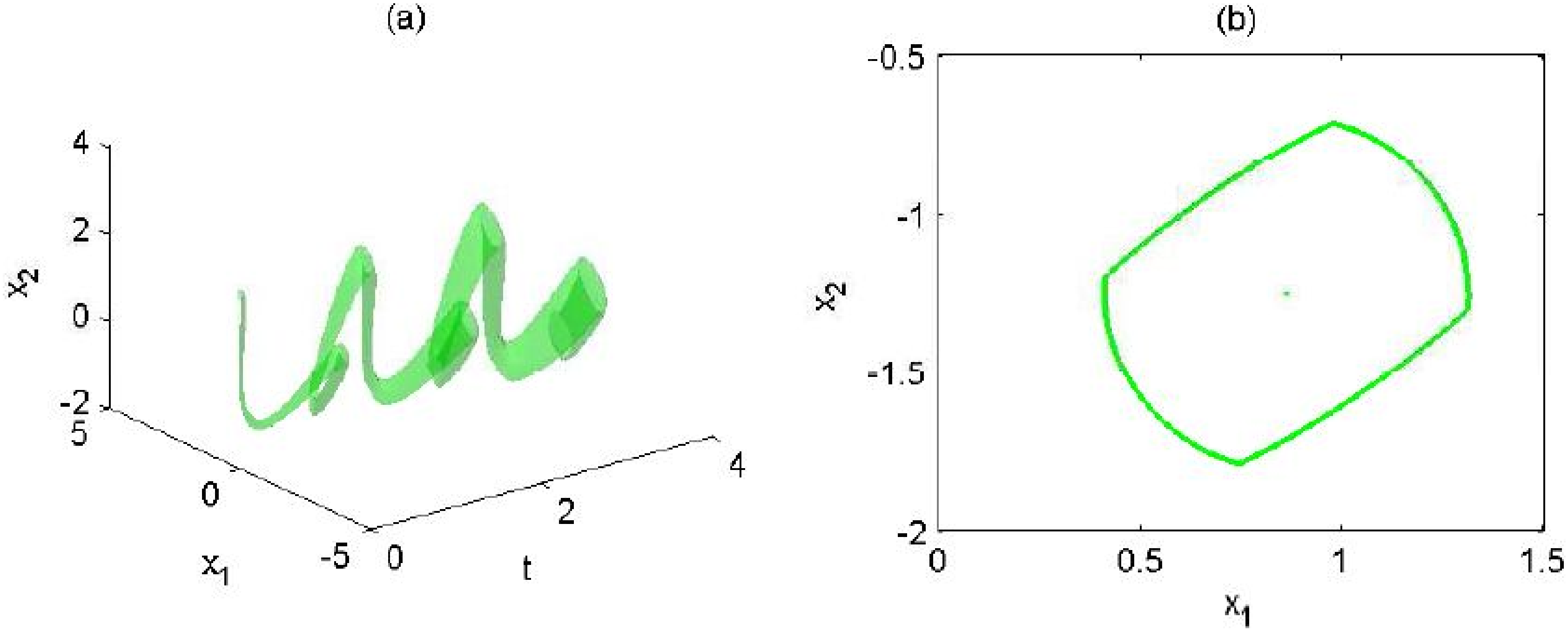
$$x_1(0) = 0,\ x_2(0) = 2.$$

$$[u_1 \ \ u_2]^T \in \mathcal{E}(0, I).$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_1+k_2}{m_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$



Spring-mass system without disturbance



System with Disturbance

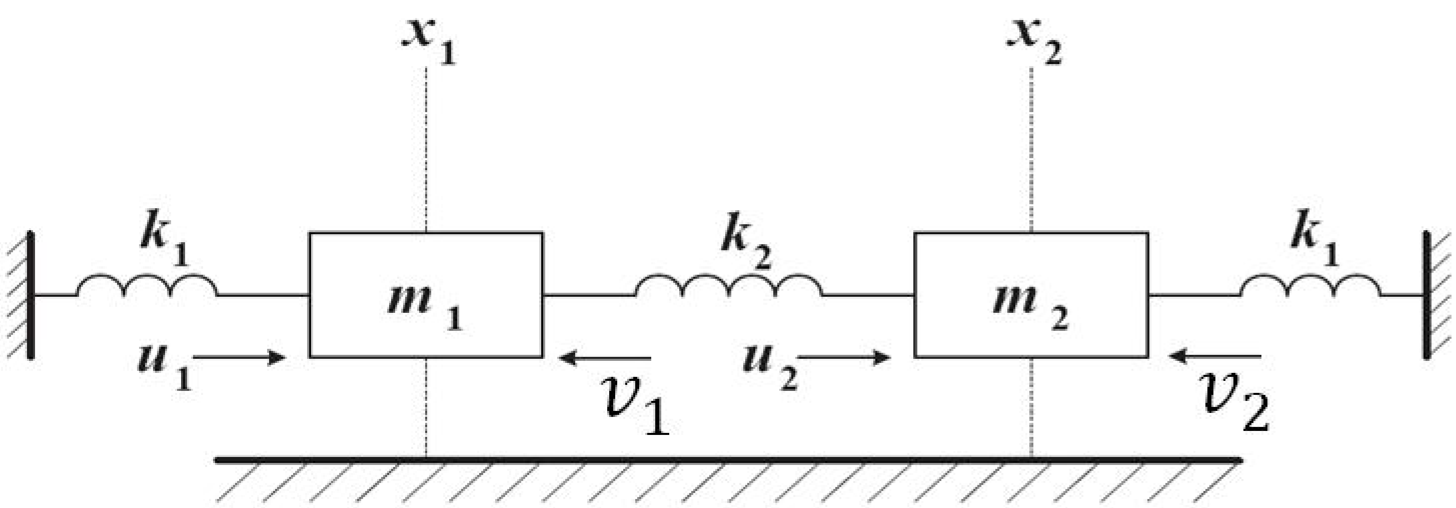
$$\begin{aligned} m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 &= u_1 + v_1 \\ m_2\ddot{x}_2 - k_2x_1 + (k_1+k_2)x_2 &= u_2 + v_2 \end{aligned}$$

$x_1(0) = 0,\ x_2(0) = 2.$

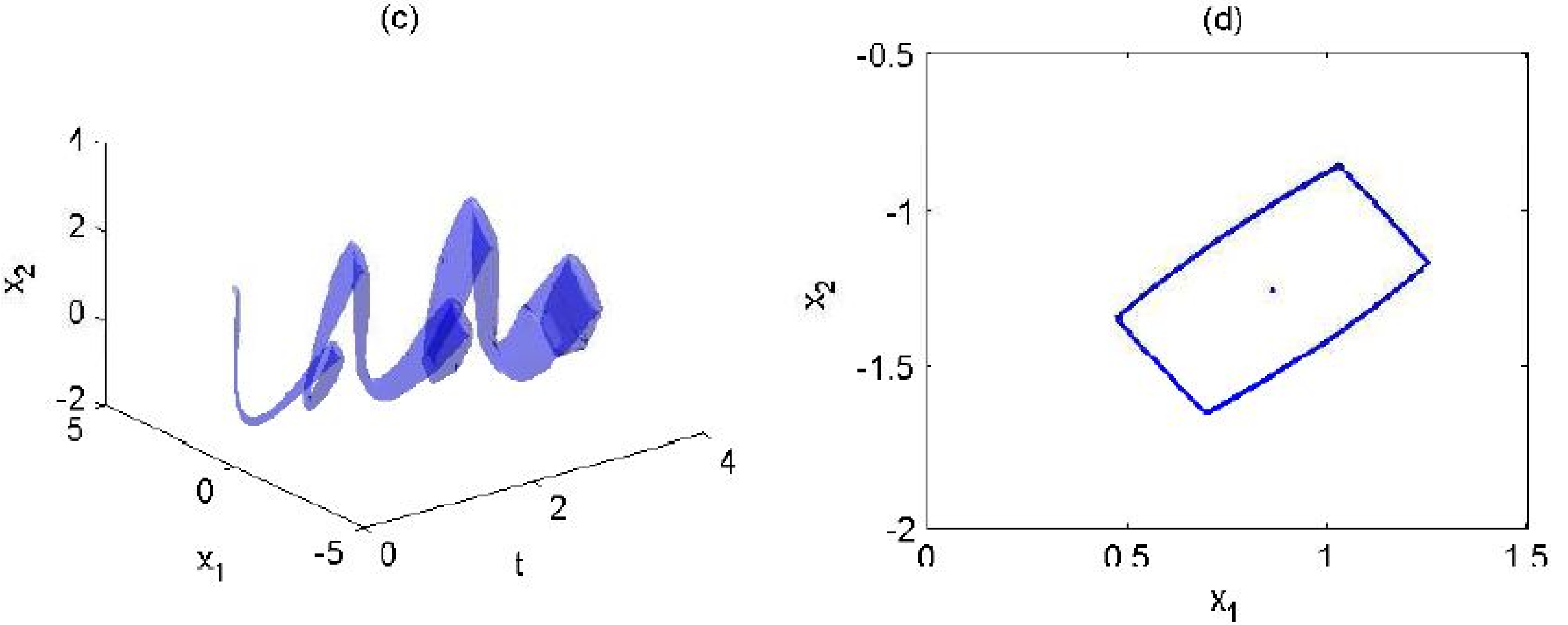
$[u_1 \ \ u_2]^T \in \mathcal{E}(0, I).$

$[v_1 \ \ v_2]^T \in \mathcal{E}(0, \frac{1}{4}I)$

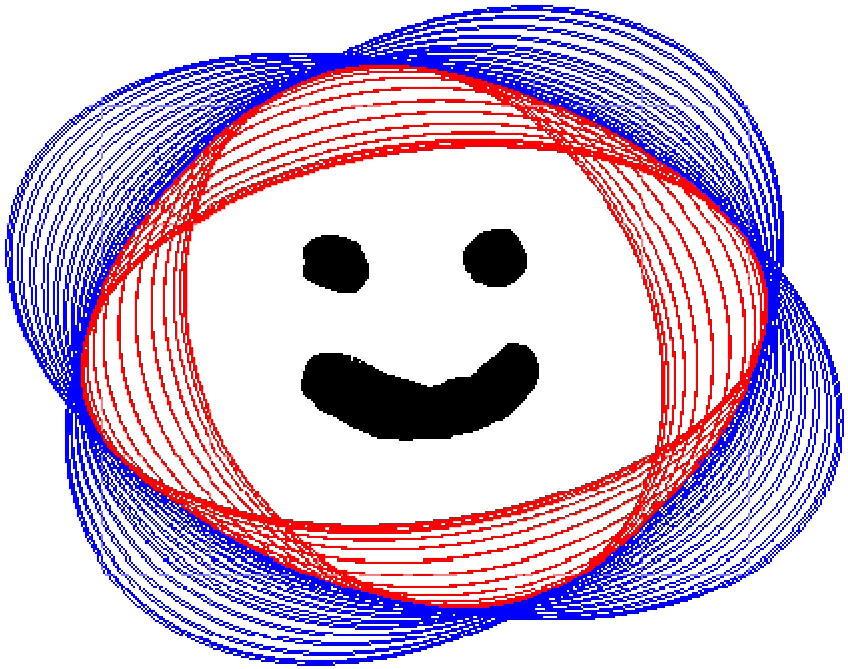
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & 0 & 0 \\ \frac{k_2}{m_2} & -\frac{k_1+k_2}{m_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$



Spring-mass system with disturbance



Thanks for attention



Enjoy ellipsoids!